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THE LAGUERRE-SOBOLEV-TYPE ORTHOGONAL POLYNOMIALS. HOLONOMIC EQUATION AND ELECTROSTATIC INTERPRETATION

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ABSTRACT. In this paper we find the second order linear differential equation satisfied by orthogonal polynomials with respect to the inner product

$$\langle p, q\rangle = \int_0^\infty p(x)q(x)x^\alpha e^{-x}\,dx + Np'(0)q'(0)$$

where $\alpha > -1$, $N \in \mathbf{R}_+$ and p, q are polynomials with real coefficients. We also find some numerical results concerning the distribution of their zeros and their electrostatic interpretation in terms of a logarithmic potential with an external field. We deduce the hypergeometric expression of these polynomials. Finally, the analysis of asymptotic behavior of such polynomials is presented.

1. Introduction. The Laguerre orthogonal polynomials are defined as the polynomials orthogonal with respect to the inner product

$$(1) \qquad \langle p, q\rangle = \int_0^\infty pqx^\alpha e^{-x}dx, \quad \alpha > -1,\ p, q \in \mathbf{P}.$$

If $\{L_n^\alpha\}_{n\geq 0}$ are the Laguerre polynomials and μ the corresponding orthogonality measure, we define an inner Sobolev-type product as follows

$$(2) \qquad \langle p, q\rangle_S = \int_0^\infty pq\,d\mu + Np'(0)q'(0), \quad N \in \mathbf{R}_+.$$

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Let $\{\widetilde{L}_n^\alpha\}_{n\geq 0}$ be the Laguerre-Sobolev-type orthogonal polynomials corresponding to the above inner product. Such a kind of Sobolev-type orthogonal polynomials have been introduced in [**10**].

The main goal of the paper is to find a second order linear differential equation which is satisfied by $\widetilde{L}_n^\alpha$ for every $n \in \mathbf{N}$. This differential equation will allow us to give an electrostatic interpretation of the zeros of $\widetilde{L}_n^\alpha$. Additionally, we will give a hypergeometric expression of $\widetilde{L}_n^\alpha$. On the other hand, we analyze the asymptotic behavior of the Laguerre-Sobolev-type orthogonal polynomials.

In Section 2, we present some preliminary results about Laguerre orthogonal polynomials. In Section 3, we find a connection formula which expresses the Laguerre-Sobolev-type orthogonal polynomials as a combination of three consecutive Laguerre polynomials. This formula will be our principal tool in this paper.

In Section 4, we find a holonomic equation that $\widetilde{L}_n^\alpha$ satisfies. The hypergeometric representation of $\widetilde{L}_n^\alpha$ is done in Section 5. The location of the zeros is analyzed in Section 6. In particular, some numerical experiments using Maple Software are presented. An electrostatic interpretation is done in Section 7. Finally, the asymptotic behavior of the Laguerre-Sobolev-Type orthogonal polynomials is studied in Section 8.

2. Preliminaries. Let $\{\mu_n\}_{n\geq 0}$ be a sequence of complex numbers and μ a linear functional defined in the linear space $\mathbf{P}$ of the polynomials with complex coefficients, such that

$$\langle \mu, x^n \rangle = \mu_n, \quad n = 0, 1, 2 \dots .$$

μ is said to be a *moment functional* associated with $\{\mu_n\}_{n\geq 0}$. Moreover μ_n is said to be the *moment of order* n of the functional μ.

Given a moment functional μ, a sequence of polynomials $\{P_n\}_{n\geq 0}$ is said to be a sequence of *orthogonal polynomials* with respect to μ if

(i) The degree of P_n is n.

(ii) $\langle \mu, P_n(x)P_m(x)\rangle = 0$, $m \neq n$.

(iii) $\langle \mu, P_n^2(x)\rangle \neq 0$, $n = 0, 1, 2, \dots$.

If every polynomial $P_n(x)$ has 1 as leading coefficient, then $\{P_n\}_{n\geq 0}$ is said to be a sequence of *monic orthogonal polynomials*. It is clear that for every sequence of orthogonal polynomials there exists the corresponding family of monic orthogonal polynomials. In the sequel we will work with monic polynomials

The next theorem, whose proof appears in [**5**], gives necessary and sufficient conditions for the existence of a sequence of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ with respect to a moment functional μ associated with $\{\mu_n\}_{n\geq 0}$.

Theorem 1. *Let μ be a moment functional associated with $\{\mu_n\}_{n\geq 0}$. There exists a sequence of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ associated with μ if and only if the leading principal submatrices of the Hankel matrix $[\mu_{i+j}]_{i,j\in\mathbf{N}}$ are non singular.*

A moment functional such that there exists the corresponding sequence of orthogonal polynomials is said to be *regular* or *quasi-definite* ([**5**]). If $\phi(x)$ is a complex polynomial, we define the moment functional $\phi\mu$, the left multiplication by a polynomial ϕ and $D\mu$, the usual distributional derivative of μ, as follows

$$\langle \phi\mu, p(x)\rangle = \langle \mu, \phi\,(x)\,p(x)\rangle\,, \quad \langle D\mu, p(x)\rangle = -\,\langle \mu, p'(x)\rangle\,.$$

A sequence of orthogonal polynomials $\{P_n\}_{n\geq 0}$ is said to be *classical* if there exist polynomials ϕ and ψ, with $\deg\phi \leq 2$ and $\deg\psi = 1$, such that μ satisfies the Pearson differential equation:

$$D\,(\phi\mu) = \psi\mu.$$

Classical orthogonal polynomials (Hermite, Laguerre, Jacobi and Bessel) are extensively used in the literature taking into account their applications in Mathematical Physics. Indeed, one of the most popular applications is the study of problems involving hypergeometric differential equations (see [**3, 8, 11, 16, 18**]).

The sequence of polynomials $\{P_n\}_{n\geq 0}$ is said to be *semiclassical* if there exist polynomials ϕ and ψ with $\deg\psi \geq 1$, so that the corresponding moment functional μ satisfies $D(\phi\mu) = \psi\mu$.

In order to find the second order linear differential equation that the Laguerre-Sobolev type polynomials satisfy, we will need to summarize some properties of the Laguerre monic orthogonal polynomials that we will use in the sequel. The details of the proof can be founded in [**3, 5, 8, 18**].

Proposition 1. *Let $\{L_n^\alpha\}_{n\geq 0}$ be the sequence of Laguerre monic orthogonal polynomials.*

1. *For every $n \in \mathbf{N}$,*

$$xL_n^\alpha(x) = L_{n+1}^\alpha(x) + (2n+1+\alpha)\,L_n^\alpha(x) + n\,(n+\alpha)\,L_{n-1}^\alpha(x) \tag{3}$$

with $L_0^\alpha(x) = 1$, $L_1^\alpha(x) = x - (\alpha+1)$.

2. *For every $n \in \mathbf{N}$,*

$$x\,(L_n^\alpha(x))' = nL_n^\alpha(x) + n\,(n+\alpha)\,L_{n-1}^\alpha(x). \tag{4}$$

3. *For every $n \in \mathbf{N}$, $L_n^\alpha(x)$ satisfies the differential equation*

$$xy'' + (\alpha+1-x)\,y' = \lambda_n y \tag{5}$$

with $\lambda_n = -n$.

4. *For every $n \in \mathbf{N}$,*

$$L_n^\alpha(x) = (-1)^n(\alpha+1)_n \sum_{k=0}^{\infty} \frac{(-n)_k}{(\alpha+1)_k}\frac{x^k}{k!}. \tag{6}$$

5. *For every $n \in \mathbf{N}$,*

$$L_n^\alpha(x) = L_n^{\alpha+1}(x) + nL_{n-1}^{\alpha+1}(x). \tag{7}$$

6. *For every $n \in \mathbf{N}$,*

$$\|L_n^\alpha\|_\alpha^2 = n!\Gamma(n+\alpha+1). \tag{8}$$

7. *For every* $n \in \mathbf{N}$

$$L_n^\alpha(0) = (-1)^n \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}. \tag{9}$$

8. *If*

$$K_n(x,y) = \sum_{j=0}^{n} \frac{L_j^\alpha(y) L_j^\alpha(x)}{\left\| L_j^\alpha \right\|_\alpha^2}$$

denotes the nth kernel polynomial then, for every $n \in \mathbf{N}$,

$$K_n(x,y) = \frac{L_{n+1}^\alpha(x) L_n^\alpha(y) - L_{n+1}^\alpha(y) L_n^\alpha(x)}{x-y} \frac{1}{\left\| L_n^\alpha \right\|_\alpha^2} \tag{10}$$

and

$$K_n(x,0) = \frac{L_n^\alpha(0)}{n!\Gamma(n+\alpha+1)} L_n^{\alpha+1}(x). \tag{11}$$

9. *Let* J_α *be the Bessel function defined by*

$$J_\alpha(x) = \sum_{j=0}^{\infty} \frac{(-1)^j (x/2)^{2j+\alpha}}{j!\Gamma(j+\alpha+1)}.$$

Then

(a) (*Asymptotic formula of Hilb's type*). *For every* $n \in \mathbf{N}$,

$$\begin{aligned} & e^{-x/2} x^{\alpha/2} \frac{(-1)^n}{n!} L_n^\alpha(x) \\ &\quad = \frac{\Gamma(n+\alpha+1)}{n!} \left(n + \frac{\alpha+1}{2} \right)^{-\alpha/2} J_\alpha \left(2 \sqrt{\left(n + \frac{\alpha+1}{2} \right) x} \right) \\ &\quad\quad + \mathcal{O}(n^{\alpha/2-3/4}), \end{aligned} \tag{12}$$

uniformly on compact subsets of $(0,\infty)$.

(b) (*Perron's formula*). *For every* $n \in \mathbf{N}$

$$\begin{aligned} & 2\sqrt{\pi} e^{-x/2} (-x)^{\alpha/2+1/4} \frac{(-1)^n}{n!} L_n^\alpha(x) \\ &\quad = n^{\alpha/2-1/4} \exp\left(2(-nx)^{1/2} \right) \left[\sum_{j=0}^{p-1} C_j(x) n^{-j/2} + \mathcal{O}(n^{-p/2}) \right] \end{aligned} \tag{13}$$

where $C_j(x)$ is independent of n and regular in $\mathbf{C}\backslash[0,\infty)$. This formula holds uniformly when $x \in \mathbf{C}\backslash[0,\infty)$.

We will use the following notation for the partial derivatives of $K_n(x,y)$ evaluated in $y=0$

$$\left.\frac{\partial\left(K_n(x,y)\right)}{\partial x}\right|_{y=0} = K_n^{(1,0)}(x,0),$$
$$\left.\frac{\partial\left(K_n(x,y)\right)}{\partial y}\right|_{y=0} = K_n^{(0,1)}(x,0),$$
$$\left.\frac{\partial^2\left(K_n(x,y)\right)}{\partial x \partial y}\right|_{y=0} = K_n^{(1,1)}(x,0).$$

If $p(x)$ is a polynomial with $\deg p \leq n$, we can write it as a linear combination of the Laguerre polynomials as follows

$$p(x) = \sum_{j=0}^{n} \frac{\left\langle L_j^\alpha(x), p(x)\right\rangle}{\left\|L_j^\alpha\right\|_\alpha^2} L_j^\alpha(x).$$

As a consequence,

$$p'(y) = \sum_{j=0}^{n} \frac{\left\langle L_j^\alpha(x), p(x)\right\rangle}{\left\|L_j^\alpha\right\|_\alpha^2} \left(L_j^\alpha\right)'(y),$$

and, taking into account that

$$\begin{aligned}\left\langle K_n^{(0,1)}(x,y), p(x)\right\rangle &= \left\langle \sum_{j=0}^{n} \frac{L_j^\alpha(x)\left(L_j^\alpha\right)'(y)}{\left\|L_j^\alpha\right\|_\alpha^2}, p(x)\right\rangle \\ &= \sum_{j=0}^{n} \frac{\left\langle L_j^\alpha(x), p(x)\right\rangle}{\left\|L_j^\alpha\right\|_\alpha^2} \left(L_j^\alpha\right)'(y),\end{aligned}$$

then

$$\left\langle K_n^{(0,1)}(x,y), p(x)\right\rangle = p'(y). \tag{14}$$

Using (11) we have

$$K_{n-1}(0,0)=\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(\alpha+2)(n-1)!}. \tag{15}$$

If we compute the partial derivative with respect to x in (11), we obtain

$$K_{n-1}^{(1,0)}(0,0)=\frac{L_{n-1}^{\alpha}(0)}{(n-2)!\Gamma(n+\alpha)}L_{n-2}^{\alpha+2}(0).$$

Therefore,

$$K_{n-1}^{(1,0)}(0,0)=-\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(\alpha+3)(n-2)!}. \tag{16}$$

In the next section, the values of $K_{n-1}^{(1,1)}(0,0)$ will be deduced.

3. Connection formula. In order to find the second order linear differential equation that the Laguerre-Sobolev-type polynomials $\widetilde{L}_n^{\alpha}(x)$ satisfy, we will write them as a linear combination of some monic Laguerre polynomials. Notice that this is an alternative approach to the connection formula described in [**1, 10**]. Indeed, we get

$$\widetilde{L}_n^{\alpha}(x)=L_n^{\alpha}(x)+\sum_{j=0}^{n-1}a_{n,j}L_j^{\alpha}(x)$$

where for $j=1,2,\dots,n-1$

$$\begin{aligned} a_{n,j}&=\frac{\left\langle \widetilde{L}_n^{\alpha}(x),L_j^{\alpha}(x)\right\rangle}{\left\|L_j^{\alpha}\right\|_{\alpha}^{2}}\\ &=\frac{-N\left(\widetilde{L}_n^{\alpha}\right)'(0)\left(L_j^{\alpha}\right)'(0)}{\left\|L_j^{\alpha}\right\|_{\alpha}^{2}}. \end{aligned}$$

Thus,

$$\widetilde{L}_n^{\alpha}(x)=L_n^{\alpha}(x)-N\left(\widetilde{L}_n^{\alpha}\right)'(0)\sum_{j=0}^{n-1}\frac{\left(L_j^{\alpha}\right)'(0)L_j^{\alpha}(x)}{\left\|L_j^{\alpha}\right\|_{\alpha}^{2}}.$$

In other words,

$$\widetilde{L}_n^\alpha(x) = L_n^\alpha(x) - N\left(\widetilde{L}_n^\alpha\right)'(0)K_{n-1}^{(0,1)}(x,0). \tag{17}$$

We know that $K_{n-1}(x,0)$ is a monic Laguerre polynomial up to a constant factor. So, we want to find a similar expression for $K_{n-1}^{(0,1)}(x,0)$. If we denote $\langle p,q\rangle_{\alpha+2}$ the inner product associated with the Laguerre weight $x^{\alpha+2}e^{-x}dx$ and using the fact that

$$\frac{\left\|L_{n-1}^\alpha\right\|_\alpha^2}{\left(L_{n-1}^\alpha\right)'(0)}K_{n-1}^{(0,1)}(x,0) = L_{n-1}^{\alpha+2}(x) + \sum_{j=0}^{n-2} b_{n-1,j}L_j^{\alpha+2}(x)$$

where

$$\begin{aligned} b_{n-1,j} &= \frac{\left\langle \left\|L_{n-1}^\alpha\right\|_\alpha^2/\left(L_{n-1}^\alpha\right)'(0)K_{n-1}^{(0,1)}(x,0), L_j^{\alpha+2}(x)\right\rangle_{\alpha+2}}{\left\|L_j^{\alpha+2}\right\|_{\alpha+2}^2} \\ &= \frac{\left\|L_{n-1}^\alpha\right\|_\alpha^2}{\left(L_{n-1}^\alpha\right)'(0)\left\|L_j^{\alpha+2}\right\|_{\alpha+2}^2}\left\langle K_{n-1}^{(0,1)}(x,0), L_j^{\alpha+2}(x)\right\rangle_{\alpha+2}, \end{aligned}$$

according to (14), for $0 \le j \le n-3$ we get

$$\begin{aligned} \left\langle K_{n-1}^{(0,1)}(x,0), L_j^{\alpha+2}(x)\right\rangle_{\alpha+2} &= \int_0^\infty K_{n-1}^{(0,1)}(x,0)L_j^{\alpha+2}(x)x^{\alpha+2}e^{-x}dx \\ &= \int_0^\infty K_{n-1}^{(0,1)}(x,0)x^2L_j^{\alpha+2}(x)x^\alpha e^{-x}dx \\ &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} K_{n-1}^{(0,1)}(x,0) = \frac{\left(L_{n-1}^\alpha\right)'(0)}{\left\|L_{n-1}^\alpha\right\|_\alpha^2}L_{n-1}^{\alpha+2}(x) \\ + \frac{\left\langle K_{n-1}^{(0,1)}(x,0), L_{n-2}^{\alpha+2}(x)\right\rangle_{\alpha+2}}{\left\|L_{n-2}^{\alpha+2}\right\|_{\alpha+2}^2}L_{n-2}^{\alpha+2}(x). \end{aligned}$$

On the other hand,

$$
\begin{aligned}
&\left\langle K_{n-1}^{(0,1)}(x,0), L_{n-2}^{\alpha+2}(x)\right\rangle_{\alpha+2} \\
&= \left\langle K_{n}^{(0,1)}(x,0) - \frac{L_n^\alpha(x)\left(L_n^\alpha\right)'(0)}{\|L_n^\alpha\|^2}, L_{n-2}^{\alpha+2}(x)\right\rangle_{\alpha+2} \\
&= \left\langle K_{n}^{(0,1)}(x,0), L_{n-2}^{\alpha+2}(x)\right\rangle_{\alpha+2} - \left\langle \frac{L_n^\alpha(x)\left(L_n^\alpha\right)'(0)}{\|L_n^\alpha\|_\alpha^2}, L_{n-2}^{\alpha+2}(x)\right\rangle_{\alpha+2} \\
&= \left\langle K_{n}^{(0,1)}(x,0), x^2 L_{n-2}^{\alpha+2}(x)\right\rangle_{\alpha} - \frac{\left(L_n^\alpha\right)'(0)}{\|L_n^\alpha\|_\alpha^2}\left\langle L_n^\alpha(x), x^2 L_{n-2}^{\alpha+2}(x)\right\rangle_\alpha \\
&= -\left(L_n^\alpha\right)'(0).
\end{aligned}
$$

As a consequence, we get

Proposition 2. *For every* $n \in \mathbf{N}$,

$$
\begin{aligned}
K_{n-1}^{(0,1)}(x,0) &= a_{n-1} L_{n-1}^{\alpha+2}(x) + b_{n-1} L_{n-2}^{\alpha+2}(x) \\
&= a_{n-1}\left(L_{n-1}^{\alpha+2}(x) + n L_{n-2}^{\alpha+2}(x)\right)
\end{aligned}
\tag{18}
$$

where

$$
a_{n-1} = \frac{\left(L_{n-1}^\alpha\right)'(0)}{\left\|L_{n-1}^\alpha\right\|_\alpha^2} = \frac{(-1)^n}{(n-2)!\Gamma(\alpha+2)}
$$

and

$$
b_{n-1} = -\frac{\left(L_n^\alpha\right)'(0)}{\left\|L_{n-2}^{\alpha+2}\right\|_{\alpha+2}^2} = \frac{(-1)^n n}{(n-2)!\Gamma(\alpha+2)} = n a_{n-1}.
$$

Taking into account (18) we obtain

$$
K_{n-1}^{(1,1)}(x,0) = (n-1)a_{n-1} L_{n-2}^{\alpha+3}(x) + (n-2) b_{n-1} L_{n-3}^{\alpha+3}(x).
$$

On the other hand, using (8) and (9)

$$\begin{aligned}
K_{n-1}^{(1,1)}(0,0) &= (n-1)a_{n-1}L_{n-2}^{\alpha+3}(0) + (n-2)b_{n-1}L_{n-3}^{\alpha+3}(0) \\
&= \frac{(-1)^n(n-1)}{(n-2)!\Gamma(\alpha+2)}\frac{(-1)^{n-2}\Gamma(n+\alpha+2)}{\Gamma(\alpha+4)} \\
&\quad + \frac{(-1)^n n(n-2)}{(n-2)!\Gamma(\alpha+2)}\frac{(-1)^{n-3}\Gamma(n+\alpha+1)}{\Gamma(\alpha+4)} \\
&= \frac{\Gamma(n+\alpha+1)\left(n(\alpha+2)-(\alpha+1)\right)}{(n-2)!\Gamma(\alpha+2)\Gamma(\alpha+4)}.
\end{aligned}$$

As a conclusion,

$$K_{n-1}^{(1,1)}(0,0) = \frac{\Gamma(n+\alpha+1)\left(n(\alpha+2)-(\alpha+1)\right)}{(n-2)!\Gamma(\alpha+2)\Gamma(\alpha+4)}. \tag{19}$$

On the other hand, using (7) and (18), we get

$$\begin{aligned}
\widetilde{L}_n^{\alpha}(x) &= L_n^{\alpha}(x) - N\left(\widetilde{L}_n^{\alpha}\right)'(0)\left(a_{n-1}L_{n-1}^{\alpha+2}(x) + b_{n-1}L_{n-2}^{\alpha+2}(x)\right) \\
&= L_n^{\alpha+1}(x) + nL_{n-1}^{\alpha+1}(x) \\
&\quad - N\left(\widetilde{L}_n^{\alpha}\right)'(0)\left(a_{n-1}L_{n-1}^{\alpha+2}(x) + b_{n-1}L_{n-2}^{\alpha+2}(x)\right) \\
&= L_n^{\alpha+2}(x) + nL_{n-1}^{\alpha+2}(x) + n\left(L_{n-1}^{\alpha+2}(x) + (n-1)L_{n-2}^{\alpha+2}(x)\right) \\
&\quad - N\left(\widetilde{L}_n^{\alpha}\right)'(0)\left(a_{n-1}L_{n-1}^{\alpha+2}(x) + b_{n-1}L_{n-2}^{\alpha+2}(x)\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\widetilde{L}_n^{\alpha}(x) = L_n^{\alpha+2}(x) &+ \left[2n - N\left(\widetilde{L}_n^{\alpha}\right)'(0)a_{n-1}\right]L_{n-1}^{\alpha+2}(x) \\
&+ \left[n(n-1) - N\left(\widetilde{L}_n^{\alpha}\right)'(0)b_{n-1}\right]L_{n-2}^{\alpha+2}(x).
\end{aligned} \tag{20}$$

We want to find an explicit expression of $(\widetilde{L}_n^{\alpha})'(0)$. In order to do that, we can take the derivative in (17) and evaluate in 0

$$\left(\widetilde{L}_n^{\alpha}\right)'(0) = nL_{n-1}^{\alpha+1}(0) - N\left(\widetilde{L}_n^{\alpha}\right)'(0)K_{n-1}^{(1,1)}(0,0).$$

Thus,

$$\left(\widetilde{L}_n^{\alpha}\right)'(0) = \frac{nL_{n-1}^{\alpha+1}(0)}{1+NK_{n-1}^{(1,1)}(0,0)}$$
$$= \frac{n(-1)^{n-1}\Gamma(\alpha+4)\Gamma(n+\alpha+1)(n-2)!}{(n-2)!\Gamma(\alpha+2)\Gamma(\alpha+4)+N\Gamma(n+\alpha+1)(n(\alpha+2)-(\alpha+1))}, \quad n \geq 2.$$

In such a way, the coefficients in (20) become

$$2n - N\left(\widetilde{L}_n^{\alpha}\right)'(0)a_{n-1}$$
$$= 2n - N\frac{n(-1)^{n-1}\Gamma(\alpha+4)\Gamma(n+\alpha+1)(n-2)!}{(n-2)!\Gamma(\alpha+2)\Gamma(\alpha+4)+N\Gamma(n+\alpha+1)(n(\alpha+2)-(\alpha+1))}\frac{(-1)^n}{(n-2)!\Gamma(\alpha+2)}$$
$$= \frac{2n(n-2)!\Gamma(\alpha+2)\Gamma(\alpha+4)+N\Gamma(n+\alpha+1)\left(2n^2\alpha+4n^2+n\alpha^2+4n+3n\alpha\right)}{(n-2)!\Gamma(\alpha+2)\Gamma(\alpha+4)+N\Gamma(n+\alpha+1)(n(\alpha+2)-(\alpha+1))},$$

and

$$n(n-1) - N\left(\widetilde{L}_n^{\alpha}\right)'(0)b_{n-1}$$
$$= n(n-1) - N\frac{n(-1)^{n-1}\Gamma(\alpha+4)\Gamma(n+\alpha+1)(n-2)!}{(n-2)!\Gamma(\alpha+2)\Gamma(\alpha+4)+N\Gamma(n+\alpha+1)(n(\alpha+2)-(\alpha+1))}$$
$$\times \frac{(-1)^n n}{(n-2)!\Gamma(\alpha+2)}$$
$$= \frac{n!\Gamma(\alpha+2)\Gamma(\alpha+4)+nN\Gamma(n+\alpha+1)\left(n^2\alpha+2n^2+n\alpha^2+3n\alpha+3n+\alpha+1\right)}{(n-2)!\Gamma(\alpha+2)\Gamma(\alpha+4)+N\Gamma(n+\alpha+1)(n(\alpha+2)-(\alpha+1))}$$

but

$$(n-1)\left(n(\alpha+2)-(\alpha+1)\right)+n(\alpha+3)(\alpha+2)$$
$$= 3n+\alpha+3n\alpha+2n^2+n\alpha^2+n^2\alpha+1.$$

Thus, we have proved

Theorem 2. *Let* $\{L_n^{\alpha}\}_{n\geq 0}$ *be the monic Laguerre orthogonal polynomials and* $\{\widetilde{L}_n^{\alpha}\}_{n\geq 0}$ *the monic Laguerre-Sobolev-type orthogonal polynomials. Then*

$$\widetilde{L}_n^{\alpha}(x) = L_n^{\alpha+2}(x) + A_n L_{n-1}^{\alpha+2}(x) + B_n L_{n-2}^{\alpha+2}(x), \tag{21}$$

where

$$A_n = \frac{2n(n-2)!\Gamma(\alpha+2)\Gamma(\alpha+4)+N\Gamma(n+\alpha+1)\left(2n^2\alpha+4n^2+n\alpha^2+4n+3n\alpha\right)}{(n-2)!\Gamma(\alpha+2)\Gamma(\alpha+4)+N\Gamma(n+\alpha+1)(n(\alpha+2)-(\alpha+1))}$$
$$B_n = \frac{n!\Gamma(\alpha+2)\Gamma(\alpha+4)+nN\Gamma(n+\alpha+1)\left(n^2\alpha+2n^2+n\alpha^2+3n\alpha+3n+\alpha+1\right)}{(n-2)!\Gamma(\alpha+2)\Gamma(\alpha+4)+N\Gamma(n+\alpha+1)(n(\alpha+2)-(\alpha+1))}$$

4. Holonomic equation. In order to find the second order differential equation that the Laguerre-Sobolev-type polynomials satisfy, we use (5). Thus,

$$\begin{aligned}
x\left(L_n^{\alpha+2}(x)\right)'' + (\alpha+3-x)\left(L_n^{\alpha+2}(x)\right)' &= -nL_n^{\alpha+2}(x)\\
x\left(L_{n-1}^{\alpha+2}(x)\right)'' + (\alpha+3-x)\left(L_{n-1}^{\alpha+2}(x)\right)' &= -(n-1)L_{n-1}^{\alpha+2}(x)\\
x\left(L_{n-2}^{\alpha+2}(x)\right)'' + (\alpha+3-x)\left(L_{n-2}^{\alpha+2}(x)\right)' &= -(n-2)L_{n-2}^{\alpha+2}(x)
\end{aligned}$$

and, as a consequence,

$$\begin{aligned}
(22)\quad & x\left(\widetilde{L}_n^{\alpha}(x)\right)'' + (\alpha+3-x)\left(\widetilde{L}_n^{\alpha}(x)\right)'\\
&\qquad = -nL_n^{\alpha+2}(x) - A_n(n-1)L_{n-1}^{\alpha+2}(x) - B_n(n-2)L_{n-2}^{\alpha+2}(x).
\end{aligned}$$

Our goal is to express $L_n^{\alpha+2}(x), L_{n-1}^{\alpha+2}(x)$ and $L_{n-2}^{\alpha+2}(x)$ as a combination of $\widetilde{L}_n^{\alpha}(x)$ and $(\widetilde{L}_n^{\alpha})'(x)$ with rational functions as coefficients. Then we take derivatives in (21) and multiply by x in both sides. Using (4) we get

$$\begin{aligned}
x\left(\widetilde{L}_n^{\alpha}(x)\right)' &= nL_n^{\alpha+2}(x) + n(n+\alpha+2)L_{n-1}^{\alpha+2}(x)\\
&\quad + A_n\left((n-1)L_{n-1}^{\alpha+2}(x) + (n-1)(n+\alpha+1)L_{n-2}^{\alpha+2}(x)\right)\\
&\quad + B_n\left((n-2)L_{n-2}^{\alpha+2}(x) + (n-2)(n+\alpha)L_{n-3}^{\alpha+2}(x)\right)\\
&= nL_n^{\alpha+2}(x) + \left(n(n+\alpha+2) + A_n(n-1) - B_n\right)L_{n-1}^{\alpha+2}(x)\\
&\quad + \Big(A_n(n-1)(n+\alpha+1)\\
&\quad + B_n(n-2) + B_nx - (2n+\alpha-1)B_n\Big)L_{n-2}^{\alpha+2}(x).
\end{aligned}$$

If we denote

$$\begin{aligned}
H_n &= n(n+\alpha+2) + A_n(n-1) - B_n\\
C_1(x;n) &= B_nx + (n+\alpha+1)\left(A_n(n-1) - B_n\right),
\end{aligned}$$

then

$$x\left(\widetilde{L}_n^{\alpha}(x)\right)' = nL_n^{\alpha+2}(x) + H_nL_{n-1}^{\alpha+2}(x) + C_1(x;n)L_{n-2}^{\alpha+2}(x).$$

Taking into account (3)

$$L_{n-2}^{\alpha+2}(x) = \frac{(x-(2n+\alpha+1))L_{n-1}^{\alpha+2}(x) - L_n^{\alpha+2}(x)}{(n-1)(n+\alpha+1)}. \tag{23}$$

Thus, we can write

$$\begin{aligned} x\left(\widetilde{L}_n^{\alpha}(x)\right)' = &\left[n - \frac{C_1(x;n)}{(n-1)(n+\alpha+1)}\right] L_n^{\alpha+2}(x) \\ &+ \left[H_n + \frac{C_1(x;n)(x-(2n+\alpha+1))}{(n-1)(n+\alpha+1)}\right] L_{n-1}^{\alpha+2}(x). \end{aligned}$$

If we define

$$\begin{aligned} K_n &= 1 - \frac{B_n}{(n-1)(n+\alpha+1)}, \\ g_1(x) &= A_n + \frac{(x-(2n+\alpha+1))B_n}{(n-1)(n+\alpha+1)}, \\ f_1(x) &= n - \frac{C_1(x)}{(n-1)(n+\alpha+1)}, \\ f_2(x) &= H_n + \frac{C_1(x)(x-(2n+\alpha+1))}{(n-1)(n+\alpha,+1)}, \end{aligned}$$

then, we obtain the following system

$$\begin{cases} \widetilde{L}_n^{\alpha}(x) = K_n L_n^{\alpha+2}(x) + g_1(x) L_{n-1}^{\alpha+2}(x) \\ x\left(\widetilde{L}_n^{\alpha}(x)\right)' = f_1(x) L_n^{\alpha+2}(x) + f_2(x) L_{n-1}^{\alpha+2}(x). \end{cases}$$

Solving it

$$L_n^{\alpha+2}(x) = \frac{f_2(x)\widetilde{L}_n^{\alpha}(x) - x g_1(x)\left(\widetilde{L}_n^{\alpha}(x)\right)'}{K_n f_2(x) - f_1(x) g_1(x)}, \tag{24}$$

$$L_{n-1}^{\alpha+2}(x) = \frac{K_n x\left(\widetilde{L}_n^{\alpha}(x)\right)' - f_1(x)\widetilde{L}_n^{\alpha}(x)}{K_n f_2(x) - f_1(x) g_1(x)}. \tag{25}$$

From (22) and using (23) we get

$$\begin{aligned}
x\left(\widetilde{L}_n^{\alpha}(x)\right)'' &+ (\alpha+3-x)\left(\widetilde{L}_n^{\alpha}(x)\right)' \\
&= nL_n^{\alpha+2}(x) - A_n(n-1)L_{n-1}^{\alpha+2}(x) \\
&\quad - B_n(n-2)\left[\frac{(x-(2n+\alpha+1))L_{n-1}^{\alpha+2}(x) - L_n^{\alpha+2}(x)}{(n-1)(n+\alpha+1)}\right] \\
&= \left[n + \frac{B_n(n-2)}{(n-1)(n+\alpha+1)}\right] L_n^{\alpha+2}(x) \\
&\quad - \left[(n-1)A_n + \frac{(n-2)B_n(x-(2n+\alpha+1))}{(n-1)(n+\alpha+1)}\right] L_{n-1}^{\alpha+2}(x),
\end{aligned}$$

and, using (25),

$$\begin{aligned}
x\left(\widetilde{L}_n^{\alpha}(x)\right)'' &+ (\alpha+3-x)\left(\widetilde{L}_n^{\alpha}(x)\right)' \\
&= \left(-n + \frac{B_n(n-2)}{(n-1)(n+\alpha+1)}\right)\frac{f_2(x)\widetilde{L}_n^{\alpha}(x) - xg_1(x)\left(\widetilde{L}_n^{\alpha}(x)\right)'}{K_nf_2(x) - f_1(x)g_1(x)} \\
&\quad - \left(A_n(n-1) + \frac{B_n(n-2)(x-(2n+\alpha+1))}{(n-1)(n+\alpha+1)}\right) \\
&\quad \times \frac{K_nx\left(\widetilde{L}_n^{\alpha}(x)\right)' - f_1(x)\widetilde{L}_n^{\alpha}(x)}{K_nf_2(x) - f_1(x)g_1(x)};
\end{aligned}$$

thus

$$\begin{aligned}
&x\left(\widetilde{L}_n^{\alpha}(x)\right)'' \left[(\alpha+3-x) + \left(-n + \tfrac{B_n(n-2)}{(n-1)(n+\alpha+1)}\right)\tfrac{xg_1(x)}{K_nf_2(x)-f_1(x)g_1(x)}\right. \\
&\left. + \left(A_n(n-1) + \tfrac{B_n(n-2)(x-(2n+\alpha+1))}{(n-1)(n+\alpha+1)}\right)\tfrac{K_nx}{K_nf_2(x)-f_1(x)g_1(x)}\right]\left(\widetilde{L}_n^{\alpha}(x)\right)' \\
&- \left[\left(-n + \tfrac{B_n(n-2)}{(n-1)(n+\alpha+1)}\right)\tfrac{f_2(x)}{K_nf_2(x)-f_1(x)g_1(x)}\right. \\
&\quad + \left(A_n(n-1) + \tfrac{B_n(n-2)(x-(2n+\alpha+1))}{(n-1)(n+\alpha+1)}\right) \\
&\quad \left.\times \tfrac{f_1(x)\widetilde{L}_n^{\alpha}(x)}{K_nf_2(x)-f_1(x)g_1(x)}\right]\widetilde{L}_n^{\alpha}(x) \\
&= 0.
\end{aligned}$$

As a consequence, we have proved

Theorem 3. *Let $\{L_n^\alpha\}_{n\geq 0}$ be the Laguerre monic orthogonal polynomials and $\{\widetilde{L}_n^\alpha\}_{n\geq 0}$ the monic orthogonal polynomials associated with the Sobolev-type inner product $\langle p,q\rangle_{\widetilde{\mu}} = \int_0^\infty pq d\mu + Np'(0)q'(0)$, where p and q are real polynomials. Then*

$$A(x;n)\left(\widetilde{L}_n^\alpha(x)\right)'' + B(x;n)\left(\widetilde{L}_n^\alpha(x)\right)' - C(x;n)\widetilde{L}_n^\alpha(x) = 0, \tag{26}$$

where

$$\begin{aligned}
A(x;n) &= x\left(K_n f_2(x) - f_1(x)g_1(x)\right),\\
B(x;n) &= (\alpha+3-x)\left(K_n f_2(x) - f_1(x)g_1(x)\right)\\
&\quad + \left(-n + \frac{B_n(n-2)}{(n-1)(n+\alpha+1)}\right)xg_1(x)\\
&\quad + \left(A_n(n-1) + \frac{B_n(n-2)(x-(2n+\alpha+1))}{(n-1)(n+\alpha+1)}\right)K_n x\\
C(x;n) &= \left(-n + \frac{B_n(n-2)}{(n-1)(n+\alpha+1)}\right)f_2(x)\\
&\quad + \left(A_n(n-1) + \frac{B_n(n-2)(x-(2n+\alpha+1))}{(n-1)(n+\alpha+1)}\right)f_1(x).
\end{aligned}$$

5. Hypergeometric representation. In this section we will show that the Laguerre-Sobolev-type orthogonal polynomials are hypergeometric functions up to a constant factor.

Using (6) in (21) we have

$$\begin{aligned}
\widetilde{L}_n^\alpha(x) &= (-1)^n(\alpha+3)_n \sum_{k=0}^\infty \frac{(-n)_k}{(\alpha+3)_k}\frac{x^k}{k!}\\
&\quad + A_n(-1)^{n-1}(\alpha+3)_{n-1}\sum_{k=0}^\infty \frac{(-n+1)_k}{(\alpha+3)_k}\frac{x^k}{k!}\\
&\quad + B_n(-1)^{n-2}(\alpha+3)_{n-2}\sum_{k=0}^\infty \frac{(-n+2)_k}{(\alpha+3)_k}\frac{x^k}{k!}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{(-1)^n(\alpha+3)_n(-n)_k}{(\alpha+3)_k} \\
&\quad \times \left[1 - \tfrac{A_n(-n+k)}{(n+\alpha+2)(-n)} + \tfrac{B_n(-n+k)(-n+k+1)}{(n+\alpha+2)(n+\alpha+1)(-n)(-n+1)}\right] \\
&= (-1)^n(\alpha+3)_n \sum_{k=0}^{\infty} \frac{(-n)_k}{(\alpha+3)_k} \\
&\quad \times \left[1 - \tfrac{A_n(-n+k)}{(n+\alpha+2)(-n)} + \tfrac{B_n(-n+k)(-n+k+1)}{(n+\alpha+2)(n+\alpha+1)(-n)(-n+1)}\right].
\end{aligned}
$$

If we introduce the polynomial $\pi_2(k)$ of degree 2 in k as

$$
\begin{aligned}
\pi_2(k) &= 1 - \tfrac{A_n(-n+k)}{(n+\alpha+2)(-n)} + \tfrac{B_n(-n+k)(-n+k+1)}{(n+\alpha+2)(n+\alpha+1)(-n)(-n+1)} \\
&= D_n(k+c_0)(k+c_1)
\end{aligned}
$$

where

$$
D_n = \frac{B_n}{(n+\alpha+2)(n+\alpha+1)(-n)(-n+1)},
$$

then

$$
\widetilde{L}_n^{\alpha}(x) = (-1)^n(\alpha+3)_n D_n c_0 c_1 \sum_{k=0}^{\infty} \frac{(-n)_k(1+c_0)_k(1+c_1)_k}{(c_0)_k(c_1)_k(\alpha+3)_k} \frac{x^k}{k!}.
$$

Thus,

Theorem 4. *For every $n \in \mathbf{N}$,*

$$
\begin{aligned}
(27)\quad &\widetilde{L}_n^{\alpha}(x) \\
&= (-1)^n(\alpha+3)_n D_n c_0 c_1 \; {}_3F_3\left(-n, 1+c_0, 1+c_1; c_0, c_1, \alpha+3; x\right).
\end{aligned}
$$

6. Zeros. In order to make the study of the behavior of the zeros of the Laguerre-Sobolev-type orthogonal polynomials, we are going to need the next two propositions. The proof of the first one can be found in [**1**], and for the second one, it follows the proof of the Proposition 3.2 of the same paper.

Proposition 3. *If $n \geq 3$, the polynomial $\widetilde{L}_n^{\alpha}$ has at least $n-2$ different zeros with odd multiplicity in $(0, \infty)$.*

Proposition 4. *The zeros of* $\widetilde{L}_n^\alpha$ *are real, simple, and at least* $n-1$ *of them belong to* $(0,\infty)$.

Proof. Assume that $\xi_{n,1}, \xi_{n,2}, \dots, \xi_{n,k}$ are the zeros of $\widetilde{L}_n^\alpha(x)$ located in $(0,\infty)$. From the last proposition, we have that $k \geq n-2$.

Let $\phi(x) = (x-\xi_{n,1})(x-\xi_{n,2})\cdots(x-\xi_{n,k})$. Then the polynomials $\phi(x)\widetilde{L}_n^\alpha(x)$ and $x\phi(x)\widetilde{L}_n^\alpha(x)$ are positive in $(0,\infty)$. Assuming that $k = n-2$, we have

$$\begin{aligned}\left\langle x\phi(x), \widetilde{L}_n^\alpha(x)\right\rangle_{\widetilde{\mu}} &= \int_0^\infty x\phi(x)\widetilde{L}_n^\alpha(x)x^\alpha e^{-x}dx \\ &\quad + N\phi(0)\left(\widetilde{L}_n^\alpha\right)'(0) = 0 \\ \left\langle \phi(x), \widetilde{L}_n^\alpha(x)\right\rangle_{\widetilde{\mu}} &= \int_0^\infty \phi(x)\widetilde{L}_n^\alpha(x)x^\alpha e^{-x}dx \\ &\quad + N\left(\phi\right)'(0)\left(\widetilde{L}_n^\alpha\right)'(0) = 0\end{aligned}$$

and given that

$$\int_0^\infty x\phi(x)\widetilde{L}_n^\alpha(x)x^\alpha e^{-x}dx \geq 0, \qquad \int_0^\infty \phi(x)\widetilde{L}_n^\alpha(x)x^\alpha e^{-x}dx \geq 0,$$

and $(\widetilde{L}_n^\alpha)'(0) > 0$ (see the proof of Theorem 2), then $\phi(0) < 0$ and $\phi'(0) < 0$ which is a contradiction. $\square$

Using (21), we consider some examples of Laguerre-Sobolev-type orthogonal polynomials in order to show the behavior of their zeros. First we analyze the cases $\widetilde{L}_2^3(x)$ and $\widetilde{L}_2^5(x)$. In the first one,

$$\widetilde{L}_2^3(x) = x^2 - \frac{240}{N+24}x - 20\frac{N-24}{N+24}.$$

Thus, for $N > 24$, $\widetilde{L}_2^3(x)$ has a negative zero; for $N = 24$, $x = 0$ is a zero of $\widetilde{L}_2^3(x)$; and for $N < 24$, $\widetilde{L}_2^3(x)$ does not have negative zeros. Furthermore, their zeros are

$$\frac{2}{N+24}\left(60 \pm \sqrt{5N^2+720}\right).$$

In a similar way we get

$$\widetilde{L}_2^5(x) = x^2 - \frac{10\,080}{N+720}x + \frac{30\,240 - 42N}{N+720},$$

and, as a consequence, the zeros of $\widetilde{L}_2^5(x)$ are

$$\frac{5040 \pm \sqrt{42}\sqrt{N^2 + 86\,400}}{N+720}.$$

Thus, $\widetilde{L}_2^5(x)$ has a negative zero for $N > 720$.

On the other hand, using (21) and Maple software we get

$$\begin{aligned}\widetilde{L}_2^\alpha(x) = 7\alpha - 8x - 2x\alpha + x^2 + \alpha^2 + 12 \\ + \frac{\left(2\Gamma(\alpha+2)\Gamma(\alpha+4)+\Gamma(\alpha+3)\left(30N+22N\alpha+4N\alpha^2\right)\right)}{\Gamma(\alpha+2)\Gamma(\alpha+4)+\Gamma(\alpha+3)(3N+N\alpha)} \\ + \frac{(x-\alpha-3)\left(4\Gamma(\alpha+2)\Gamma(\alpha+4)+\Gamma(\alpha+3)\left(24N+14N\alpha+2N\alpha^2\right)\right)}{\Gamma(\alpha+2)\Gamma(\alpha+4)+\Gamma(\alpha+3)(3N+N\alpha)},\end{aligned}$$

and

$$\begin{aligned}\lim_{N\to\infty} \widetilde{L}_2^\alpha(x) &= x^2 - 3\alpha - \alpha^2 - 2 \\ &= \left(x - \sqrt{(\alpha+2)(\alpha+1)}\right)\left(x + \sqrt{(\alpha+2)(\alpha+1)}\right).\end{aligned}$$

Thus,

$$l = \pm\sqrt{(\alpha+2)(\alpha+1)},$$

are the limit points of the least and the largest zero of $\widetilde{L}_2^\alpha(x)$ when $N \to \infty$.

When $n = 3$, taking the limit when $N \to \infty$ and using Maple, we get

$$\begin{aligned}\lim_{N\to\infty} \widetilde{L}_3^\alpha(x) = 60x - 47\alpha + 27x\alpha - 15x^2 + x^3 - 12\alpha^2 - \alpha^3 \\ + 3x\alpha^2 - 3x^2\alpha \\ + \frac{1}{2\alpha+5}\left(84x - 255\alpha + 57x\alpha - 84\alpha^2 - 9\alpha^3 + 9x\alpha^2 - 252\right) \\ + \frac{1}{2\alpha+5}\left(\begin{matrix}660\alpha - 384x - 312x\alpha + 48x^2 + 273\alpha^2 + 48\alpha^3 + 3\alpha^4 \\ -78x\alpha^2 + 27x^2\alpha - 6x\alpha^3 + 3x^2\alpha^2 + 576\end{matrix}\right) - 60,\end{aligned}$$

but, an expression showing the dependence of α for the zeros of this polynomial cannot be deduced in a straightforward way. In order to see how the zeros of $\widetilde{L}_3^\alpha(x)$ behave, we present some computations for $\widetilde{L}_3^2(x)$ and $\widetilde{L}_3^5(x)$.

	$\widetilde{L}_3^2(x)$	Zeros
$N=1$	$24x-11x^2+x^3$	0; 3; 8
$N=5$	$7.058\,8x-9.117\,6x^2+x^3+28.235$	7.7325; 2.725 1; −1.340 0
$N=100$	$0.397\,35x-8.377\,5x^2+x^3+39.338$	7.654 1; 2.657 4; −1.934
$N=1000$	$3.997\,3\times10^{-2}x-8.337\,8x^2+x^3+39.933$	7.650 3; 2.654 2; −1.966 6
$N=5000$	$7.998\,9\times10^{-3}x-8.334\,2x^2+x^3+39.987$	7.649 9, 2.653 9; −1.969 6
$N=10^6$	$4.000\,0\times10^{-5}x-8.333\,3x^2+x^3+40.000$	7.649 8; 2.653 9; −1.970 3
$N=10^{10}$	$4.000\,0\times10^{-9}x-8.333\,3x^2+x^3+40.000$	7.649 8; 2.653 9; −1.970 3

	$\widetilde{L}_3^5(x)$	Zeros
$N=1$	$164.57x-23.771x^2+x^3-325.03$	13.107; 7.237 4; 3.426 3
$N=20$	$118.59x-20.706x^2+x^3-177.88$	12.151; 6.189 8; 2.365 0
$N=50$	$82.286x-18.286x^2+x^3-61.714$	11.708; 5.644; 0.933 92
$N=80$	$63x-17x^2+x^3$	11.541; 5.458 6; 0
$N=1000$	$7.694\,7x-13.313x^2+x^3+176.98$	11.222; 5.152 1; −3.061 1
$N=5000$	$1.597\,5x-12.906x^2+x^3+196.49$	11.196; 5.130 9; −3.420 6
$N=10^{10}$	$8.064\,0\times10^{-7}x-12.8x^2+x^3+201.60$	11.190; 5.125 2; −3.515 2

Notice that $\widetilde{L}_3^2(x)$ has a negative zero for $N>1$ and $\widetilde{L}_3^5(x)$ has a negative zero for $N>80$. For $n=3$ and other choices of α, we can conjecture that the least zero decreases when N increases.

Our following goal is to determine, for fixed n and α, the values of N such that the polynomial $\widetilde{L}_n^\alpha(x)$ has a negative zero. Thus, let Π and N_0 be defined as follows

$$\Pi=\{N\in\mathbf{R}_+ : \widetilde{L}_n^\alpha(x) \text{ has a negative zero}\} \text{ and } N_0=\inf\Pi.$$

For $n=2$, $n=3$ and different choices of the parameter α, we obtain the approximate values of N such that $\widetilde{L}_n^\alpha(x)$ has a negative zero.

	Π	$\widetilde{L}_2^{\alpha}(x)$ for N_0
$\alpha = -0.99$	$N \gtrapprox 0.994\,4$	$x^2 - 1.010\,0x - 3.765\,7 \times 10^{-7}$
$\alpha = -1/2$	$N \gtrapprox 0.886\,3$	$x^2 - 1.499\,9x - 3.092\,0 \times 10^{-5}$
$\alpha = -0.1$	$N \gtrapprox 0.961\,78$	$x^2 - 1.900\,0x - 1.259\,5 \times 10^{-5}$
$\alpha = 0.1$	$N \gtrapprox 1.0465$	$x^2 - 2.1x - 1.562\,1 \times 10^{-5}$
$\alpha = 1/2$	$N \gtrapprox 1.329\,5$	$x^2 - 2.499\,8x - 2.251\,1 \times 10^{-4}$
$\alpha = 1$	$N > 2$	$x^2 - 3x$
$\alpha = 3$	$N > 24$	$x^2 - 5x$
$\alpha = 5$	$N > 720$	$x^2 - 7x$
$\alpha = 10$	$N > 39\,916\,800$	$x^2 - 12x$

	Π	$\widetilde{L}_3^{\alpha}(x)$ for N_0
$\alpha = -0.99$	$N \gtrapprox 0.3304$	$1.815\,4 \times 10^{-6} + 3.039\,8x - 5.019\,9x^2 + x^3$
$\alpha = -1/2$	$N \gtrapprox 0.2533$	$3.189\,2 \times 10^{-4} + 5.249\,0x - 5.999\,7x^2 + x^3$
$\alpha = -0.1$	$N \gtrapprox 0.2467$	$8.415\,5 \times 10^{-4} + 7.408\,5x - 6.799\,7x^2 + x^3$
$\alpha = 0.1$	$N \gtrapprox 0.2552$	$7.364\,6 \times 10^{-4} + 8.608\,9x - 7.199\,8x^2 + x^3$
$\alpha = 1/2$	$N \gtrapprox 0.2955$	$+1.732\,9 \times 10^{-3} + 11.248x - 7.999\,7x^2 + x^3$
$\alpha = 1$	$N > 0.4$	$15x - 9x^2 + x^3$
$\alpha = 3$	$N \gtrapprox 3.4286$	$3.888\,9 \times 10^{-4} + 35x - 13x^2 + x^3$
$\alpha = 5$	$N > 80$	$63x - 17x^2 + x^3$
$\alpha = 10$	$N > 2851200$	$168x - 27x^2 + x^3$

It is important to remark that, according the numerical experiments, we can find the exact value of N_0. Notice that the polynomial expression of the third column vanishes at $x = 0$, and, thus, for $N > N_0$, $\widetilde{L}_n^{\alpha}(x)$ has a negative zero.

Another approach is based in a different choice of the parameters. Indeed, we fix N and α, and we ask for the values of n such that $\widetilde{L}_n^{\alpha}(x)$ has a negative zero. If we denote n_0 the lowest n satisfying such a condition and we make the computation for some values of N, we obtain (28) and (29).

Finally, from (28) and (29), we see that if N is fixed, then the values of n_0 increase as α increase, and if we fix α the values of n_0 decrease when N increases.

$N=0.1$	$n \geq n_0$	$N=0.5$	$n > n_0$	$N=1$	$n \geq n_0$
$\alpha=-0.99$	$n \geq 5$	$\alpha=-0.99$	$n \geq 3$	$\alpha=-0.99$	$n \geq 2$
$\alpha=-0.1$	$n \geq 5$	$\alpha=-0.1$	$n \geq 3$	$\alpha=-0.1$	$n \geq 2$
$\alpha=0.1$	$n \geq 5$	$\alpha=0.1$	$n \geq 3$	$\alpha=0.1$	$n \geq 3$
$\alpha=1/2$	$n \geq 5$	$\alpha=1/2$	$n \geq 5$	$\alpha=1/2$	$n \geq 3$
$\alpha=1$	$n \geq 5$	$\alpha=1$	$n \geq 5$	$\alpha=1$	$n \geq 3$
$\alpha=3$	$n \geq 7$	$\alpha=3$	$n \geq 7$	$\alpha=3$	$n \geq 4$
$\alpha=5$	$n \geq 10$	$\alpha=5$	$n \geq 10$	$\alpha=5$	$n \geq 7$
$\alpha=10$	$n \geq 22$	$\alpha=10$	$n \geq 22$	$\alpha=8$	$n \geq 10$

(28)

$N=5$	n_0	$N=10$	n_0
$\alpha=-0.99$	$n \geq 2$	$\alpha=-0.99$	$n \geq 2$
$\alpha=-0.1$	$n \geq 2$	$\alpha=-0.1$	$n \geq 2$
$\alpha=0.1$	$n \geq 2$	$\alpha=0.1$	$n \geq 2$
$\alpha=1/2$	$n \geq 2$	$\alpha=1/2$	$n \geq 2$
$\alpha=1$	$n \geq 2$	$\alpha=1$	$n \geq 2$
$\alpha=3$	$n \geq 3$	$\alpha=3$	$n \geq 3$
$\alpha=5$	$n \geq 5$	$\alpha=5$	$n \geq 5$
$\alpha=8$	$n \geq 10$	$\alpha=10$	$n \geq 14$

(29)

7. Electrostatic Interpretation. We will analyze an electrostatic model that the zeros of the Laguerre-Sobolev-type orthogonal polynomials satisfy. Assume that $\{x_{n,k}^{(N)}\}_{k\geq 1}$ are the zeros of $\widehat{L}_n^\alpha(x)$ and we evaluate (26) in these zeros. Thus

$$A(x_{n,k}^{(N)};n)\left(\widetilde{L}_n^\alpha(x_{n,k}^{(N)})\right)'' + B(_{n,k}^{(N)};n)\left(\widetilde{L}_n^\alpha(x_{n,k}^{(N)})\right)' = 0.$$

As a consequence,

$$\frac{B(x_{n,k}^{(N)};n)}{A(x_{n,k}^{(N)};n)} = -\frac{\left(\widetilde{L}_n^\alpha(x_{n,k}^{(N)})\right)''}{\left(\widetilde{L}_n^\alpha(x_{n,k}^{(N)})\right)'}. \tag{30}$$

But, using the explicit expression of $A(x_{n,k}^{(N)};n)$ and $B(x_{n,k}^{(N)};n)$

$$\begin{aligned}\frac{B(x_{n,k}^{(N)};n)}{A(x_{n,k}^{(N)};n)} &= \frac{\alpha+3}{x_{n,k}^{(N)}} - 1 + \frac{\left(-n+\frac{B_n(n-2)}{(n-1)(n+\alpha+1)}\right) g_1(x_{n,k}^{(N)})}{K_n f_2(x_{n,k}^{(N)}) - f_1(x_{n,k}^{(N)}) g_1(x_{n,k}^{(N)})} \\ &\quad + \frac{\left(A_n(n-1) + \dfrac{B_n(n-2)(x_{n,k}^{(N)} - (2n+\alpha+1))}{(n-1)(n+\alpha+1)}\right) K_n}{K_n f_2(x_{n,k}^{(N)}) - f_1(x_{n,k}^{(N)}) g_1(x_{n,k}^{(N)})}.\end{aligned}$$

In order to find the electrostatic model, we will need some information concerning the zeros of $K_n f_2(x) - f_1(x) g_1(x)$. So

$$\begin{aligned}&K_n f_2(x) - f_1(x) g_1(x) \\ &\quad = K_n H_n + \tfrac{K_n[B_n x + (n+\alpha+1)(A_n(n-1)-B_n)](x-(2n+\alpha+1))}{(n-1)(n+\alpha+1)} \\ &\quad\quad - \left(n - \tfrac{B_n x + (n+\alpha+1)(A_n(n-1)-B_n)}{(n-1)(n+\alpha+1)}\right)\left(A_n + \tfrac{(x-(2n+\alpha+1))B_n}{(n-1)(n+\alpha+1)}\right) \\ &\quad = R(n,\alpha)x^2 + S(n,\alpha)x + T(n,\alpha)\end{aligned}$$

where

$$\begin{aligned}R(n,\alpha) &= \frac{K_n B_n}{(n-1)(n+\alpha+1)} + \frac{B_n^2}{(n-1)^2(n+\alpha+1)^2} \\ S(n,\alpha) &= \frac{K_n(n+\alpha+1)\left(A_n(n-1)-B_n\right) - B_n K_n(2n+\alpha+1)}{(n-1)(n+\alpha+1)} \\ &\quad - \frac{nB_n + A_n B_n}{(n-1)(n+\alpha+1)} \\ &\quad + \frac{B_n(n+\alpha+1)\left(A_n(n-1)-B_n\right) - B_n^2(2n+\alpha+1)}{(n-1)^2(n+\alpha+1)^2} \\ T(n,\alpha) &= K_n H_n - \frac{\left(A_n(n-1)-B_n\right)(2n+\alpha+1)K_n}{(n-1)} - A_n n \\ &\quad + \frac{nB_n(2n+\alpha+1)}{(n-1)(n+\alpha+1)} \\ &\quad + \frac{A_n\left(A_n(n-1)-B_n\right)}{(n-1)} \\ &\quad - \frac{(2n+\alpha+1)\left(A_n(n-1)-B_n\right)B_n}{(n-1)^2(n+\alpha+1)}.\end{aligned}$$

Assume that $x_1^{(\alpha,n)}$ and $x_2^{(\alpha,n)}$ are the zeros of $K_n f_2(x) - f_1(x) g_1(x)$. Then

$$\begin{aligned}\frac{B(x_{n,k}^{(N)};n)}{A(x_{n,k}^{(N)};n)} &= \frac{\alpha+3}{x_{n,k}^{(N)}} - 1 \\ &\quad + \frac{1/R(n,\alpha)(-n+(B_n(n-2)/(n-1)(n+\alpha+1)))g_1(x_{n,k}^{(N)})}{\left(x_{n,k}^{(N)}-x_1^{(\alpha,n)}\right)\left(x_{n,k}^{(N)}-x_2^{(\alpha,n)}\right)} \\ &\quad + \frac{1/R(n,\alpha)(A_n(n-1)+(B_n(n-2)(x-(2n+\alpha+1))/(n-1)(n+\alpha+1)))K_n}{\left(x_{n,k}^{(N)}-x_1^{(\alpha,n)}\right)\left(x_{n,k}^{(N)}-x_2^{(\alpha,n)}\right)}.\end{aligned}$$

But

$$\begin{aligned}&\frac{1}{R(n,\alpha)}\left(-n+\frac{B_n(n-2)}{(n-1)(n+\alpha+1)}\right)g_1(x) \\ &= \frac{(n-1)^2(n+\alpha+1)^2}{K_nB_n(n-1)(n+\alpha+1)+B_n^2} \\ &\quad \times\left(-n+\frac{B_n(n-2)}{(n-1)(n+\alpha+1)}\right)\left(A_n+\frac{(x-(2n+\alpha+1))B_n}{(n-1)(n+\alpha+1)}\right) \\ &= \left(\frac{B_n^2(n-2)-n(n-1)(n+\alpha+1)B_n}{K_nB_n(n-1)(n+\alpha+1)+B_n^2}\right)x \\ &\quad + \left(\frac{B_n(n-2)-n(n-1)(n+\alpha+1)}{K_nB_n(n-1)(n+\alpha+1)+B_n^2}\right) \\ &\quad \times\left(A_n(n-1)(n+\alpha+1)-(2n+\alpha+1)B_n\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\frac{1}{R(n,\alpha)}\left(A_n(n-1)+\frac{B_n(n-2)(x-(2n+\alpha+1))}{(n-1)(n+\alpha+1)}\right)K_n \\ &\qquad = \frac{(n-1)^2(n+\alpha+1)^2K_n}{K_nB_n(n-1)(n+\alpha+1)+B_n^2} \\ &\qquad\quad \times\left(A_n(n-1)+\frac{B_n(n-2)(x-(2n+\alpha+1))}{(n-1)(n+\alpha+1)}\right) \\ &\qquad = \frac{(n-1)(n-2)(n+\alpha+1)K_n}{K_n(n-1)(n+\alpha+1)+B_n}x \\ &\qquad\quad + \frac{(n-1)^3(n+\alpha+1)^2K_nA_n}{K_nB_n(n-1)(n+\alpha+1)+B_n^2} \\ &\qquad\quad - \frac{(n-1)(n-2)(n+\alpha+1)(2n+\alpha+1)K_n}{K_n(n-1)(n+\alpha+1)+B_n}.\end{aligned}$$

If we denote

$$\frac{1}{R(n,\alpha)}\left(-n+\frac{B_n(n-2)}{(n-1)(n+\alpha+1)}\right)g_1(x)=\kappa(n,\alpha)x+\lambda(n,\alpha)$$

and

$$\frac{1}{R(n,\alpha)}\left(A_n(n-1)+\frac{B_n(n-2)(x-(2n+\alpha+1))}{(n-1)(n+\alpha+1)}\right)K_n \\ =\nu(n,\alpha)x+\xi(n,\alpha)$$

where

$$\begin{aligned}
\kappa(n,\alpha)&=\frac{B_n^2(n-2)-n(n-1)(n+\alpha+1)B_n}{K_nB_n(n-1)(n+\alpha+1)+B_n^2},\\
\lambda(n,\alpha)&=\left(\frac{B_n(n-2)-n(n-1)(n+\alpha+1)}{K_nB_n(n-1)(n+\alpha+1)+B_n^2}\right)\\
&\quad\times\left(A_n(n-1)(n+\alpha+1)-(2n+\alpha+1)B_n\right),\\
\nu(n,\alpha)&=\frac{(n-1)(n-2)(n+\alpha+1)K_n}{K_n(n-1)(n+\alpha+1)+B_n},\\
\xi(n,\alpha)&=\frac{(n-1)^3(n+\alpha+1)^2K_nA_n}{K_nB_n(n-1)(n+\alpha+1)+B_n^2}\\
&\quad-\frac{(n-1)(n-2)(n+\alpha+1)(2n+\alpha+1)K_n}{K_n(n-1)(n+\alpha+1)+B_n},
\end{aligned}$$

then

$$\begin{aligned}
\frac{B(x_{n,k}^{(N)};n)}{A(x_{n,k}^{(N)};n)}&=\frac{\alpha+3}{x_{n,k}^{(N)}}-1+\frac{\kappa(n,\alpha)x_{n,k}^{(N)}+\lambda(n,\alpha)}{\left(x_{n,k}^{(N)}-x_1^{(\alpha,n)}\right)\left(x_{n,k}^{(N)}-x_2^{(\alpha,n)}\right)}\\
&\quad+\frac{\nu(n,\alpha)x_{n,k}^{(N)}+\xi(n,\alpha)}{\left(x_{n,k}^{(N)}-x_1^{(\alpha,n)}\right)\left(x_{n,k}^{(N)}-x_2^{(\alpha,n)}\right)}\\
&=\frac{\alpha+3}{x_{n,k}^{(N)}}-1+\frac{(\kappa(n,\alpha)x_1^{(\alpha,n)}+\lambda(n,\alpha))/(x_1^{(\alpha,n)}-x_2^{(\alpha,n)})}{\left(x_{n,k}^{(N)}-x_1^{(\alpha,n)}\right)}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(\kappa(n,\alpha)x_2^{(\alpha,n)}+\lambda(n,\alpha))/(x_2^{(\alpha,n)}-x_1^{(\alpha,n)})}{\left(x_{n,k}^{(N)}-x_2^{(\alpha,n)}\right)}\\
&+\frac{(\nu(n,\alpha)x_1^{(\alpha,n)}+\xi(n,\alpha))/(x_1^{(\alpha,n)}-x_2^{(\alpha,n)})}{\left(x_{n,k}^{(N)}-x_1^{(\alpha,n)}\right)}\\
&+\frac{(\nu(n,\alpha)x_2^{(\alpha,n)}+\xi(n,\alpha))/(x_2^{(\alpha,n)}-x_1^{(\alpha,n)})}{\left(x_{n,k}^{(N)}-x_2^{(\alpha,n)}\right)}\\
=&\ \frac{\alpha+3}{x_{n,k}^{(N)}}-1\\
&+\frac{(\kappa(n,\alpha)x_1^{(\alpha,n)}+\lambda(n,\alpha)+\nu(n,\alpha)x_1^{(\alpha,n)}+\xi(n,\alpha))/(x_1^{(\alpha,n)}-x_2^{(\alpha,n)})}{x_{n,k}^{(N)}-x_1^{(\alpha,n)}}\\
&+\frac{(\kappa(n,\alpha)x_2^{(\alpha,n)}+\lambda(n,\alpha)+\nu(n,\alpha)x_2^{(\alpha,n)}+\xi(n,\alpha))/(x_2^{(\alpha,n)}-x_1^{(\alpha,n)})}{x_{n,k}^{(N)}-x_2^{(\alpha,n)}}.
\end{aligned}
$$

Taking into account that

$$
\frac{\left(\widetilde{L}_n^a(x_{n,k}^{(N)})\right)''}{\left(\widetilde{L}_n^a(x_{n,k}^{(N)})\right)'}=-2\sum_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{x_{n,j}^{(N)}-x_{n,k}^{(N)}},
$$

and using the last expression of $B(x_{n,k}^{(N)};n)/A(x_{n,k}^{(N)};n)$ we get

$$
\begin{aligned}
0=&\sum_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{x_{n,j}^{(N)}-x_{n,k}^{(N)}}-\frac{\alpha+3-x_{n,k}^{(M)}}{2x_{n,k}^{(N)}}\\
&+\frac{(\kappa(n,\alpha)x_1^{(\alpha,n)}+\lambda(n,\alpha)+\nu(n,\alpha)x_1^{(\alpha,n)}+\xi(n,\alpha))/(x_1^{(\alpha,n)}-x_2^{(\alpha,n)})}{2\left(x_{n,k}^{(N)}-x_1^{(\alpha,n)}\right)}\\
&+\frac{(\kappa(n,\alpha)x_2^{(\alpha,n)}+\lambda(n,\alpha)+\nu(n,\alpha)x_2^{(\alpha,n)}+\xi(n,\alpha))/(x_2^{(\alpha,n)}-x_1^{(\alpha,n)})}{2\left(x_{n,k}^{(N)}-x_2^{(\alpha,n)}\right)}.
\end{aligned}
$$

If we denote

$$
\begin{aligned}
C_n&=\frac{\kappa(n,\alpha)x_1^{(\alpha,n)}+\lambda(n,\alpha)+\nu(n,\alpha)x_1^{(\alpha,n)}+\xi(n,\alpha)}{(x_1^{(\alpha,n)}-x_2^{(\alpha,n)})}\\
D_n&=\frac{\kappa(n,\alpha)x_2^{(\alpha,n)}+\lambda(n,\alpha)+\nu(n,\alpha)x_2^{(\alpha,n)}+\xi(n,\alpha)}{x_2^{(\alpha,n)}-x_1^{(\alpha,n)}},
\end{aligned}
$$

then

$$\sum_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{x_{n,j}^{(N)}-x_{n,k}^{(N)}}-\frac{\alpha+3-x_{n,k}^{(N)}}{2x_{n,k}^{(N)}}+\frac{C_n}{2(x_{n,k}^{(N)}-x_1^{(\alpha,n)})}+\frac{D_n}{2\left(x_{n,k}^{(N)}-x_2^{(\alpha,n)}\right)}=0. \tag{31}$$

Thus, the following electrostatic interpretation for the location of zeros of $\widetilde{L}_n^{\alpha}$ can be stated. If we consider n charges located in the real line under a logarithmic interaction with an external field

$$\varphi(x)=-(1/2)\ln(x^{\alpha+3}e^{-x})+\frac{C_n}{2}\ln\left|x-x_1^{(\alpha,n)}\right|+\frac{D_n}{2}\ln\left|x-x_2^{(\alpha,n)}\right|,$$

(31) means that the gradient of the total energy

$$E(X)=-\sum_{1\leq k<j\leq n}\ln|x_k-x_j|+\sum_{j=1}^{n}\varphi(x_j)$$

with $X=(x_1,x_2,\dots,x_n)$ vanishes at $(x_{n,1}^{(M)},x_{n,2}^{(M)},\dots,x_{n,n}^{(M)})$. In other words, it is a critical point. See [**7, 9, 11**] for a more general study of the electrostatic interpretation of standard orthogonal polynomials.

The zeros of $R(n,\alpha)x^2+S(n,\alpha)x+T(n,\alpha)$, $x_1^{(\alpha,n)}$ and $x_2^{(\alpha,n)}$, play an important role in the electrostatic interpretation. So, we would like to see what is the behavior of these zeros. Therefore, we are going to make some numerical analysis about it. At the first time, we fix α and N, and we will see the behavior when $n\to\infty$.

$\alpha=1, N=1$	$R(n,\alpha)x^2+S(n,\alpha)x+T(n,\alpha)$	Zeros
$n=2$	$1.5x^2-21.0x+1.0$	$13.952;\ 4.7782\times10^{-2}$
$n=5$	$1.5139x^2-43.772x-0.47258$	$28.924;\ -1.0792\times10^{-2}$
$n=10$	$1.2295x^2-59.668x-0.10128$	$48.532;\ -1.6973\times10^{-3}$
$n=100$	$1.0203x^2-416.32x-8.1896\times10^{-4}$	$408.04;\ -1.9671\times10^{-6}$
$n=1000$	$1.002x^2-4016.0x-8.0187\times10^{-6}$	$4008.0;\ -1.9967\times10^{-9}$

(32)

$\alpha = 1/2,\ N = 1$	$R(n,\alpha)x^2 + S(n,\alpha)x + T(n,\alpha)$	Zeros
$n = 2$	$1.798x^2 - 24.683x + 0.325\,17$	$13.715;\ 1.318\,7 \times 10^{-2}$
$n = 5$	$1.512\,3x^2 - 41.886x - 0.304\,09$	$27.704;\ -7.258 \times 10^{-3}$
$n = 10$	$1.228\,5x^2 - 58.245x - 6.542\,5 \times 10^{-2}$	$47.413;\ -1.123\,2 \times 10^{-3}$
$n = 100$	$1.020\,3x^2 - 415.29x - 5.367\,2 \times 10^{-4}$	$407.03;\ -1.292\,4 \times 10^{-6}$
$n = 1000$	$1.002x^2 - 4015.0x - 5.261\,6 \times 10^{-6}$	$4007.0;\ -1.310\,5 \times 10^{-9}$

(33)

$\alpha = -1/2, N = 1$	$R(n,\alpha)x^2 + S(n,\alpha)x + T(n,\alpha)$	Zeros
$n = 2$	$2.072\,4x^2 - 24.647x - 0.028\,78$	$11.894;\ -1.167\,6 \times 10^{-3}$
$n = 5$	$1.489\,8x^2 - 37.437x - 6.128\,9 \times 10^{-2}$	$25.131;\ -1.637 \times 10^{-3}$
$n = 10$	$1.224\,1x^2 - 55.245x - 1.450\,5 \times 10^{-2}$	$45.131;\ -2.625\,6 \times 10^{-4}$
$n = 100$	$1.020\,2x^2 - 413.21x - 1.271\,0 \times 10^{-4}$	$405.03;\ -3.075\,9 \times 10^{-7}$
$n = 1000$	$1.002x^2 - 4013.0x - 1.252\,1 \times 10^{-6}$	$4005.0;\ -3.120\,1 \times 10^{-10}$

(34)

From the tables (32), (33) and (34), we can infer that the largest zero of $R(n,\alpha)x^2 + S(n,\alpha)x + T(n,\alpha)$ grows when n grows, and the lowest zero tends to 0 when n tends to ∞.

On the other hand, if we fix α and n, then we will analyze the behavior of $x_1^{(\alpha,n)}$ and $x_2^{(\alpha,n)}$ when N tends to ∞.

$\alpha = 1;\ n = 2$	$R(n,\alpha)x^2 + S(n,\alpha)x + T(n,\alpha)$	Zeros
$N = 1$	$1.5x^2 - 21.0x + 1.0$	$13.952;\ 4.778\,2 \times 10^{-2}$
$N = 10$	$3.0x^2 - 54.0x - 5.0$	$18.092;\ -9.212\,1 \times 10^{-2}$
$N = 100$	$3.441\,2x^2 - 65.419x - 8.477\,5$	$19.139;\ -0.128\,72$
$N = 1000$	$3.494x^2 - 66.838x - 8.946\,2$	$19.262;\ -0.132\,93$
$N = 10^5$	$3.499\,9x^2 - 66.998x - 8.999\,5$	$19.276;\ -0.133\,40$
$N = 10^7$	$3.500\,0x^2 - 67.000x - 9.000\,0$	$19.276;\ -0.133\,40$

(35)

$\alpha = 1$; $n = 5$	$R(n,\alpha)x^2 + S(n,\alpha)x + T(n,\alpha)$	Zeros
$N = 1$	$1.513\,9x^2 - 43.772x - 0.472\,58$	28.924; $-1.079\,2 \times 10^{-2}$
$N = 10$	$1.535\,9x^2 - 44.688x - 0.526\,26$	29.107; $-1.177\,2 \times 10^{-2}$
$N = 100$	$1.538\,2x^2 - 44.782x - 0.531\,91$	29.125; $-1.187\,3 \times 10^{-2}$
$N = 1000$	$1.538\,4x^2 - 44.792x - 0.532\,48$	29.128; $-1.188\,3 \times 10^{-2}$
$N = 10^5$	$1.538\,5x^2 - 44.793x - 0.532\,54$	29.127; $-1.188\,4 \times 10^{-2}$
$N = 10^7$	$1.538\,5x^2 - 44.793x - 0.532\,54$	29.127; $-1.188\,4 \times 10^{-2}$

(36)

From (35) and (36) we can infer that there exist real constants a and b such that

$$\lim_{N\to\infty} x_1^{(\alpha,n)} = a \quad \text{and} \quad \lim_{N\to\infty} x_2^{(\alpha,n)} = b.$$

In fact, using the Maple software, we obtain

$$\begin{aligned}
&\lim_{N\to\infty} \big(R(2,\alpha)x^2 + S(2,\alpha)x + T(2,\alpha)\big)\\
&\quad = \Bigg[x - \frac{1}{2\alpha+5}\bigg(\frac{49}{2}\alpha + \frac{7}{2}\alpha^2\\
&\qquad - \frac{1}{2}\sqrt{7836\alpha + 3657\alpha^2 + 746\alpha^3 + 57\alpha^4 + 6164} + 39\bigg)\Bigg]\\
&\qquad \times \Bigg[x - \frac{1}{2\alpha+5}\bigg(\frac{49}{2}\alpha + \frac{7}{2}\alpha^2\\
&\qquad\quad + \frac{1}{2}\sqrt{7836\alpha + 3657\alpha^2 + 746\alpha^3 + 57\alpha^4 + 6164} + 39\bigg)\Bigg].
\end{aligned}$$

8. Asymptotic behavior. First, we will study the behavior of the sequences of real numbers A_n and B_n. Using (21) and making some computations, we get

$$\begin{aligned}
\frac{A_n}{n} &= \tfrac{2(n-2)!\Gamma(\alpha+2)\Gamma(\alpha+4)+N\Gamma(n+\alpha+1)\left(2n(\alpha+2)+\alpha^2+4+3\alpha\right)}{(n-2)!\Gamma(\alpha+2)\Gamma(\alpha+4)+N\Gamma(n+\alpha+1)(n(\alpha+2)-(\alpha+1))}\\
&= 2 + \tfrac{\alpha+3}{n} + \tfrac{(\alpha+3)(\alpha+1)N\Gamma(n+\alpha+1)-(\alpha+3)(n-2)!\Gamma(\alpha+2)\Gamma(\alpha+4)}{nN\Gamma(n+\alpha+1)(n(\alpha+2)-(\alpha+1))+(n-2)!\Gamma(\alpha+2)\Gamma(\alpha+4)}.
\end{aligned}$$

Thus,

$$\frac{A_n}{n} = 2 + \frac{\alpha+3}{n} + \frac{(\alpha+1)(\alpha+3)}{(\alpha+2)\,n^2} + \mathcal{O}\left(\frac{1}{n^3}\right). \tag{37}$$

On the other hand, for B_n we have

$$\begin{aligned}
&\frac{B_n}{n(n-1)} \\
&\quad = \frac{(n-2)!\Gamma(\alpha+2)\Gamma(\alpha+4)+\frac{N}{n-1}\Gamma(n+\alpha+1)\big(n^2\alpha+2n^2+n\big(\alpha^2+3\alpha+3\big)+(\alpha+1)\big)}{(n-2)!\Gamma(\alpha+2)\Gamma(\alpha+4)+N\Gamma(n+\alpha+1)(n(\alpha+2)-(\alpha+1))} \\
&\quad = \frac{(n-1)!\Gamma(\alpha+2)\Gamma(\alpha+4)+N\Gamma(n+\alpha+1)\big(n^2\alpha+2n^2+n\big(\alpha^2+3\alpha+3\big)+(\alpha+1)\big)}{(n-1)!\Gamma(\alpha+2)\Gamma(\alpha+4)+N\Gamma(n+\alpha+1)(n(\alpha+2)-(\alpha+1))(n-1)} \\
&\quad = 1 + \frac{\alpha+3}{n} + \frac{(\alpha+3)(2\alpha+3)}{(\alpha+2)n^2} + \mathcal{O}\left(\frac{1}{n^3}\right),
\end{aligned}$$

as a consequence

$$\frac{B_n}{n(n-1)} = 1 + \frac{\alpha+3}{n} + \frac{(\alpha+3)(2\alpha+3)}{(\alpha+2)\,n^2} + \mathcal{O}\left(\frac{1}{n^3}\right). \tag{38}$$

From (37) and (38) we get

Proposition 5. *For $n \in \mathbf{N}$,*

$$\lim_{n\to\infty} \frac{A_n}{n} = 2 \quad \textit{and} \quad \lim_{n\to\infty} \frac{B_n}{n(n-1)} = 1. \tag{39}$$

In [**2**] the authors analyze the asymptotic behavior of the Laguerre-Sobolev-type orthogonal polynomials $L_n^{(\alpha,M,N)}(x)$ with leading coefficient $(-1)^n/n!$. In the sequel we will present an alternative approach to such results for $\widetilde{L}_n^\alpha(x)$ when $M=0$. If we denote by $\widehat{L}_n^\alpha(x)$ the Laguerre orthogonal polynomial of degree n with leading coefficient $(-1)^n/n!$ and by $\widehat{\widetilde{L}}_n^\alpha(x)$ the Laguerre-Sobolev-type orthogonal polynomial of degree n with the same leading coefficient, taking into account(21), we have

$$\widehat{\widetilde{L}}_n^\alpha(x) = \widehat{L}_n^{\alpha+2}(x) - \frac{A_n}{n}\widehat{L}_{n-1}^{\alpha+2}(x) + \frac{B_n}{n(n-1)}\widehat{L}_{n-2}^{\alpha+2}(x). \tag{40}$$

From (37) and (38) we obtain

$$\begin{aligned}\widehat{\widetilde{L}}_n^{\alpha}(x) &= \widehat{L}_n^{\alpha+2}(x) - \left(2 + \frac{\alpha+3}{n}\right)\widehat{L}_{n-1}^{\alpha+2}(x)\\ &\quad + \left(1 + \frac{\alpha+3}{n}\right)\widehat{L}_{n-2}^{\alpha+2}(x)\\ &\quad - \widehat{L}_{n-1}^{\alpha+2}(x)\mathcal{O}\left(\frac{1}{n^2}\right) + \widehat{L}_{n-2}^{\alpha+2}(x)\mathcal{O}\left(\frac{1}{n^2}\right)\\ &= \widehat{L}_n^{\alpha}(x) - \frac{\alpha+3}{n}\widehat{L}_{n-1}^{\alpha+2}(x) + \frac{\alpha+3}{n}\widehat{L}_{n-2}^{\alpha+2}(x)\\ &\quad - \widehat{L}_{n-1}^{\alpha+2}(x)\mathcal{O}\left(\frac{1}{n^2}\right) + \widehat{L}_{n-2}^{\alpha+2}(x)\mathcal{O}\left(\frac{1}{n^2}\right).\end{aligned}$$

Dividing on both sides by $\widehat{L}_n^{\alpha}(x)$, we get

$$\begin{aligned}\frac{\widehat{\widetilde{L}}_n^{\alpha}(x)}{\widehat{L}_n^{\alpha}(x)} &= \frac{\widetilde{L}_n^{\alpha}(x)}{L_n^{\alpha}(x)}\\ &= 1 - \frac{\alpha+3}{n}\frac{\widehat{L}_{n-1}^{\alpha+2}(x)}{\widehat{L}_n^{\alpha}(x)} + \frac{\alpha+3}{n}\frac{\widehat{L}_{n-2}^{\alpha+2}(x)}{\widehat{L}_n^{\alpha}(x)}\\ &\quad - \frac{\widehat{L}_{n-1}^{\alpha+2}(x)}{\widehat{L}_n^{\alpha}(x)}\mathcal{O}\left(\frac{1}{n^2}\right) + \frac{\widehat{L}_{n-2}^{\alpha+2}(x)}{\widehat{L}_n^{\alpha}(x)}\mathcal{O}\left(\frac{1}{n^2}\right).\end{aligned}\tag{41}$$

But, if $h, k \in \mathbf{Z}$, then

$$\lim_{n\to\infty}\frac{\widehat{L}_{n+k}^{\alpha+2}(x)}{n\widehat{L}_{n+h}^{\alpha}(x)} = -\frac{1}{x},\tag{42}$$

uniformly on compact subsets of $\mathbf{C}\backslash[0,\infty)$ (see [**2, 18**]).

Using (41), we have

$$\lim_{n\to\infty}\frac{\widetilde{L}_n^{\alpha}(x)}{L_n^{\alpha}(x)} = 1 + \frac{(\alpha+3)}{x} - \frac{(\alpha+3)}{x} = 1,$$

and, as a consequence, the outer relative asymptotic formula follows.

Proposition 6. *For $n \in \mathbf{N}$,*

$$\lim_{n\to\infty} \frac{\widetilde{L}_n^{\alpha}(x)}{L_n^{\alpha}(x)} = 1 \tag{43}$$

uniformly on compact subsets of $\mathbf{C}\backslash[0,\infty)$.

It is known (see [**2, 18**])that

$$\frac{\widehat{L}_n^{\alpha}(x)}{n^{\alpha/2}} = e^{x/2}x^{-\alpha/2}J_{\alpha}\left(2\sqrt{nx}\right) + \mathcal{O}\left(n^{-3/4}\right) \tag{44}$$

uniformly on compact subsets of $(0,\infty)$. We will use this result in order to find the following result.

From (37) and (38) in (40) we have

$$\begin{aligned}\widehat{\widehat{L}}_n^{\alpha}(x) &= \widehat{L}_n^{\alpha}(x) - \frac{\alpha+3}{n}\widehat{L}_{n-1}^{\alpha+1}(x)\\ &\quad - \frac{(\alpha+3)}{(\alpha+2)\,n^2}\left((\alpha+1)\widehat{L}_{n-1}^{\alpha+2}(x) - (2\alpha+3)\widehat{L}_{n-2}^{\alpha+2}(x)\right)\\ &\quad - \widehat{L}_{n-1}^{\alpha+2}(x)\mathcal{O}\left(\frac{1}{n^3}\right) + \widehat{L}_{n-2}^{\alpha+2}(x)\mathcal{O}\left(\frac{1}{n^3}\right)\end{aligned}$$

dividing on both sides for $n^{\alpha/2}$

$$\begin{aligned}\frac{\widehat{\widehat{L}}_n^{\alpha}(x)}{n^{\alpha/2}} &= \frac{\widehat{L}_n^{\alpha}(x)}{n^{\alpha/2}} - \frac{\alpha+3}{n^{1/2}}\frac{\widehat{L}_{n-1}^{\alpha+1}(x)}{n^{(\alpha+1)/2}}\\ &\quad - \frac{(\alpha+3)}{(\alpha+2)\,n}\left((\alpha+1)\frac{\widehat{L}_{n-1}^{\alpha+2}(x)}{n^{(\alpha+2)/2}} - (2\alpha+3)\frac{\widehat{L}_{n-2}^{\alpha+2}(x)}{n^{(\alpha+2)/2}}\right)\\ &\quad - \frac{\widehat{L}_{n-1}^{\alpha+2}(x)}{n^{(\alpha+2)/2}}\mathcal{O}\left(\frac{1}{n^2}\right) + \frac{\widehat{L}_{n-2}^{\alpha+2}(x)}{n^{(\alpha+2)/2}}\mathcal{O}\left(\frac{1}{n^2}\right),\end{aligned}$$

and using (44) we get

Proposition 7. *For* $n \in \mathbf{N}$,

$$\begin{aligned}\frac{\widehat{\widehat{L}}_n^{\alpha}(x)}{n^{\alpha/2}} = e^{x/2}x^{-\alpha/2}\Bigg(& J_\alpha\left(2\sqrt{nx}\right) - \frac{\alpha+3}{\sqrt{nx}} J_{\alpha+1}\left(2\sqrt{(n-1)x}\right) \\ & - \frac{(\alpha+3)}{(\alpha+2)\,n}\Big[(\alpha+1)J_{\alpha+2}\left(2\sqrt{(n-1)x}\right) \\ & -(2\alpha+3)J_{\alpha+2}\left(2\sqrt{(n-2)x}\right)\Big]\Bigg) \\ & + \mathcal{O}\left(n^{-3/4}\right)\end{aligned} \tag{45}$$

uniformly on compact subsets of $(0, \infty)$.

In order to find a scaled strong asymptotic formula, we use (40) and introduce the change of variable nx in $\widehat{\widehat{L}}_n^{\alpha}(x)$. As a consequence,

$$\widehat{\widehat{L}}_n^{\alpha}(nx) = \widehat{L}_n^{\alpha+2}(nx) - \frac{A_n}{n}\widehat{L}_{n-1}^{\alpha+2}(nx) + \frac{B_n}{n(n-1)}\widehat{L}_{n-2}^{\alpha+2}(nx),$$

and taking into account (37) and (38) we deduce

$$\begin{aligned}\widehat{\widehat{L}}_n^{\alpha}(nx) = \widehat{L}_n^{\alpha}(nx) - \frac{(\alpha+3)}{n}\widehat{L}_{n-1}^{\alpha+2}(nx) + \frac{(\alpha+3)}{n}\widehat{L}_{n-2}^{\alpha+2}(nx) \\ - \widehat{L}_{n-1}^{\alpha+2}(nx)\mathcal{O}\left(\frac{1}{n^2}\right) + \widehat{L}_{n-2}^{\alpha+2}(nx)\mathcal{O}\left(\frac{1}{n^2}\right).\end{aligned}$$

Thus

$$\begin{aligned}\frac{\widehat{\widehat{L}}_n^{\alpha}(nx)}{\widehat{L}_n^{\alpha}(nx)} = 1 - \frac{(\alpha+3)}{n}\frac{\widehat{L}_{n-1}^{\alpha+2}(nx)}{\widehat{L}_n^{\alpha}(nx)} + \frac{(\alpha+3)}{n}\frac{\widehat{L}_{n-2}^{\alpha+2}(nx)}{\widehat{L}_n^{\alpha}(nx)} \\ - \frac{\widehat{L}_{n-1}^{\alpha+2}(nx)}{\widehat{L}_n^{\alpha}(nx)}\mathcal{O}\left(\frac{1}{n^2}\right) + \frac{\widehat{L}_{n-2}^{\alpha+2}(nx)}{\widehat{L}_n^{\alpha}(nx)}\mathcal{O}\left(\frac{1}{n^2}\right).\end{aligned} \tag{46}$$

We want to find the limit when n tends to ∞ on the left hand side of the previous identity. Using that (see [**2, 18**]

$$\lim_{n\to\infty} \frac{\widehat{L}_{n-1}^{\alpha}(nx)}{\widehat{L}_n^{\alpha}(nx)} = -\frac{1}{\varphi\left((x-2)/2\right)} \tag{47}$$

uniformly on compact subsets of $\mathbf{C}\backslash[0,4]$, where φ is the mapping of $\mathbf{C}\backslash[-1,1]$ onto the exterior of the unit circle given by

$$\varphi(x) = x + \sqrt{x^2-1},$$

Alvarez-Nodarse and Moreno-Balcazar proved in [**2**] that

$$\lim_{n\to\infty} \frac{\widehat{L}_n^{\alpha}(nx)}{\widehat{L}_{n-1}^{\alpha+2}(nx)} = -\frac{(\varphi\left((x-2)/2\right)+1)^2}{\varphi\left((x-2)/2\right)}. \tag{48}$$

Then, using (47) and (48) we can conclude

$$\begin{aligned}
\lim_{n\to\infty} \frac{\widehat{L}_{n-2}^{\alpha+2}(nx)}{\widehat{L}_n^{\alpha}(nx)} &= \lim_{n\to\infty} \frac{\widehat{L}_{n-2}^{\alpha+2}(nx)}{\widehat{L}_{n-1}^{\alpha+2}(nx)} \frac{\widehat{L}_{n-1}^{\alpha+2}(nx)}{\widehat{L}_n^{\alpha}(nx)} \\
&= \left(-\frac{1}{\varphi\left((x-2)/2\right)}\right)\left(-\frac{\varphi\left((x-2)/2\right)}{(\varphi\left((x-2)/2\right)+1)^2}\right) \\
&= \frac{1}{(\varphi\left((x-2)/2\right)+1)^2}
\end{aligned}$$

uniformly on compact subsets of $\mathbf{C}\backslash[0,4]$. As a conclusion, from (46) we get the relative asymptotics for the scaled Laguerre-Sobolev-type orthogonal polynomials

Proposition 8. *For $n \in \mathbf{N}$,*

$$\lim_{n\to\infty} \frac{\widehat{\widetilde{L}}{}_n^{\alpha}(nx)}{\widehat{L}_n^{\alpha}(nx)} = \lim_{n\to\infty} \frac{\widetilde{L}_n^{\alpha}(nx)}{L_n^{\alpha}(nx)} = 1 \tag{49}$$

uniformly on compact subsets of $\mathbf{C}\backslash[0,4]$.

On the other hand, taking into account the Mehler-Heine type formula

$$\lim_{n\to\infty} \frac{\widehat{L}_n^{\alpha}(x/(n+j))}{n^{\alpha}} = x^{-\alpha/2} J_{\alpha}(2\sqrt{x}) \tag{50}$$

that holds uniformly on compact subsets of $\mathbf{C}\backslash[0,4]$ then, from (37) and (38), expression (40) becomes

$$\begin{aligned}\widehat{\widehat{L}}_n^{\alpha}(x) &= \widehat{L}_n^{\alpha}(x) - \frac{\alpha+3}{n}\left(\widehat{L}_{n-1}^{\alpha+2}(x) - \widehat{L}_{n-2}^{\alpha+2}(x)\right)\\ &\quad - \frac{(\alpha+1)(\alpha+3)}{(\alpha+2)\,n^2}\widehat{L}_{n-1}^{\alpha+2}(x) + \frac{(\alpha+3)(2\alpha+3)}{(\alpha+2)\,n^2}\widehat{L}_{n-2}^{\alpha+2}(x)\\ &\quad - \widehat{L}_{n-1}^{\alpha+2}(x)\mathcal{O}\left(\frac{1}{n^3}\right) + \widehat{L}_{n-2}^{\alpha+2}(x)\mathcal{O}\left(\frac{1}{n^3}\right)\\ &= \widehat{L}_n^{\alpha}(x) - \frac{\alpha+3}{n}\widehat{L}_{n-1}^{\alpha+1}(x)\\ &\quad + \frac{(\alpha+3)}{(\alpha+2)\,n^2}\left((2\alpha+3)\,\widehat{L}_{n-2}^{\alpha+2}(x) - (\alpha+1)\,\widehat{L}_{n-1}^{\alpha+2}(x)\right)\\ &\quad - \widehat{L}_{n-1}^{\alpha+2}(x)\mathcal{O}\left(\frac{1}{n^3}\right) + \widehat{L}_{n-2}^{\alpha+2}(x)\mathcal{O}\left(\frac{1}{n^3}\right).\end{aligned}$$

If we replace x/n in the above identity, dividing in both sides by n^{α},

$$\begin{aligned}\frac{\widehat{\widehat{L}}_n^{\alpha}(x/n)}{n^{\alpha}} &= \frac{\widehat{L}_n^{\alpha}(nx)}{n^{\alpha}} - (\alpha+3)\frac{\widehat{L}_{n-1}^{\alpha+1}(x/n)}{n^{\alpha+1}}\\ &\quad + \frac{(\alpha+3)}{(\alpha+2)}\left((2\alpha+3)\,\frac{\widehat{L}_{n-2}^{\alpha+2}(x)}{n^{\alpha+2}} - (\alpha+1)\,\frac{\widehat{L}_{n-1}^{\alpha+2}(x)}{n^{\alpha+2}}\right)\\ &\quad - \frac{\widehat{L}_{n-1}^{\alpha+2}(x)}{n^{\alpha+2}}\mathcal{O}\left(\frac{1}{n}\right) + \frac{\widehat{L}_{n-2}^{\alpha+2}(x)}{n^{\alpha+2}}\mathcal{O}\left(\frac{1}{n}\right),\end{aligned}$$

and taking the limit when n tends to ∞, from (50) we get

$$\begin{aligned}\lim_{n\to\infty}\frac{\widehat{\widehat{L}}_n^{\alpha}(x/n)}{n^{\alpha}} &= x^{-\alpha/2}J_{\alpha}(2\sqrt{x}) - (\alpha+3)x^{-(\alpha+1)/2}J_{\alpha+1}(2\sqrt{x})\\ &\quad + \frac{(\alpha+3)}{(\alpha+2)}\Big((2\alpha+3)\,x^{-(\alpha+2)/2}J_{\alpha+2}(2\sqrt{x})\\ &\quad - (\alpha+1)\,x^{-(\alpha+2)/2}J_{\alpha+2}(2\sqrt{x})\Big)\end{aligned}$$

uniformly on compact subsets of $\mathbf{C}$. Then

Proposition 9. *For $n \in \mathbf{N}$,*

$$\lim_{n\to\infty} \frac{\widehat{\widetilde{L}}_n^{\alpha}(x/n)}{n^{\alpha}} = x^{-\alpha/2}\left[J_{\alpha}(2\sqrt{x}) - \frac{(\alpha+3)}{\sqrt{x}}J_{\alpha+1}(2\sqrt{x}) + \frac{(\alpha+3)}{x}J_{\alpha+2}(2\sqrt{x})\right] \tag{51}$$

uniformly on compact subsets of $\mathbf{C}$.

Finally, we present some results about the behavior of the norm of the Laguerre Sobolev-type orthogonal polynomials. Taking into account that

$$\frac{\left\|\widetilde{L}_n^{\alpha}\right\|_S^2}{n(n-1)\left\|L_{n-2}^{\alpha+2}\right\|_{\alpha+2}^2} = \frac{B_n}{n(n-1)} \longrightarrow 1, \tag{52}$$

we have

$$\frac{\left\|\widetilde{L}_n^{\alpha}\right\|_S^2}{n(n-1)\left\|L_n^{\alpha}\right\|_{\alpha}^2}\frac{\left\|L_n^{\alpha}\right\|_{\alpha}^2}{\left\|L_{n-2}^{\alpha+2}\right\|_{\alpha+2}^2} \longrightarrow 1.$$

This means that

$$\frac{\left\|\widetilde{L}_n^{\alpha}\right\|_S^2}{\left\|L_n^{\alpha}\right\|_{\alpha}^2} \to 1. \tag{53}$$

But

$$\begin{aligned}\left\|L_n^{\alpha}\right\|_{\alpha}^{2/n} &= \left(n!\Gamma(n+\alpha+1)\right)^{1/n}\\ &\simeq n^2e^{-2}\end{aligned}$$

then

$$n^{-1}\left\|L_n^{\alpha}\right\|_{\alpha}^{1/n} \simeq e^{-1}. \tag{54}$$

As a conclusion, from (53) and (54), we obtain

Proposition 10. *For $n \in \mathbf{N}$,*

$$n^{-1}\left\|\widetilde{L}_n^{\alpha}\right\|_S^{1/n} \longrightarrow e^{-1}. \tag{55}$$

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