ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 41, Number 1, 2011

A CHARACTERIZATION OF NORMED SPACES AMONG METRIC SPACES

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ABSTRACT. We obtain an improvement of a metric characterization of real normed spaces obtained by Oikhberg and Rosenthal and provide a much shorter self-contained proof.

1. Introduction and statement of the main result. If X is a real normed space, then X is a metric space with the metric d defined by $d(x,y) = \|x-y\|$, $x, y \in X$. In [**1**], Oikhberg and Rosenthal initiated the study of the following problem. Assume that X is a real linear space equipped with a metric d. What properties of metric d guarantee that d is induced by some norm on X? They considered the following three conditions:

(i) the metric d is translation invariant, that is, $d(x+z, y+z) = d(x,y)$ for any $x, y, z \in X$.

(ii) multiplication by real scalars is separately continuous on the unit interval, that is, for every $x \in X$ the map from $[0,1]$ to X defined by $t \mapsto tx$ is continuous.

(iii) each one-dimensional affine subspace of X is isometric to $\mathbf{R}$.

Obviously, if a metric d is induced by some norm then it has all the above three properties. It should be mentioned that, for a translation invariant metric d, the assumption (iii) is equivalent to the requirement that each one-dimensional linear subspace of X is isometric to $\mathbf{R}$.

The main result in [**1**] states that if (X, d) is a real linear space equipped with a metric d satisfying the above conditions then there exists a norm $\|\cdot\|$ on X such that $d(x,y) = \|x-y\|$, $x, y \in X$. In [**2**] we showed that condition (ii) is superfluous when $\dim X \geq 2$.

Assumption (iii) in the result of Oikhberg and Rosenthal is an "outer" condition. Namely, it states that for every one-dimensional affine

This work was partially supported by a grant from the Ministry of Science of Slovenia.
Received by the editors on June 12, 2008, and in revised form on August 19, 2008.
DOI:10.1216/RMJ-2011-41-1-293

subspace $A \subset X$ there exists a bijective isometry $\xi_A : A \to \mathbf{R}$. We would like to replace (iii) by a property that can be defined with the algebraic and metric structure of X without using mathematical objects outside X. We believe that the most natural approach is to consider algebraic and metric midpoints. For an arbitrary pair of vectors $x, y \in X$ we call $(x+y)/2$ the algebraic midpoint of x and y. A point z is called a metric midpoint of x and y if $d(x,y) = 2d(z,x) = 2d(z,y)$. Clearly, if d is induced by a norm, then the algebraic midpoint is always a metric midpoint (note that there might be infinitely many metric midpoints). So, the natural candidate for an "inner" condition replacing (iii) is

(iii′) for all $x, y \in X$ their algebraic midpoint is a metric midpoint.

It is not difficult to see that for a translation invariant metric d this new condition is strictly weaker than the original condition (iii).

Proposition 1.1. *Let X be a real linear space equipped with a translation invariant metric d. If d satisfies* (iii)*, then it satisfies* (iii′)*. For each real linear space $Y \neq \{0\}$, there exists a translation invariant metric d on Y satisfying* (iii′) *but not* (iii).

In fact, we will prove even slightly more. Namely, we will show that if d is a translation invariant metric on a real linear space X such that for each one-dimensional affine subspace A of X there exists an isometry ξ_A of A into $\mathbf{R}$ (we do not assume that ξ_A is onto), then (iii′) holds true.

We will improve the result of Oikhberg and Rosenthal by replacing (iii) by a weaker condition (iii′) and also (ii) by the following weaker assumption:

(ii′) for every $x \in X$ the set $\{tx : t \in [0,1]\}$ is bounded in (X, d).

Theorem 1.2. *Let X be a real linear space equipped with a metric d satisfying* (i), (ii′)*, and* (iii′)*. Then $\|x\| = d(x, 0)$, $x \in X$, is a norm on X and $d(x,y) = \|x - y\|$, $x, y \in X$.*

The proof of the main result of Oikhberg and Rosenthal [**1**, Theorem 1.1] was based on the use of Mazur-Ulam theorem. The use of algebraic and metric midpoints gives rise to a much shorter self-contained proof of this theorem (all we need to do is to combine the proofs of the first part of Proposition 1.1 and Theorem 1.2).

We observe that in Theorem 1.2 the condition (ii$'$) is indispensable. Indeed, let $X \neq \{0\}$ be any real normed space and $f : X \to X$ an injective additive non-linear map. Then the metric d on X defined by $d(x, y) = \|f(x - y)\|$, $x, y \in X$, satisfies (i) and (iii$'$), but is obviously not induced by a norm.

2. Proofs.

Proof of Proposition 1.1. Assume that X is a real linear space equipped with a translation invariant metric d. Assume further that for each one-dimensional affine subspace A of X there exists an isometry ξ_A of A into $\mathbf{R}$ equipped with the usual metric. We have to prove that (iii$'$) holds true. Let $x, y \in X$, $x \neq y$, and denote by A the one-dimensional affine subspace containing x and y. Then, of course, $(x + y)/2 \in A$ as well. Set $a = d((x + y)/2, x)$. We have

$$\begin{aligned}\left|\xi_A\left(\frac{x+y}{2}\right) - \xi_A(x)\right| &= d\left(\frac{x+y}{2}, x\right) = a \\ &= d\left(\frac{x+y}{2} - \frac{x-y}{2}, x - \frac{x-y}{2}\right) \\ &= d\left(y, \frac{x+y}{2}\right) \\ &= \left|\xi_A\left(\frac{x+y}{2}\right) - \xi_A(y)\right|.\end{aligned}$$

It follows that

$$\xi_A(x), \xi_A(y) \in \left\{\xi_A\left(\frac{x+y}{2}\right) + a, \xi_A\left(\frac{x+y}{2}\right) - a\right\}.$$

As ξ_A is an isometry, it is injective, and consequently, one of the real numbers $\xi_A(x)$ and $\xi_A(y)$ is equal to $\xi_A((x+y)/2) + a$, while the other one is equal to $\xi_A((x + y)/2) - a$. In particular,

$$|\xi_A(x) - \xi_A(y)| = 2a,$$

which yields that $d(x,y) = 2a = 2d((x+y)/2, x) = 2d((x+y)/2, y)$. Hence, $(x+y)/2$ is a metric midpoint of x and y, as desired.

To prove the second part of the statement we take any nonzero real linear space Y. Take two nonzero vectors $f_1, f_2 \in Y$ such that $f_2 = tf_1$ for some real $t \notin \mathbf{Q}$. In the sequel we will consider Y as a vector space over $\mathbf{Q}$. Then f_1 and f_2 are linearly independent. Using Zorn's lemma we can find in Y a maximal $\mathbf{Q}$-linearly independent set $\mathcal{B}$ containing f_1 and f_2, $\mathcal{B} = \{f_1, f_2\} \cup \{e_\alpha : \alpha \in J\}$. In other words, $\mathcal{B}$ is a basis of Y where we consider Y as a vector space over $\mathbf{Q}$. Take any real Banach space Z containing a $\mathbf{Q}$-linearly independent set of cardinality no smaller than the cardinality of J and choose a $\mathbf{Q}$-linearly independent set $\{d_\alpha : \alpha \in J\} \subset Z$. Let $\mathbf{R}^2 \oplus Z$ be a Banach space with the norm $\|(x,y) \oplus z\| = \max\{\|(x,y)\|_{\mathbf{R}^2}, \|z\|_Z\}$, $(x,y) \in \mathbf{R}^2$, $z \in Z$, and define a map $\tau : \{f_1, f_2\} \cup \{e_\alpha : \alpha \in J\} \to \{(1,0) \oplus 0, (0,1) \oplus 0\} \cup \{0 \oplus d_\alpha : \alpha \in J\}$ by

$$\tau(f_1) = (1,0) \oplus 0, \quad \tau(f_2) = (0,1) \oplus 0, \quad \tau(e_\alpha) = 0 \oplus d_\alpha, \quad \alpha \in J.$$

Then τ can be uniquely extended to a $\mathbf{Q}$-linear map from Y into $\mathbf{R}^2 \oplus Z = W$. This extension will be again denoted by τ. It is obviously an injective additive map from Y into W.

For an arbitrary pair of vectors $x, y \in Y$ we define

$$d(x,y) = \|\tau(x) - \tau(y)\|_W = \|\tau(x-y)\|_W.$$

It is straightforward to check that d is a metric on Y. Moreover, it is translation invariant. Next,

$$d\left(\frac{x+y}{2}, x\right) = \left\|\tau\left(\frac{x+y}{2}\right) - \tau(x)\right\|_W = \frac{1}{2}\|\tau(y-x)\|_W = \frac{1}{2}\, d(x,y)$$

and similarly,

$$d\left(\frac{x+y}{2}, y\right) = \frac{1}{2}\, d(x,y)$$

for every pair $x, y \in Y$. Thus, (iii$'$) holds true.

Finally, denote by $[f_1]$ the one-dimensional $\mathbf{R}$-linear span of f_1. If the condition (iii) was fulfilled then we would have an isometry σ of $[f_1]$ into $\mathbf{R}$ equipped with the usual metric. Since

$$d(f_1, 0) = \|\tau(f_1) - \tau(0)\|_W = \|(1,0) \oplus 0\|_W = 1,$$

and similarly

$$d(-f_1, 0) = d(f_2, 0) = d(-f_2, 0) = 1,$$

we would then have

$$\begin{aligned}|\sigma(f_1) - \sigma(0)| = |\sigma(-f_1) - \sigma(0)| &= |\sigma(f_2) - \sigma(0)| \\ &= |\sigma(-f_2) - \sigma(0)| = 1,\end{aligned}$$

which would further yield

$$\sigma(f_1), \sigma(-f_1), \sigma(f_2), \sigma(-f_2) \in \{\sigma(0) - 1, \sigma(0) + 1\},$$

contradicting the injectivity of the isometry σ. □

Proof of Theorem 1.2. We define $\|\cdot\| : X \to \mathbf{R}$ by

$$\|x\| = d(x, 0), \quad x \in X.$$

Obviously, $\|x\| \geq 0$ for every $x \in X$ and $\|x\| = 0$ if and only if $x = 0$. Moreover, $\|-x\| = d(-x, 0) = d(-x + x, x) = \|x\|$, $x \in X$. For every pair $x, y \in X$ we have $\|x + y\| = d(x + y, 0) = d(x, -y) \leq d(x, 0) + d(0, -y) = \|x\| + \|-y\| = \|x\| + \|y\|$. In particular, we have $\|mx\| \leq m\|x\|$ for every positive integer m and every $x \in X$. Clearly, $d(x, y) = d(x - y, 0) = \|x - y\|$, $x, y \in X$. So, it remains to prove that $\|tx\| = |t|\,\|x\|$ for all real t and $x \in X$.

Choose and fix a nonzero $x \in X$. Clearly, x is the algebraic midpoint of 0 and $2x$, and by (iii$'$) we have $\|x\| = d(x, 0) = d(2x, 0)/2 = \|2x\|/2$, or equivalently,

$$\|2x\| = 2\|x\|.$$

We prove by induction that $\|2^n x\| = 2^n\|x\|$, $n = 1, 2, \ldots$.

Let k be an arbitrary positive integer. Choose a positive integer n such that $k < 2^n$. Then

$$2^n\|x\| = \|2^n x\| \leq \|kx\| + \|(2^n - k)x\| \leq k\|x\| + (2^n - k)\|x\| = 2^n\|x\|,$$

and therefore, all the above inequalities are in fact equalities. In particular,

$$\|kx\| = k\|x\|, \quad k = 1, 2, \ldots .$$

It follows that $\|(1/k)x\| = (1/k)(k\|(1/k)x\|) = (1/k)\|x\|$, and consequently, $\|rx\| = r\|x\|$ for every nonnegative $r \in \mathbf{Q}$. Applying the fact that $\|-x\| = \|x\|$ we conclude that

$$\|rx\| = |r|\,\|x\|, \quad r \in \mathbf{Q}.$$

By (ii′) we know that there exists a positive real number M such that $d(tx, 0) \le M$ for every $t \in [0, 1]$, or equivalently,

$$\|tx\| \le M, \quad t \in [0, 1].$$

Hence,

$$\|sx\| = \frac{1}{n}\|(ns)x\| \le \frac{M}{n}, \quad s \in [0, (1/n)].$$

Let t be any real number. For every $r \in \mathbf{Q}$, $r < t$, we have by the triangle inequality

$$\|tx\| \ge \|rx\| - \|(t-r)x\| = |t|\,\|x\| + (|r| - |t|)\|x\| - \|(t-r)x\|.$$

As we can choose a rational number $r < t$ as close to t as we want we have

$$\|tx\| \ge |t|\,\|x\|.$$

In the same way we show that

$$\|tx\| \le |t|\,\|x\|.$$

Thus, $\|\cdot\|$ is a norm. $\square$

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