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A NOTE ON MORITA EQUIVALENCE OF GROUP ACTIONS ON PRO-C^*-ALGEBRAS

MARIA JOIŢA

ABSTRACT. In this paper, we prove that two continuous inverse limit actions α and β of a locally compact group G on two pro-C^*-algebras A and B are strongly Morita equivalent if and only if there is a pro-C^*-algebra C such that A and B appear as two complementary full corners in C and there is a continuous inverse limit action γ of G on C which leaves A and B invariant and such that $\gamma|_A = \alpha$ and $\gamma|_B = \beta$. This generalizes a result of Combes [**3**].

1. Introduction and preliminaries. Pro-C^*-algebras are generalizations of C^*-algebras. Instead of being given by a single C^*-norm, the topology on a pro-C^*-algebra is defined by a directed family of C^*-seminorms. The $*$-algebra $C_{cc}([0,1])$ of all complex-valued continuous functions on $[0,1]$ with the topology of uniform convergence on the countable compact subsets of $[0,1]$ is a pro-C^*-algebra which is not topologically isomorphic with any C^*-algebra [**4**]. If X is a Hausdorff countably compactly generated topological space (that is, there is a countable family of compact spaces $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n \subseteq \cdots$ such that $X = \lim_{\substack{\rightarrow \\ n\in\mathbf{N}}} K_n$ [**9**]), then the $*$-algebra $C(X)$ of all continuous complex-valued functions on X equipped with the topology defined by the family of C^*-seminorms $\{p_{K_n}\}_n$, where

$$p_{K_n}(f) = \sup\{|f(x)|, x \in K_n\},$$

is a unital commutative metrizable pro-C^*-algebra [**9**, Proposition 5.7]. Other very nice examples of pro-C^*-algebras are presented in [**9**, Section 1]. In the literature, pro-C^*-algebras have been given different names such as b^*-algebras (Apostol), LMC^*-algebras (Lassner and Schmüdgen) or locally C^*-algebras (Inoue and Fragoulopoulou).

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By a *morphism of pro-C^*-algebras* we always mean a continuous morphism and by an *isomorphism of pro-C^*-algebras* a bijective morphism φ of pro-C^*-algebras such that φ^{-1} is also a morphism of pro-C^*-algebras.

In fact a pro-C^*-algebra can be identified up to an isomorphism of pro-C^*-algebras with an inverse limit of C^*-algebras. For a given pro-C^*-algebra A, the set $S(A)$ of all continuous C^*-seminorms on A is directed with the order $p \geq q$ if $p(a) \geq q(a)$ for all $a \in A$, and for each $p \in S(A)$, $\ker p = \{a \in A; p(a) = 0\}$ is a two-sided $*$-ideal of A. The quotient $*$-algebra $A/\ker p$, denoted by A_p, is a C^*-algebra in the C^*-norm induced by p, for each $p \in S(A)$ (see, for example, [**9**]). For $p, q \in S(A)$ with $p \geq q$ there is a canonical surjective morphism of C^*-algebras $\pi_{pq}^A : A_p \to A_q$ such that $\pi_{pq}^A \circ \pi_p^A = \pi_q^A$, where π_p^A is the canonical map from A to A_p. Then $\{A_p; \pi_{pq}^A\}_{p,q \in S(A), p \geq q}$ is an inverse system of C^*-algebras, and moreover, the map $\varphi : A \to \lim\limits_{\substack{\leftarrow \\ p \in S(A)}} A_p$ defined by $\varphi(a) = (\pi_p^A(a))_p$ is an isomorphism of pro-C^*-algebras.

Let G be a locally compact group and let A be a pro-C^*-algebra. An *action* of G on A is a morphism of groups $t \mapsto \alpha_t$ from G to $\mathrm{Aut}\,(A)$, the group of all isomorphisms of pro-C^*-algebras from A to A. The action α is *continuous* if the function $t \mapsto \alpha_t(a)$ from G to A is continuous for each $a \in A$.

The study of the group actions on pro-C^*-algebras is motivated by the following example. If (G, X) is a transformation group with X a Hausdorff compactly countably generated topological space (this means that there is a continuous map $(t, x) \mapsto t \cdot x$ from $G \times X$ to X such that $e \cdot x = x$ and $s \cdot (t \cdot x) = (st) \cdot x$ for all $s, t \in G$ and for all $x \in X$), then the map $\alpha_t : C(X) \to C(X)$ defined by $\alpha_t(f)(x) = f(t^{-1} \cdot x)$ is an isomorphism of pro-C^*-algebras for each $t \in G$, and moreover, the map $t \to \alpha_t(f)$ from G to $C(X)$ is a continuous action of G on $C(X)$. Therefore, the transformation group (G, X) induces a continuous action of G on $C(X)$.

An action α of G on A is an *inverse limit action* if we can write A as inverse limit $\lim\limits_{\substack{\leftarrow \\ \lambda \in \Lambda}} A_\lambda$ of C^*-algebras in such a way that there are actions α^λ of G on A_λ, $\lambda \in \Lambda$ such that $\alpha_t = \lim\limits_{\substack{\leftarrow \\ \lambda \in \Lambda}} \alpha_t^\lambda$ for all t in G [**10**, Definition 5.1].

A transformation group (G, X) with X a Hausdorff countably compactly generated topological space and G a compact group induces a continuous inverse limit action of G on $C(X)$ [**10**].

In [**5**] we introduced the notion of strong Morita equivalence for pro-C^*-algebras and proved that two metrizable pro-C^*-algebras A and B both possessing countable approximate unit are strongly Morita equivalent if and only if they are stably isomorphic. The notion of strong Morita equivalence for group actions on pro-C^*-algebras was introduced in [**8**]. In [**6**, Theorem 2.9] we showed that two pro-C^*-algebras A and B are strongly Morita equivalent if and only if there is a pro-C^*-algebras C such that A and B appear as complementary full corners in C. This extends a well-known result of Brown, Green and Rieffel [**2**, Theorem 1.1]. It is known that two continuous actions α and β of a locally compact group G on two C^*-algebras A and B are strongly Morita equivalent if and only if there is a C^*-algebra C such that A and B appear as two complementary full corners in C and there is a continuous action γ of G on C which leaves A and B invariant and such that $\gamma|_A = \alpha$ and $\gamma|_B = \beta$ [**3**]. In this paper we extend this result to the case of group continuous inverse limit actions on pro-C^*-algebras.

A *Hilbert A-module* is a complex vector space E which is also a right A-module, compatible with the complex algebra structure, equipped with an A-valued inner product $\langle\cdot,\cdot\rangle : E \times E \to A$ which is $\mathbf{C}$- and A-linear in its second variable and satisfies the following relations:

(1) $\langle\xi,\eta\rangle^* = \langle\eta,\xi\rangle$ for every $\xi, \eta \in E$;

(2) $\langle\xi,\xi\rangle \geq 0$ for every $\xi \in E$;

(3) $\langle\xi,\xi\rangle = 0$ if and only if $\xi = 0$

and which is complete with respect to the topology determined by the family of seminorms $\{\overline{p}_E\}_{p\in S(A)}$, where $\overline{p}_E(\xi) = \sqrt{p(\langle\xi,\xi\rangle)}, \xi \in E$.

A Hilbert A-module E is *full* if the linear space $\langle E, E\rangle$ generated by $\{\langle\xi,\eta\rangle; \xi,\eta \in E\}$ is dense in A.

Let E be a Hilbert A-module. For $p \in S(A)$, $\ker\overline{p}_E = \{\xi \in E; \overline{p}_E(\xi) = 0\}$ is a closed submodule of E and $E_p = E/\ker\overline{p}_E$ is a Hilbert A_p-module with $(\xi + \ker\overline{p}_E)\pi_p^A(a) = \xi a + \ker\overline{p}_E$ and $\langle\xi + \ker\overline{p}_E, \eta + \ker\overline{p}_E\rangle = \pi_p^A(\langle\xi,\eta\rangle)$. Moreover, the map $U : E \to$

$\lim\limits_{\substack{\leftarrow \\ p\in S(A)}} E_p$ defined by $U(\xi) = (\sigma_p^E(\xi))_p$ is an isomorphism of Hilbert modules (that is, U is a surjective linear map with the property that $\langle U(\xi), U(\eta)\rangle = \langle \xi, \eta\rangle$ for all $\xi, \eta \in E$).

A module morphism $T : E \to E$ is adjointable if there is a module morphism $T^* : E \to E$ such that $\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle$ for all $\xi, \eta \in E$. If T is an adjointable module morphism on E, then, for each $p \in S(A)$, there is a positive constant M_p such that $\overline{p}_E(T\xi) \leq M_p\overline{p}_E(\xi)$ for all $\xi \in E$.

The $*$-algebra $L(E)$ of all adjointable module morphisms on E is a pro-C^*-algebra with respect to the topology defined by the family of C^*-seminorms $\{\widetilde{p}_{L(E)}\}_{p\in S(A)}$, where

$$\widetilde{p}_{L(E)}(T) = \sup\{\overline{p}_E\,(T(\xi))\,; \xi \in E, \overline{p}_E\,(\xi) \leq 1\}.$$

For $\xi, \eta \in E$ the map $\theta_{\eta,\xi} : E \to E$ defined by $\theta_{\eta,\xi}(\zeta) = \eta\langle \xi, \zeta\rangle$ is an adjointable module morphism. The linear subspace of $L(E)$ spanned by $\{\theta_{\eta,\xi}; \xi, \eta \in E\}$ is denoted by $\Theta(E)$, and the closure of $\Theta(E)$ in $L(E)$ is denoted by $K(E)$.

A *generalized morphism* of Hilbert modules from E to E is a map $u : E \to E$ with the property that there is a morphism of pro-C^*-algebras $\alpha : A \to A$ such that

$$\langle u\,(\xi)\,, u\,(\eta)\rangle = \alpha\,(\langle \xi, \eta\rangle)$$

for all $\xi, \eta \in E$. A *generalized isomorphism* of Hilbert modules is a bijective map $u : E \to E$ such that u and u^{-1} are generalized morphisms of Hilbert modules.

An *action* of a locally compact group G on E is a morphism of groups $t \mapsto u_t$ from G to $\mathrm{Aut}\,(E)$, the group of all generalized isomorphisms of Hilbert modules from E to E. The action $t \mapsto u_t$ of G on E is *continuous* if the map $t \mapsto u_t(\xi)$ from G to E is continuous for each $\xi \in E$.

An action $t \mapsto u_t$ of G on E is an *inverse limit action* if we can write E as an inverse limit of Hilbert C^*-modules $\lim\limits_{\substack{\leftarrow \\ \lambda\in\Lambda}} E_\lambda$ in such a way that for each $t \in G$, $u_t = \lim\limits_{\substack{\leftarrow \\ \lambda\in\Lambda}} u_t^\lambda$, where $t \mapsto u_t^\lambda$ is an action of G on E_λ, $\lambda \in \Lambda$.

An action $t \mapsto u_t$ of a locally compact group G on a full Hilbert A-module E induces two actions $t \mapsto \alpha_t^u$ and $t \mapsto \beta_t^u$ of G on the pro-C^*-algebras A and $K(E)$ defined by

$$\alpha_t^u(\langle \xi, \eta \rangle) = \langle u_t(\xi), u_t(\eta) \rangle$$

for all $t \in G$ and for all $\xi, \eta \in E$, respectively

$$\beta_t^u(\theta_{\xi,\eta}) = \theta_{u_t(\xi), u_t(\eta)}$$

for all $t \in G$ and for all $\xi, \eta \in E$. Moreover, if $t \mapsto u_t$ is a continuous inverse limit action of G on E, then the actions of G on A and $K(E)$ induced by u are continuous inverse limit actions.

Two continuous actions $t \mapsto \alpha_t$ and $t \mapsto \beta_t$ of G on two pro-C^*-algebras A and B are *conjugate* if there is an isomorphism of pro-C^*-algebras $\varphi : A \to B$ such that $\alpha_t = \varphi^{-1} \circ \beta_t \circ \varphi$ for all $t \in G$.

Two continuous actions $t \mapsto \alpha_t$ and $t \mapsto \beta_t$ of a locally compact group G on two pro-C^*-algebras A and B are *strongly Morita equivalent* if there is a full Hilbert module E over A, and there is a continuous action $t \mapsto u_t$ of G on E such that the actions of G on A and $K(E)$ induced by u are conjugate with α, respectively β.

2. Main results. A pro-C^*-algebra A is a *full corner* in a given pro-C^*-algebra C, if there is a projection e in the multiplier algebra $M(C)$ of C such that $A = eCe$ and CeC is dense in C.

Proposition 2.1. *Let G be a locally compact group, C a pro-C^*-algebra and $t \mapsto \gamma_t$ a continuous action of G on C. If A is a full corner in C invariant under γ, then the actions $t \mapsto \gamma_t$ and $t \mapsto \gamma_t|_A$ of G on C and A are strongly Morita equivalent.*

Proof. By [**6**, Proposition 2.8], the pro-C^*-algebras A and C are strongly Morita equivalent. Moreover, if e is a projection in $M(C)$ such that $A = eCe$ and CeC is dense in C, then the Hilbert A-module Ce implements a strong Morita equivalence between A and C.

Let $t \in G$. Since $\gamma_t \in \mathrm{Aut}(C)$, there is $\overline{\gamma_t} \in \mathrm{Aut}(M(C))$ such that $\overline{\gamma_t}|_C = \gamma_t$. We will show that $\overline{\gamma_t}(e) \in A$. If $\{e_i\}_{i \in I}$ is an approximate unit of C [**9**, Proposition 3.11], then the net $\{ee_ie\}_{i\in I}$ converges to e and so the net $\{\gamma_t(ee_ie)\}_{i\in I}$ converges to $\overline{\gamma_t}(e)$. But $\gamma_t(ee_ie)\in A$, and then $\overline{\gamma_t}(e)\in A$. Thus we have

$$\gamma_t(ce)e=\gamma_t(c)\overline{\gamma_t}\,(e)\,e=\gamma_t(c)\overline{\gamma_t}\,(e)=\gamma_t(ce)$$

and so $\gamma_t(ce)\in Ce$ for each $c\in C$. Therefore, we can consider the linear map $u_t:Ce\to Ce$ defined by $u_t(ce)=\gamma_t(ce)$. Since

$$\langle u_t(ce),u_t(de)\rangle=\langle\gamma_t(ce),\gamma_t(de)\rangle=\gamma_t\left(ec^*de\right)=\gamma_t|_A\left(\langle ce,de\rangle\right)$$

for all $c,d\in C$ and since u_t is invertible and $(u_t)^{-1}=u_{t^{-1}}$, $u_t\in\operatorname{Aut}(Ce)$. It is not difficult to check that $t\mapsto u_t$ is an action of G on Ce. Moreover, since the map $t\mapsto\gamma_t(c)$ from G to C is continuous for each $c\in C$, the map $t\mapsto u_t(ce)$ from G to Ce is continuous for each $c\in C$. Therefore, $t\mapsto u_t$ is a continuous action of G on Ce. Since

$$\langle u_t(ce),u_t(de)\rangle=\gamma_t|_A\left(\langle ce,de\rangle\right)$$

for all $t\in G$ and for all $c,d\in C$, $\alpha^u=\gamma|_A$.

Let φ be the isomorphism from $K(Ce)$ onto C defined by $\varphi(\theta_{ce,de})=ced^*$. Then

$$\begin{aligned}\left(\varphi\circ\beta_t^u\right)\left(\theta_{ce,de}\right)&=\varphi\left(\theta_{u_t(ce),u_t(de)}\right)=u_t(ce)u_t(de)^*\\&=\gamma_t(ce)\gamma_t(de)^*=\gamma_t(ced^*)=\left(\gamma_t\circ\varphi\right)\left(\theta_{ce,de}\right)\end{aligned}$$

for all $t\in G$ and for all $c,d\in C$, and so the actions β^u and γ are conjugate. Thus we proved that the actions $\gamma|_A$ and γ are strongly Morita equivalent. □

Let α be a continuous inverse limit action of a locally compact group G on a pro-C^*-algebra A. The vector space $C_c(G,A)$ of all continuous functions from G to A with compact support becomes a $*$-algebra with convolution

$$\left(f\times h\right)(s)=\int_G f(t)\alpha_t\left(h(t^{-1}s)\right)\,dt,$$

where dt denotes the Haar measure on G, as product and involution defined by

$$f^\sharp(t)=\Delta(t)^{-1}\alpha_t\left(f(t^{-1})^*\right)$$

where Δ is the modular function on G [**7**].

The Hausdorff completion of $C_c(G, A)$ with respect to the topology defined by the family of submultiplicative $*$-seminorms $\{N_p\}_{p\in S(A)}$, where

$$N_p(f) = \int_G p(f(s))\, ds$$

is a complete locally m-convex $*$-algebra $L^1(G, A, \alpha)$ with bounded approximate unit [**7**]. The crossed product of A by the action α, denoted by $G \times_\alpha A$, is the enveloping pro-C^*-algebra of $L^1(G, A, \alpha)$ [**4, 7**].

Corollary 2.2. *Let G be a locally compact group, C a pro-C^*-algebra and $t \mapsto \gamma_t$ a continuous inverse limit action of G on C. If A is a full corner in C invariant under γ, then the pro-C^*-algebras $G \times_\gamma C$ and $G \times_{\gamma|_A} A$ are isomorphic.*

Proof. Clearly, $t \mapsto \gamma_t|_A$ is a continuous inverse limit action of G on A and so there is $G \times_{\gamma|_A} A$. By Proposition 2.1, the actions $t \mapsto \gamma_t$ and $t \mapsto \gamma_t|_A$ of G on C and A are strongly Morita equivalent, and then by [**8**, Theorem 5.6], the pro-C^*-algebras $G \times_\gamma C$ and $G \times_{\gamma|_A} A$ are isomorphic. $\square$

Corollary 2.3. *Let $t \mapsto \alpha_t$ be a continuous inverse limit action of G on A. Then the pro-C^*-algebras $G\times_\alpha A$ and $G\times_\gamma M_2(A)$, where $t \mapsto \gamma_t$ is the action of G on the pro-C^*-algebra $M_2(A)$ of all 2×2 matrices over A defined by $\gamma_t([a_{ij}]_{i,j=1}^2) = [\alpha_t(a_{ij})]_{i,j=1}^2$, are isomorphic.*

Proof. Clearly, A can be identified with a full corner of $M_2(A)$ invariant under γ. Then, by Proposition 2.1 the actions α and γ of G on A respectively $M_2(A)$ are strongly Morita equivalent, and since α is an inverse limit action, by [**8**, Remark 4.7], γ is an inverse limit action. Therefore, the pro-C^*-algebras $G \times_\alpha A$ and $G \times_\gamma M_2(A)$ are isomorphic. $\square$

Let E be a full Hilbert module over a pro-C^*-algebra A. The *linking algebra* $\mathcal{L}(E)$ of E is the pro-C^*-subalgebra of $L(A \oplus E)$ generated by $\{L_{a,\xi,\eta,T}; a \in A, \xi, \eta \in E, T \in K(E)\}$, where $L_{a,\xi,\eta,T}$ is the module morphism on $A \oplus E$ defined by

$$L_{a,\xi,\eta,T}(b \oplus \zeta) = (ab + \langle \xi, \zeta \rangle) \oplus (\eta b + T(\zeta)).$$

Moreover,

$$\mathcal{L}(E) = \lim_{\substack{\leftarrow \\ p \in S(A)}} \mathcal{L}(E_p)$$

where $\mathcal{L}(E_p)$ is the linking algebra of E_p for each $p \in S(A)$, up to an isomorphism of pro-C^*-algebras [**6**]. But $\mathcal{L}(E_p) = K(A_p \oplus E_p)$ for each $p \in S(A)$ [**11**], and then, by [**9**, Proposition 4.7], $\mathcal{L}(E)$ can be identified with $K(A \oplus E)$.

Proposition 2.4. *Let G be a locally compact group, and E a full Hilbert module over a pro-C^*-algebra A. Any action $t \mapsto u_t$ of G on E induces a unique action $t \mapsto \gamma_t^u$ of G on the linking algebra $\mathcal{L}(E)$ of E such that*

$$\gamma_t^u (L_{a,\xi,\eta,T}) = L_{\alpha_t^u(a), u_t(\xi), u_t(\eta), \beta_t^u(T)}$$

for all $a \in A$, $\xi, \eta \in E$, $T \in K(E)$ and for all $t \in G$. Moreover, if $t \mapsto u_t$ is a continuous inverse limit action, then $t \mapsto \gamma_t^u$ is a continuous inverse limit action.

Proof. Clearly, if there is an action $t \mapsto \gamma_t^u$ of G on $\mathcal{L}(E)$ such that

$$\gamma_t^u (L_{a,\xi,\eta,T}) = L_{\alpha_t^u(a), u_t(\xi), u_t(\eta), \beta_t^u(T)}$$

for all $a \in A$, $\xi, \eta \in E$, $T \in K(E)$ and for all $t \in G$, then this action is unique.

Let $t \in G$. The map $w_t^u : A \oplus E \to A \oplus E$ defined by

$$w_t^u (a \oplus \xi) = \alpha_t^u(a) \oplus u_t (\xi)$$

is a generalized morphism of Hilbert modules, since

$$\begin{aligned} \langle w_t^u (a \oplus \xi), w_t^u (b \oplus \eta) \rangle &= \langle \alpha_t^u(a), \alpha_t^u(b) \rangle + \langle u_t (\xi), u_t (\eta) \rangle \\ &= \alpha_t^u (\langle a \oplus \xi, b \oplus \eta \rangle) \end{aligned}$$

for all $a, b \in A$ and for all $\xi, \eta \in E$, and since α_t^u is an isomorphism of pro-C^*-algebras. Moreover, since w_t^u is invertible and $(w_t^u)^{-1} = w_{t^{-1}}^u$,

w_t^u is a generalized isomorphism of Hilbert modules. It is not difficult to check that $t \mapsto w_t^u$ is an action of G on $A \oplus E$.

Since $A \oplus E$ is a full Hilbert A-module, the action $t \mapsto w_t^u$ of G on $A \oplus E$ induces an action $t \mapsto \gamma_t^u$ of G on $K(A \oplus E)$ such that

$$\gamma_t^u \left(\theta_{a\oplus\xi, b\oplus\eta}\right) = \theta_{w_t^u(a\oplus\xi), w_t^u(b\oplus\eta)} = \theta_{\alpha_t^u(a)\oplus u_t(\xi), \alpha_t^u(b)\oplus u_t(\eta)}$$

for all $a, b \in A$, for all $\xi, \eta \in E$, and for all $t \in G$.

Let $t \in G$, $a \in A$, $\xi, \eta \in E$ and $T \in K(E)$. We will show that

$$\gamma_t^u \left(L_{a,\xi,\eta,T}\right) = L_{\alpha_t^u(a), u_t(\xi), u_t(\eta), \beta_t^u(T)}.$$

For this, let $\{e_i\}_i$ be an approximate unit for A. From

$$\widetilde{p}_{L(A\oplus E)}(L_{a,0,0,0} - \theta_{a\oplus 0, e_i\oplus 0}) \leq p\,(a - ae_i)$$
$$\widetilde{p}_{L(A\oplus E)}\left(L_{0,\xi,0,0} - \theta_{e_i\oplus 0, 0\oplus\xi}\right) \leq \overline{p}_E\left(\xi - \xi e_i\right)$$

and

$$\widetilde{p}_{L(A\oplus E)}\left(L_{0,0,\eta,0} - \theta_{0\oplus\eta, e_i\oplus 0}\right) \leq \overline{p}_E\left(\eta - \eta e_i\right)$$

for all $p \in S(A)$ and for all $i \in I$, and taking into account that γ_t^u, α_t^u and u_t are continuous, $ae_i \to a$, $\xi e_i \to \xi$ and $\eta e_i \to \eta$, we conclude that

$$\gamma_t^u \left(L_{a,\xi,\eta,0}\right) = L_{\alpha_t^u(a), u_t(\xi), u_t(\eta), 0}.$$

If $T \in K(E)$, then there is a net $\{\sum_{k\in I_j} \theta_{\xi_k,\eta_k}\}_j$ in $\Theta(E)$ which converges to T. From

$$\widetilde{p}_{L(A\oplus E)}\left(L_{0,0,0,T} - \sum_{k\in I_j} \theta_{0\oplus\xi_k, 0\oplus\eta_k}\right) \leq \widetilde{p}_{L(E)}\left(T - \sum_{k\in I_j} \theta_{\xi_k,\eta_k}\right)$$

for all $p \in S(A)$, and taking into account that γ_t^u and β_t^u are continuous and

$$\begin{aligned}\sum_{k\in I_j} \gamma_t^u \left(\theta_{0\oplus\xi_k, 0\oplus\eta_k}\right) &= \sum_{k\in I_j} \theta_{0\oplus u_t(\xi_k), 0\oplus u_t(\eta_k)} \\ &= \sum_{k\in I_j} L_{0,0,0,\theta_{u_t(\xi_k), u_t(\eta_k)}} \\ &= \sum_{k\in I_j} L_{0,0,0,\beta_t^u\left(\theta_{\xi_k,\eta_k}\right)},\end{aligned}$$

we deduce that

$$\gamma_t^u\left(L_{0,0,0,T}\right) = L_{0,0,0,\beta_t^u(T)}.$$

Thus we have

$$\begin{aligned}\gamma_t^u\left(L_{a,\xi,\eta,T}\right) &= \gamma_t^u\left(L_{a,\xi,\eta,0}\right) + \gamma_t^u\left(L_{0,0,0,T}\right)\\ &= L_{\alpha_t^u(a),u_t(\xi),u_t(\eta),0} + L_{0,0,0,\beta_t^u(T)}\\ &= L_{\alpha_t^u(a),u_t(\xi),u_t(\eta),\beta_t^u(T)}.\end{aligned}$$

If u is a continuous inverse limit action, then we can suppose that $u_t = \lim\limits_{\substack{\leftarrow\\ p\in S(A)}} u_t^p$ [**8**, Remark 3.6], where $t \mapsto u_t^p$ is a continuous action of G on E_p for each $p \in S(A)$. Let $p \in S(A)$, and let $t \mapsto w_t^{u^p}$ be the action of G on $A_p \oplus E_p$ induced by u^p. Since

$$\begin{aligned}&\left\| w_t^{u^p}\left(a_p \oplus \xi_p\right) - a_p \oplus \xi_p \right\|_{A_p\oplus E_p}\\ &\qquad = \left\| \alpha_t^{u^p}\left(a_p\right) \oplus u_t^p\left(\xi_p\right) - a_p \oplus \xi_p \right\|_{A_p\oplus E_p}\\ &\qquad \leq \left\| \alpha_t^{u^p}\left(a_p\right) - a_p \right\|_{A_p} + \left\| u_t^p\left(\xi_p\right) - \xi_p \right\|_{E_p}\end{aligned}$$

and since the actions $t \mapsto u_t^p$ and $t \mapsto \alpha_t^{u^p}$ of G on E_p and A_p are continuous, the map $t \mapsto w_t^{u^p}(a_p \oplus \xi_p)$ from G to $A_p \oplus E_p$ is continuous for each $a_p \oplus \xi_p \in A_p \oplus E_p$. Therefore, $t \mapsto w_t^{u^p}$ is a continuous action of G on $A_p \oplus E_p$.

It is not difficult to check that $(w_t^{u^p})_p$ is an inverse system of generalized isomorphisms of Hilbert C^*-modules for each $t \in G$ and $t \mapsto \lim\limits_{\substack{\leftarrow\\ p\in S(A)}} w_t^{u^p}$ is a continuous inverse limit action of G on $A \oplus E$. Moreover, $w_t^u = \lim\limits_{\substack{\leftarrow\\ p\in S(A)}} w_t^{u^p}$ for each $t \in G$. By [**8**, Proposition 3.8], the action γ^u of G on $K(A \oplus E)$ induced by w^u is a continuous inverse limit action, and moreover, $\gamma_t^u = \lim\limits_{\substack{\leftarrow\\ p\in S(A)}} \gamma_t^{u^p}$ for each $t \in G$, where γ^{u^p} is the action of G of $\mathcal{L}(E_p)$ induced by u^p. □

Remark 2.5. Let G be a locally compact group, E a full Hilbert A-module, and $t \mapsto u_t$ an action of G on E.

(1) Since the map $a \mapsto L_{a,0,0,0}$ from A to $\mathcal{L}(E)$ identifies A with a pro-C^*-subalgebra of $\mathcal{L}(E)$ and $\gamma_t^u(L_{a,0,0,0}) = L_{\alpha_t^u(a),0,0,0}$ for all $a \in A$

and for all $t \in G$, the restriction of γ^u to A can be identified with the action of G on A induced by u.

(2) Since the map $T \mapsto L_{0,0,0,T}$ from $K(E)$ to $\mathcal{L}(E)$ identifies $K(E)$ with a pro-C^*-subalgebra of $\mathcal{L}(E)$ and $\gamma_t^u(L_{0,0,0,T}) = L_{0,0,0,\beta_t^u(T)}$ for all $T \in K(E)$ and for all $t \in G$, the restriction of γ^u to $K(E)$ can be identified with the action of G on $K(E)$ induced by u.

Recall that two corners eCe and fCf in the pro-C^*-algebra C are *complementary* if $e + f = 1_{M(C)}$.

The following theorem is a version of [**6**, Theorem 2.9] for continuous inverse limit group action on pro-C^*-algebras.

Theorem 2.6. *Let G be a locally compact group, and let $t \mapsto \alpha_t$ and $t \mapsto \beta_t$ be two continuous inverse limit actions of G on two pro-C^*-algebras A and B. Then the actions α and β are strongly Morita equivalent if and only if there is a pro-C^*-algebra C such that A and B appear as two complementary full corners in C and there is a continuous inverse limit action $t \mapsto \gamma_t$ of G on C such that A and B are invariant under γ and the actions $t \mapsto \gamma_t|_A$ and $t \mapsto \gamma_t|_B$ of G on A and B can be identified with α, respectively β.*

Proof. First we suppose that α and β are strongly Morita equivalent. Let (E, u) be the pair consisting of a full Hilbert A-module and a continuous action of G on E which implements a strong Morita equivalence between α and β. Let $C = \mathcal{L}(E)$, and let γ^u be the action of G on C induced by u. By [**6**, Theorem 2.9], A and B are isomorphic with two complementary full corners in C, and by Proposition 2.4 and Remark 2.5, $t \mapsto \gamma_t^u$ is a continuous inverse limit action of G on C such that identifying A and B with corners in C, $\gamma^u|_A = \alpha$ and $\gamma^u|_B = \beta$.

Conversely, we suppose that there is a pro-C^*-algebra C such that A and B appear as two complementary full corners in C, and there is a continuous inverse limit action $t \mapsto \gamma_t$ of G on C such that A and B are invariant under γ and the actions $t \mapsto \gamma_t|_A$ and $t \mapsto \gamma_t|_B$ of G on A and B can be identified with α, respectively β. Then, by Proposition 2.1 the actions $t \mapsto \gamma_t$ and $t \mapsto \gamma_t|_A$ of G on C respectively A are strongly Morita equivalent as well as the actions $t \mapsto \gamma_t$ and $t \mapsto \gamma_t|_B$, and since the strong Morita equivalence is an equivalence relation [**8**, Proposition 4.13], the actions $t \mapsto \gamma_t|_A$ and $t \mapsto \gamma_t|_B$ are strongly Morita equivalent. Therefore, the actions α and β are strongly Morita equivalent. □

Corollary 2.7. *Let G be a compact group, and let $t \mapsto \alpha_t$ and $t \mapsto \beta_t$ be two actions of G on two pro-C^*-algebras A and B such that the maps $(t, a) \mapsto \alpha_t(a)$ from $G \times A$ to A and $(t, b) \mapsto \beta_t(b)$ from $G \times B$ to B are jointly continuous. Then the actions α and β are strongly Morita equivalent if and only if there is a pro-C^*-algebra C such that A and B appear as two complementary full corners in C and there is an action $t \mapsto \gamma_t$ of G on C with the property that the map $(t, c) \mapsto \gamma_t(c)$ from $G \times C$ to C is jointly continuous and such that A and B are invariant under γ and the actions $t \mapsto \gamma_t|_A$ and $t \mapsto \gamma_t|_B$ of G on A and B can be identified with α, respectively β.*

Proof. By [**10**, Lemma 5.2], the actions α, β and γ of G on A, B and C are continuous inverse limit actions and apply Theorem 2.6. □

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Department of Mathematics, Faculty of Chemistry, University of Bucharest, Bd. Regina Elisabeta nr. 4-12, Bucharest, Romania
Email address: m_joita@yahoo.com, maria.joita@g.unibuc.ro