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STEADY STATE SOLUTIONS OF A LOGISTIC EQUATION WITH NONLINEAR BOUNDARY CONDITIONS

ROBERT S. CANTRELL, CHRIS COSNER AND SALOMÉ MARTÍNEZ

ABSTRACT. We consider the diffusive logistic equation with nonlinear boundary conditions

$$\begin{cases} \Delta u + \lambda u(1-u) = 0 & x \in \Omega, \\ \alpha(u)(\partial u/\partial \nu) + (1-\alpha(u))u = 0 & x \in \partial\Omega, \\ u \geq 0 & \text{for } x \in \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^N$ with ν its outer unit normal and λ is a positive parameter. The boundary conditions of the equation considered are nonlinear, with the function α satisfying $\alpha([0,1]) \subset [0,1]$ and increasing. In this work we will study the case $\alpha(0) = 0$, $\alpha'(0) > 0$, which implies that, as λ varies, the above equation has two continua of solutions, one having Dirichlet boundary conditions, and another one in which each solution is positive at the boundary. We show that the second continuum of solutions may contain infinitely many solutions for a fixed value of λ.

1. Introduction. The purpose of this article is to continue the study of the logistic equation with nonlinear boundary condition

$$\begin{cases} \Delta u + \lambda u(1-u) = 0 & x \in \Omega, \\ \alpha(u)(\partial u/\partial \nu) + (1-\alpha(u))u = 0 & x \in \partial\Omega, \\ u \geq 0 & \text{for } x \in \Omega. \end{cases} \tag{1.1}$$

The domain $\Omega \subset \mathbf{R}^N$ is bounded, ν denotes its outer unit normal, and λ is a positive parameter. The function α satisfies $\alpha([0,1]) \subset [0,1]$ and is smooth and increasing. The solutions of (1.1) are the steady state solutions of

$$\begin{cases} (\partial u/\partial t) = \Delta u + \lambda u(1-u) & x \in \Omega,\ t > 0 \\ \alpha(u)(\partial u/\partial \nu) + (1-\alpha(u))u = 0 & x \in \partial\Omega,\ t > 0 \\ u(x,0) \geq 0 \text{ for } x \in \Omega. \end{cases} \tag{1.2}$$

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The above evolution problem has been used to model the dynamics of a population occupying a patch Ω in which the tendency of individuals to stay in the patch when they reach the boundary increases with the population density u. In the model behavior at the patch boundary corresponds to having $\alpha(u)$ increasing. A derivation of the model and a discussion of its biological interpretation are given in [**1**]. Since the carrying capacity within the patch is scaled to 1, the model is relevant to the biological application only for $0 \leq u \leq 1$. Note that by the maximum principle solutions (1.1) can have a maximum that is greater than 1 only on $\partial\Omega$, and only if $\alpha(u) > 1$ at the point where the maximum occurs. However, having $\alpha > 1$ on part of $\partial\Omega$ implies that part of the boundary serves as a source of new individuals entering the patch, while the situation we want to model is that of an isolated patch where the rate at which individuals leave the patch depends on the density within it but where there is no immigration from elsewhere. Thus, we will focus on solutions with $0 \leq u \leq 1$.

As was shown in [**2, 3**], the structure of the set of solutions of ((1.1) is greatly affected by the behavior of the function α near $u = 0$ and $u = 1$, and in particular (1.1) may have multiple solutions for some values of λ. In contrast, the uniqueness of the positive solution has been shown for similar equations under different assumptions on the nonlinear boundary conditions; see [**6**]. As in [**2**] we note that when $\alpha(0) = 0$ then the boundary condition of (1.1) can be written as

$$u(\beta(u)\frac{\partial u}{\partial \nu} + 1 - \alpha(u)) = 0 \quad \text{on } \partial\Omega,$$

with $\beta(u) = \alpha(u)/u$, for $u > 0$ and $\beta(0) = \alpha'(0)$. Thus, we can distinguish two distinct sets of nontrivial solutions. One consists of the solutions of the Dirichlet problem, which we denote by $\mathcal{C}_D = \{(\overline{u}_\lambda, \lambda)/\lambda > \lambda_0^1\}$, where λ_0^1 is the principal eigenvalue of

$$-\Delta\psi + \lambda\psi = 0 \quad \text{in } \Omega,\ \psi = 0 \text{ on } \partial\Omega, \tag{1.3}$$

and $\overline{u}_\lambda$ is the solution of

$$-\Delta u + \lambda u(1-u) = 0 \quad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega. \tag{1.4}$$

It is well known that there is a unique positive solution for the Dirichlet problem for each $\lambda > \lambda_0^1$. The other set, if it exists, is denoted by $\mathcal{C}_1$

and consists of the solutions of

$$\text{(1.5)} \qquad \begin{cases} \Delta u + \lambda u(1-u) = 0 & x \in \Omega, \\ \beta(u)(\partial u/\partial \nu) + 1 - \alpha(u) = 0 & \text{for } x \in \partial\Omega, \\ 0 \le u \le 1 & \text{in } \Omega. \end{cases}$$

In [**2**] was proved that the branch $\mathcal{C}_D$ does not intersect $\mathcal{C}_1$ if and only if Ω is not a ball. Indeed, in the case $\Omega = B(0,1)$ set $\gamma(\lambda) = -(\partial \overline{u}_\lambda/\partial\nu)$ at $|x| = 1$. We have that γ is increasing in (λ_0^1, ∞), $\gamma(\lambda) \to 0$ as $\lambda \to \lambda_0^1$ and $\gamma(\lambda) \to \infty$ as $\lambda \to \infty$; therefore there exists a unique λ^* such that $\overline{u}_{\lambda^*}$ is a solution of (1.5), namely, $\gamma(\lambda^*) = 1/\beta(0)$.

Observe that when $\alpha(1) = 1$ and $\alpha'(0) > 0$ we have that $u \equiv 1$ is a solution of (1.5) for any λ. Moreover, as it was proved in [**2**], we have that a branch of solutions of (1.5) bifurcate from $(\lambda_0, 1)$, where the value λ_0 is defined as the principal eigenvalue of

$$\Delta\phi - \lambda_0\phi = 0, \quad \text{in } \Omega,\ \frac{\partial\phi}{\partial\nu} - \alpha'(1)\phi = 0 \text{ on } \partial\Omega.$$

Moreover, this bifurcating branch can be extended to a continuum of solutions denoted by $\mathcal{C}_1$ of (1.1) and of (1.5). The set $\mathcal{C}_1$ always meets the trivial branch $u \equiv 0$ when $\alpha(0) > 0$, as was shown in [**3**].

The purpose of this paper is to further examine the behavior of the set of solutions of (1.1) in the case $\alpha(0) = 0$. Our main results are that for λ sufficiently large the only nontrivial solutions are the Dirichlet solutions and $u \equiv 1$, but for some choices of α there can be infinitely many nontrivial solutions for some fixed value of λ.

Reaction-diffusion equations with nonlinear boundary conditions have been studied by various authors. We will not attempt to give an overall review of the literature, but some additional recent references are [**5, 7–10**].

Throughout this paper we will assume that:

(H1) $\alpha(0) = 0$ and $\alpha'(0) > 0$.

In this paper we will sometimes consider the one-dimensional case of (1.5), i.e., $\Omega = (-L, L)$. In that case we can write (1.5) as

$$
\begin{aligned}
&u'' + \lambda u(1-u) = 0 \text{ in } (-L, L), \quad 0 \le u \le 1, \\
&u'(L) = -\left(\frac{1-\alpha(u(L))}{\beta(u(L))}\right), \\
&u'(-L) = \left(\frac{1-\alpha(u(-L))}{\beta(u(-L))}\right).
\end{aligned} \tag{1.6}
$$

2. Asymptotic behavior of nontrivial solutions for λ large. In this section we will show that if $(u_{\lambda_n}, \lambda_n)$ is a sequence of solutions of (1.5) with $\lambda_n \to \infty$, then $u_{\lambda_n} \to 1$ uniformly in $\overline{\Omega}$ as $n \to \infty$. Moreover, when $\alpha(1) = 1$ we have that $u_{\lambda_n} \equiv 1$ for n large.

Proposition 1. *Suppose that u_{λ_n} is a solution of (1.5) with $\lambda_n \to \infty$ as $n \to \infty$. Then $u_{\lambda_n} \to 1$ uniformly in $\overline{\Omega}$.*

Proof. To simplify notation we denote $u_n \equiv u_{\lambda_n}$. We will proceed by contradiction, i.e., we suppose that for all sufficiently small $\delta > 0$ there exists a sequence $\lambda_n \to \infty$ and $z_n \in \overline{\Omega}$ such that $u_n(z_n) \le 1 - \delta$. Without loss of generality we can suppose that $z_n \to z$.

For $r > 0$ we set $\mu_0 > 0$ to be the principal eigenvalue of

$$\Delta\psi + \mu\psi = 0 \quad \text{in } B(0,r),\ \psi = 0 \text{ on } \partial B(0,r), \tag{2.1}$$

with normalized principal eigenvector $\psi_0 > 0$, with $\max_{y \in B(0,r)} \psi_0(y) = \psi_0(0) = 1 - \delta/2$. Observe that if n is large enough, then ψ_0 satisfies

$$\Delta\psi_0 + \lambda_n \psi_0(1-\psi_0) \ge 0, \quad \text{in } B(0,r).$$

Thus, if $B(x,r) \subset \Omega$, then by a simple application of the strong maximum principle, for n large we have $u_n(x) \ge \psi_0(0) = 1 - \delta/2$. Hence this implies that $z \in \partial\Omega$ and that there exists $x_n \in \Omega$ such that $u_n(x_n) = 1 - \delta$.

Suppose that $\sqrt{\lambda_n}\,\mathrm{dist}\,(x_n, \partial\Omega) \to \infty$. In this case set $r > 0$ and μ_0 as above, and we define $v_n(y) = u_n(x_n + \sqrt{\mu/\lambda_n}y)$ which satisfies

$$\Delta v_n + \mu v_n(1 - v_n) = 0 \quad \text{on } \sqrt{\tfrac{\lambda_n}{\mu}}\left(\Omega - \{x_n\}\right). \tag{2.2}$$

Then, proceeding as above, taking $\mu = 2\mu_0/\delta$ we obtain a contradiction since

$$B(0,r) \subset \sqrt{\frac{\lambda_n}{\mu}}\,(\Omega - \{x_n\})$$

for n large.

Suppose now that $\sqrt{\lambda_n}\operatorname{dist}(x_n, \partial\Omega) \to d_0$. Set $y_n \in \partial\Omega$ such that $|y_n - x_n| \leq d/\sqrt{\lambda_n}$ with $d \geq d_0$. We can assume, considering a subsequence if necessary, that $y_n \to \overline{y} \in \partial\Omega$. Using the smoothness of Ω there exists an $\eta > 0$ and a smooth one-to-one map $\varphi : B(\overline{y}, \eta) \to \mathbf{R}^N$, such that $\varphi(\overline{y}) = 0$, $\varphi(\partial\Omega \cap B(\overline{y}, \eta) \subset \{x_N = 0\}$, $\varphi(\Omega \cap B(\overline{y}, \eta) \subset \{x_N > 0\}$. Moreover φ can be chosen such that if $w_n(x) = u_n(\varphi^{-1}(x))$ then w_n satisfies

$$\begin{aligned} &Lw_n + \lambda_n w_n(1 - w_n) = 0 \quad \text{in } W, \\ &\qquad\qquad \frac{\partial w_n}{\partial x_N} = h(w_n) \quad \text{on } \partial(W \cap \{x_N = 0\}), \end{aligned}$$

where $W = \varphi(\Omega \cap B(\overline{y}, \eta))$, h a smooth function, and

$$L = \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i},$$

a uniformly elliptic operator in $\overline{W}$ with smooth coefficients independent of n.

Proceeding as above, we define $v_n(y) = w_n(\varphi(y_n) + y/\sqrt{\lambda_n})$, that satisfies

$$\begin{aligned} (2.3) \quad \sum_{i,j} a_{ij}^n(y) \frac{\partial^2 v_n}{\partial x_i \partial x_j} &+ \frac{1}{\sqrt{\lambda_n}} \sum_i b_i^n(y) \frac{\partial v_n}{\partial x_i} \\ &\qquad + w_n(1 - w_n) = 0 \quad \text{in } \lambda_n W, \\ \frac{\partial w_n}{\partial x_N} &= \frac{1}{\sqrt{\lambda_n}} h(w_n) \quad \text{on } \partial(\sqrt{\lambda_n} W \cap \{x_N = 0\}), \end{aligned}$$

with $a_{ij}^n(y) = a_{ij}(\varphi(y_n) + y/\sqrt{\lambda_n})$ and $b_i^n(y) = b_i(\varphi(y_n) + y/\sqrt{\lambda_n})$. Now, using Schauder estimates (see for instance [**4**]) we have that if $B_R^+ = B(0, R) \cap \{x_N \geq 0\}$ and $\gamma > 0$, then for n large we have

$$\|v_n\|_{C^{2+\gamma}(B_R^+)} \leq C\left(\frac{1}{\sqrt{\lambda_n}} \|v_n\|_{C^{2+\gamma}(B_R^+)} + \|v_n\|_{C^{\gamma}(B_R^+)}\right).$$

Using that $v_n \leq 1$ and an interpolation inequality we obtain that there exists a constant C_0 such that

$$\|v_n\|_{C^{2+\gamma}(B_R^+)} \leq C_0.$$

Therefore, as $n \to \infty$ we have that, except possibly for a subsequence, v_n converges to a solution v of

$$\sum_{ij} a_{ij}(0)\frac{\partial^2 w}{\partial x_i \partial x_j} + v(1-v) = 0 \quad \text{in } \{x_N > 0\},$$
$$\frac{\partial w}{\partial x_N} = 0 \quad \text{on } \{x_N = 0\}.$$

Therefore, after reflecting and making a change of variables, we have that $v \geq 0$ is a solution of $\Delta w + w(1-w) = 0$ in $\mathbf{R}^N$. Observe that $1 - \delta = u_n(x_n) = v_n(\sqrt{\lambda_n}(\varphi(x_n) - \varphi(y_n)))$, and since $\sqrt{\lambda_n}|x_n - y_n|$ is bounded we have that, except possibly for a subsequence, that $\sqrt{\lambda_n}(\varphi(x_n) - \varphi(y_n))$ converges. Hence, there exists $\overline{x} \in \mathbf{R}^N$ such that $v(\overline{x}) = 1 - \delta$, thus v is positive, which implies that $v \equiv 1$, which is a contradiction. □

Remark 1. We note that the result also applies to the case of solutions to the diffusive logistic equation with fixed Robin boundary conditions, with the same proof.

The next result states that in the case $\alpha(1) = 1$ and $\alpha'(1) > 0$ we have that for λ large enough the only solution of (1.5) is $u \equiv 1$.

Proposition 2. *There exists a λ_0 depending on Ω such that for all $\lambda > \lambda_0$ the equation* (1.5) *admits $u \equiv 1$ as its only nontrivial solution.*

Proof. The proof of this proposition is similar to the one of Theorem 2.1 in [**3**]. We include it here for completeness. We proceed by contradiction. Suppose that we have a sequence u_{λ_n} of solutions of (1.5), which for simplicity will be denoted by u_n, with $\lambda_n \to \infty$ as $n \to \infty$. Then by Proposition 1 we have that for δ small we have

$$u_n \geq 1 - \delta \quad \text{for } n \text{ large.} \tag{2.4}$$

As in [**3**], we define $h > 0$ and smooth satisfying

$$\begin{aligned} \nabla^2 h + 1 = 0 &\quad \text{in } \Omega \\ \frac{\partial h}{\partial \nu} + Kh = 0 &\quad \text{on } \partial\Omega. \end{aligned}$$

We let $z_n = (1 - u_n)h$. It can be checked that z_n satisfies:
(2.5)
$$\begin{cases} -\nabla \cdot \big((1/h^2)\nabla z\big) - (z/h) \cdot \nabla^2 (1/h) + \lambda_n (z/h^2) u_n = 0 & \text{in } \Omega \\ (\partial z/\partial \nu) + (K - (1 - \alpha(u_n)/1 - u_n) \cdot (u_n/\alpha(u_n)))\, z = 0 & \text{on } \partial\Omega. \end{cases}$$

Choose K large enough that the expression multiplying z in the boundary condition of (2.5) is positive. Multiplying the top equation in (2.5) by z_n, integrating by parts and using (1.1), we obtain:
(2.6)
$$\begin{aligned} \lambda_n \int_\Omega \frac{z_n^2}{h^2} u_n \, dx &= \int_\Omega z_n \nabla \cdot \left(\frac{1}{h^2} \nabla z_n\right) dx + \int_\Omega \frac{z_n^2}{h} \nabla^2 \left(\frac{1}{h}\right) dx \\ &= -\int_\Omega \frac{1}{h^2} |\nabla z_n|^2 \, dx + \int_\Omega \nabla \cdot \left(\frac{z_n}{h^2} \nabla z_n\right) dx \\ &\quad + \int_\Omega \frac{z_n^2}{h} \nabla^2 \left(\frac{1}{h}\right) dx \\ &= -\int_\Omega \frac{1}{h^2} |\nabla z_n|^2 \, dx - \int_{\partial\Omega} \left(K - \frac{1 - \alpha(u_n)}{1 - u_n} \cdot \frac{u_n}{\alpha(u_n)}\right) \frac{z_n^2}{h^2} \, dS \\ &\quad + \int_\Omega \frac{z_n^2}{h} \nabla^2 \left(\frac{1}{h}\right) dx \\ &\leq \int_\Omega \frac{z_n^2}{h^2} \left(\frac{1}{h} + \frac{2|\nabla h|^2}{h^2}\right) dx. \end{aligned}$$

Then using (2.4) and (2.6) we obtain that

$$\lambda_n \leq \frac{1}{1 - \delta} \max_{x \in \Omega} \left(\frac{1}{h(x)} + \frac{2|\nabla h(x)|^2}{h(x)^2}\right),$$

which contradicts the unboundedness of λ_n. $\square$

3. Existence of multiple equilibria. As noted in the introduction, we will now consider $\Omega = (-L, L)$. We are interested in describing the solutions of (1.1) different from the Dirichlet solutions, i.e., solutions of (1.4). We start by rescaling the problem considering $v(t) = u(t/\sqrt{\lambda})$, which satisfies

$$v'' + v(1-v) = 0 \quad \text{in } (-L\sqrt{\lambda}, L\sqrt{\lambda}), \quad v \geq 0, \tag{3.1}$$

with the boundary conditions

$$v'(L\sqrt{\lambda}) = \frac{1}{\sqrt{\lambda}}\left(\frac{1-\alpha(v(L\sqrt{\lambda}))}{\beta(v(L\sqrt{\lambda}))}\right), \tag{3.2}$$

$$v'(-L\sqrt{\lambda}) = -\frac{1}{\sqrt{\lambda}}\left(\frac{1-\alpha(v(-L\sqrt{\lambda}))}{\beta(v(-L\sqrt{\lambda}))}\right). \tag{3.3}$$

We observe that any solution v of (3.1) satisfies

$$\frac{v'2}{2} + \frac{v^2}{2} - \frac{v^3}{3} = C,$$

with $C \in (0, 1/6]$. We define $f(x,y) = (y^2/2) + (x^2/2) - (x^3/3)$. To start the study of (3.1), (3.2), (3.3), it is useful to define a time map. Set $U = \{(x,y)/f(x,y) \in (0,1/6),\ 0 \leq x \leq 1,\ y \geq 0\}$. For any $(x,y) \in U$ we define $v(\cdot\,; x,y)$ to be the solution of (3.1) with $v(0;x,y) = x$, $v'(0;x,y) = y$. We set $T(x,y)$ the first time such that $v(t;x,y)$ satisfies $v'(T(x,y),x,y) = 0$. Observe $\eta(x,y) = v(T(x,y),x,y)$ is given by $\eta(x,y) = \gamma^{-1}(f(x,y))$, where $\gamma(u) = (u^2/2) - (u^3/3)$. Then, we have the following expression for $T(x,y)$:

$$T(x,y) = \frac{1}{\sqrt{2}} \int_x^{\eta(x,y)} \frac{1}{\sqrt{f(x,y) - \gamma(z)}}\, dz.$$

With a change of variables, using $s = z/\eta(x,y)$ and the fact that $f(x,y) = \gamma(\eta(x,y))$ we obtain:

$$T(x,y) = \frac{1}{\sqrt{2}} \int_{x/\eta}^{1} \frac{1}{\sqrt{1-s}\sqrt{(s+1)/2 - [\eta(x,y)(s^2+s+1)]/3}}\, ds. \tag{3.4}$$

Proposition 3. *The following hold*:

(i) *For all* $x, y \in U$ *we have* $\partial T/\partial y > 0$.

(ii) *There exists a* $\delta > 0$ *such that for* $(x, y) \in U$, $x \leq \delta$

$$\frac{\partial T}{\partial x} < 0.$$

Proof. The proof of (i) follows easily from differentiating T and observing that by a simple computation $\partial\eta/\partial y > 0$.

To prove (ii) we differentiate to obtain:

(3.5)

$$\frac{\partial T}{\partial x} = -\frac{1}{\sqrt{2}}\frac{\partial(x/\eta)}{\partial x}\frac{1}{\sqrt{1-(x/\eta)}\sqrt{[(x/\eta)+1]/2-[\eta((x^2/\eta^2)+(x/\eta)+1)]/3}}$$
$$+\frac{1}{6\sqrt{2}}\int_{x/\eta}^{1}\frac{1}{\sqrt{1-s}}\frac{(s^2+s+1)(\partial\eta/\partial x)}{\left(\sqrt{(s+1)/2-[\eta(s^2+s+1)]/3}\right)^3}\,ds.$$

It can be checked that $\partial\eta/\partial x(0, y) = 0$ and $\eta(0, y) > 0$ for all $y > 0$. Thus we obtain

$$\frac{\partial(x/\eta)}{\partial x} = \frac{\eta - x(\partial\eta/\partial x)}{\eta^2} > 0,$$

if x is close to zero. □

Remark 2. We observe that since $\partial\eta/\partial y > 0$ if $t \in (\pi/2, \infty)$ then the equation $T(x, y) = t$ can be solved in a unique way, with $y = y(x)$ defined for $x \in [0, 1]$. Moreover, since $\partial T/\partial x < 0$ at x close to zero, we have that $y'(x) > 0$ for small x.

Going back to our problem, we observe that (v, λ) is a solution of (3.1), (3.2), (3.3), if and only if

$$T\left(x, \frac{1}{\sqrt{\lambda}}\frac{1-\alpha(x)}{\beta(x)}\right) = L\sqrt{\lambda}. \tag{3.6}$$

Proposition 4. *Fix* $L > \pi/2$ *and* $\lambda = \lambda_0$. *There exists a function* α *strictly increasing with* $\alpha(0) = 0$, $\alpha'(0) > 0$, *and* $\delta > 0$ *such that for* *all* $x \in [0,\delta]$ *we have*

$$T\left(x, \frac{1-\alpha(x)}{\beta(x)}\right) = L. \tag{3.7}$$

Thus, for $\lambda = \lambda_0$ *and the choice of* α *there are infinitely many positive solutions of the one-dimensional case of* (1.5).

Proof. Set $t = L$ and $y(x)$ as in Remark 2. We define for $0 \le x \le \delta$

$$\alpha(x) = \frac{x}{x + y(x)},$$

where $\delta > 0$ is to be chosen and we continue α to be smooth, increasing, with $\alpha(u) \in [0,1]$. Observe that $\alpha(0) = 0$ and $1 - \alpha(x)/\beta(x) = y(x)$, thus (3.7) holds. It remains to check that α is increasing in $[0,\delta]$. Since:

$$\alpha'(x) = \frac{x + y(x) - x(1 + y'(x))}{(x + y(x))^2},$$

we can chose δ small so that $\alpha'(x) > 0$ in $[0,\delta]$. □

We observe that this result gives that for some nonlinearity α in the one-dimensional case we may have that for some λ there are a continuum of solutions of (1.1) bifurcating from the Dirichlet branch $\mathcal{C}_D$.

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Department of Mathematics, The University of Miami, Coral Gables, FL 33124
Email address: rsc@math.miami.edu

Department of Mathematics, The University of Miami, Coral Gables, FL 33124
Email address: gcc@math.miami.edu

Departamento de Ingeniería Matemática, Universidad de Chile, Blanco Encalada 2120, 5º piso–Santiago, Chile and Centro de Modelamiento Matemático, UMI 2807 CNRS-UChile, Universidad de Chile, Casilla 170 Correo 3–Santiago, Chile
Email address: samartin@dim.uchile.cl