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GENERALIZED CAUCHY PROBLEM FOR HYPERBOLIC FUNCTIONAL DIFFERENTIAL SYSTEMS

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ABSTRACT. Weak solutions of nonlinear systems of functional differential equations are investigated. The generalized Cauchy problem is transformed into a system of Volterra type integral functional equations. The existence of solutions of this system is proved by using a method of successive approximations. The theory of bicharacteristics and integral inequalities are applied.

Differential systems with deviated variables and differential integral problems are particular cases of systems considered here.

1. Introduction. For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions from X into Y. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Write

$$E = [0,a] \times \mathbf{R}^n, \quad B = [-b_0, 0] \times [-b, b]$$

where $a > 0$, $b_0 \in \mathbf{R}_+$, $b = (b_1, \dots, b_n) \in \mathbf{R}^n_+$ and $\mathbf{R}_+ = [0, +\infty)$.

Suppose that $\psi_0 : [0,a] \to \mathbf{R}$ and $\psi' = (\psi_1, \dots, \psi_n) : E \to \mathbf{R}^n$ are given functions. The requirements on ψ_0 are that there is $c_0 \in \mathbf{R}_+$ such that $-c_0 \leq \psi_0(t)$ and $\psi_0(t) \leq t$ for $t \in [0,a]$. Write $\psi(t,x) = (\psi_0(t), \psi'(t,x))$ and $d_0 = c_0 + b_0$. For a function $z : [-d_0, a] \times \mathbf{R}^n \to \mathbf{R}^k$ and for a point $(t,x) \in [-c_0, a] \times \mathbf{R}^n$ we define a function $z_{(t,x)} : B \to \mathbf{R}^k$ by

$$z_{(t,x)}(\tau, y) = z(t + \tau, x + y), \quad (\tau, y) \in B.$$

Then $z_{(t,x)}$ is the restriction of z to the set $[t - b_0, t] \times [x - b, x + b]$ and this restriction is shifted to the set B. Write $\Omega = E \times C(B, \mathbf{R}^k) \times \mathbf{R}^n$ and

$$E_{0.i} = [-d_0, a_i] \times \mathbf{R}^n, \quad i = 1, \dots, k,$$

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where $0 \le a_i < a$ for $1 \le i \le k$. Suppose that $f = (f_1, \dots, f_k) : \Omega \to \mathbf{R}^k$ and $\varphi_i : E_{0.i} \to \mathbf{R}$, $i = 1, \dots, k$, are given functions. Let us denote by $z = (z_1, \dots, z_k)$ an unknown function of the variables (t, x), $x = (x_1, \dots, x_n)$. We consider the system of functional differential equations

$$\partial_t z_i(t,x) = f_i(t, x, z_{\psi(t,x)}, \partial_x z_i(t,x)), \quad i = 1, \dots, k, \tag{1}$$

with the initial condition

$$z_i(t,x) = \varphi_i(t,x) \quad \text{on } E_{0.i} \text{ for } i = 1, \dots, k, \tag{2}$$

where $\partial_x z_i = (\partial_{x_1} z_i, \dots, \partial_{x_n} z_i)$. A function $\tilde{z} : [-d_0, c] \times \mathbf{R}^n \to \mathbf{R}^k$, $\tilde{z} = (\tilde{z}_1, \dots, \tilde{z}_k)$, where $a_i < c \le a$ for $1 \le i \le k$, is a solution of (1), (2) provides

(i) $\tilde{z} \in C([-d_0, c] \times \mathbf{R}^n, \mathbf{R}^k)$ and $\partial_x \tilde{z}_i$ exist on $[a_i, c] \times \mathbf{R}^n$ for $1 \le i \le k$,

(ii) for each i, $1 \le i \le k$, and $x \in \mathbf{R}^n$, the function $\tilde{z}_i(\cdot, x) : [a_i, c] \to \mathbf{R}$ is absolutely continuous,

(iii) for each $x \in \mathbf{R}^n$ and for $1 \le i \le k$, the i-th equation in (1) is satisfied for almost all $t \in [a_i, c]$ and condition (2) holds.

System (1) with initial condition (2) is called a generalized Cauchy problem. If $a_i = 0$ for $i = 1, \dots, k$ then (1), (2) reduces to the classical Cauchy problem.

In this time numerous papers were published concerning various problems for first order partial functional differential equations. The following questions were considered: functional differential inequalities and their applications, uniqueness of solutions to initial or initial boundary value problems, existence theory of classical or generalized solutions, numerical methods of functional differential equations. It is not our aim to show a full review of papers concerning the above problems. We mention the results on the existence of solutions only. There are various concepts of solutions concerning initial or mixed problems for functional differential equations. Continuous functions satisfying integral systems obtained by integrating of original equations along bicharacteristics were considered in [**1, 12**]. Generalized solutions in the Carathéodory sense of quasilinear problems were investigated in

[**5, 7, 15**]. The method of bicharacteristics and functional differential inequalities are used for proving existence results. Difference method was adopted in [**6**] for discussing the existence of Carathéodory solutions of nonlinear equations with unknown function of two variables. Weak solutions in the Cinquini Cibrario sense have been studied in [**3, 4, 11**]. Existence results for nonlinear equations are based on a method of quasilinearization. It consists of a construction of a quasilinear system for unknown function and for their spatial derivatives. The system obtained is equivalent to a system of functional integral equations of the Volterra type. Continuous solutions of integral functional equations generate weak solutions of functional differential problems. Initial-boundary value problems for differential integral equations were considered in [**13**]. The method of semi-groups of linear operators is used. The functional dependence in equations considered in [**13**] concerns the time variable only. The spatial variable in the unknown function appears in a classical sense. Classical solutions of functional differential problems have been considered in [**2, 8, 9, 14**]. Existence results presented in these papers are based on a method of successive approximations which was introduced by Wazewski for systems without a functional dependence [**16**]. For further bibliography on first order partial functional differential equations see [**10**].

We list below examples of systems which can be derived from (1) by specializing f and ψ.

Example 1.1. Suppose that $\tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_k) : E \times \mathbf{R}^k \times \mathbf{R}^n \to \mathbf{R}^k$ is a given function. Write

$$f(t, x, w, q) = \tilde{f}(t, x, w(0, \theta), q) \text{ on } \Omega,$$

where $\theta = (0, \dots, 0) \in \mathbf{R}^n$. Then (1) reduces to the system with deviated variables

$$\partial_t z_i(t, x) = \tilde{f}_i\big(t, x, z(\psi(t, x)), \partial_x z_i(t, x)\big), \quad i = 1, \dots, k. \tag{3}$$

Example 1.2. For the above $\tilde{f}$ we put

$$f(t, x, w, q) = \tilde{f}(t, x, \int_B w(\tau, y)\, dy\, d\tau, q) \text{ on } \Omega$$

and assume that $\psi(t,x) = (t,x)$. Then (1) is the differential integral system
(4)
$$\partial_t z_i(t,x) = \tilde{f}_i\big(t,x,\int_B z(t+\tau,x+y))\,dy\,d\tau, \partial_x z_i(t,x)\big), \quad i=1,\dots,k.$$

It is clear that more complicated differential systems with deviated variables and differential integral problems can be obtained from (1) by a suitable definition of f.

Remark 1.3. Note that results on generalized Cauchy problems presented in [**14**] are not applicable to (3) and (4).

2. Bicharacteristics. Let us denote by $M_{k\times n}$ the set of all $k\times n$ matrices with real elements. If $X \in M_{k\times n}$, then X^T denotes the transpose matrix. For $x\in \mathbf{R}^n$, $p\in\mathbf{R}^k$, $X\in M_{k\times n}$ where

$$x=(x_1,\dots,x_n),\ p=(p_1,\dots,p_k),\quad X=\big[X_{ij}\big]_{\substack{i=1,\dots,k,\\ j=1,\dots,n}}$$

we define the norms

$$\|x\| = \sum_{i=1}^n |x_i|,\ \|p\|_\infty = \max\{|p_i|:\ 1\le i\le k\},$$
$$\|X\| = \max\left\{\sum_{j=1}^n |x_{ij}| : 1\le i\le k\right\}.$$

The scalar product in $\mathbf{R}^n$ will be denoted by "$\circ$." Write $E_t=[-d_0,t]\times\mathbf{R}^n$ where $0<t\le a$. For $z=(z_1,\dots,z_k)\in C([-d_0,a]\times\mathbf{R}^n,\mathbf{R}^k)$, $v\in C([-d_0,a]\times\mathbf{R}^n,\mathbf{R}^n)$ we define the seminorms

$$\|z_i\|_t = \sup\{\,|z_i(\tau,y)| : (\tau,y)\in E_t\},\quad 1\le i\le k,$$
$$\|z\|_{(t,\mathbf{R}^k)} = \sup\{\|z(\tau,y)\|_\infty : (\tau,y)\in E_t\},$$
$$\|v\|_{(t,\mathbf{R}^n)} = \sup\{\|v(\tau,y)\| : (\tau,y)\in E_t\}.$$

The norm in the space $C(B,\mathbf{R}^k)$ is given by $\|w\|_B = \max\{\|w(t,x)\|_\infty : (t,x)\in B\}$. We will denote by $\mathbf{L}([t_1,t_2],\mathbf{R}_+)$, $[t_1,t_2]\subset\mathbf{R}$, the class of all functions $\zeta : [t_1, t_2] \to \mathbf{R}_+$ which are integrable on $[t_1, t_2]$. Given $c_1, c_2 \in \mathbf{R}_+$ and $\mu_0,\ \mu_1 \in \mathbf{L}([-d_0, \tilde{a}], \mathbf{R}_+)$, $\tilde{a} = \max\{a_i : 1 \le i \le k\}$, we denote by $\mathbf{K}$ the set of all functions $\varphi = (\varphi_1, \dots, \varphi_k)$ such that for each i, $1 \le i \le k$ we have

(i) $\varphi_i \in C(E_{0.i}, \mathbf{R})$, the derivatives $\partial_x \varphi_i = (\partial_{x_1}\varphi_i, \dots, \partial_{x_n}\varphi_i)$ exist on $E_{0.i}$ and

$$\left|\varphi_i(t,x) - \varphi_i(\bar{t},x)\right| \le \left| \int_t^{\bar{t}} \mu_0(\xi)\, d\xi \right| \quad \text{on } E_{0.i},$$

(ii) the estimates

$$\begin{gathered} \|\partial_x \varphi_i(t,x)\| \le c_1, \\ \left\|\partial_x \varphi_i(t,x) - \partial_x \varphi_i(\bar{t},\bar{x})\right\| \le \left| \int_t^{\bar{t}} \mu_1(\xi)\, d\xi \right| + c_2 \|x - \bar{x}\| \end{gathered}$$

are satisfied on $E_{0.i}$.

Let $\varphi \in \mathbf{K}$ be given, and let $\tilde{a} < c \le a$. Suppose that $d \in \mathbf{R}_+$, $\lambda \in \mathbf{L}([-d_0, c], \mathbf{R}_+)$ and $d \ge c_1$, $\lambda(\tau) \ge \mu_0(\tau)$ for almost all $\tau \in [-d_0, \tilde{a}]$. We denote by $C_{\varphi.c}[d, \lambda]$ the class of all $z \in \mathbf{C}([-d_0, c] \times \mathbf{R}^n, \mathbf{R}^k)$, $z = (z_1, \dots, z_k)$, such that $z_i(t,x) = \varphi_i(t,x)$ on $E_{0.i}$ and

$$|z_i(t,x) - z_i(\bar{t},\bar{x})| \le \left| \int_t^{\bar{t}} \lambda(\xi)\, d\xi \right| + d\|x - \bar{x}\|, (t,x), (\bar{t},\bar{x}) \in [a_i, c],$$

where $1 \le i \le k$.

Suppose that $s = (s_1, s_2) \in \mathbf{R}_+^2$, $\gamma \in \mathbf{L}([-d_0, c], \mathbf{R}_+)$ and $s_1 \ge c_1$, $s_2 \ge c_2$ and $\gamma(\tau) \ge \mu_1(\tau)$ for almost all $\tau \in [-d_0, \tilde{a}]$. We denote by $C_{\partial\varphi_i, c}[s, \gamma]$ the class of all $v \in C([-d_0, c] \times \mathbf{R}^n, \mathbf{R}^n)$ such that $v(t,x) = \partial_x \varphi_i(t,x)$ on $E_{0.i}$ and

$$\begin{gathered} \|v(t,x)\| \le s_1, \\ \|v(t,x) - v(\bar{t},\bar{x})\| \le \left| \int_t^{\bar{t}} \gamma(\xi)\, d\xi \right| + s_2 \|x - \bar{x}\| \end{gathered}$$

where (t,x), $(\bar{t},\bar{x}) \in [a_i, c] \times \mathbf{R}^n$. We put $i = 1, \dots, k$ in the above definition.

Assumption $H[\partial_q f, \psi]$**.** The functions f and ψ of the variables (t, x, w, q) and (t, x) respectively satisfy the conditions:

1) the derivatives

$$\partial_q f = \left[\partial_{q_j} f_i\right]_{i=1,\dots,k,\, j=1,\dots,n}$$

exist on Ω and the function $\partial_q f(\cdot, x, w, q) : [0, a] \to M_{k\times n}$ is measurable for $(x, w, q) \in \mathbf{R}^n \times C(B, \mathbf{R}^k) \times \mathbf{R}^n$ and $\partial_q f(t, \cdot) : \mathbf{R}^n \times C(B, \mathbf{R}^k) \times \mathbf{R}^n \to M_{k\times n}$ is continuous for almost all $t \in [0, a]$,

2) there are α, $L \in \mathbf{L}([0, a], \mathbf{R}_+)$ such that

$$\|\partial_q f(t, x, w, q)\| \le \alpha(t)$$

and

(5)
$$\|\partial_q f(t, x, w, q) - \partial_q f(t, \bar{x}, \bar{w}, \bar{q})\| \le L(t)\left[\|x - \bar{x}\| + \|w - \bar{w}\|_B + \|q - \bar{q}\|\right]$$

on Ω.

3) $\psi_0 \in C([0, a], \mathbf{R})$, $\psi' \in C(E, \mathbf{R}^n)$ and

(i) $-c_0 \le \psi_0(t) \le t$ for $t \in [0, a]$,

(ii) the derivatives

$$\partial_x \psi' = \left[\partial_{x_j} \psi_i\right]_{i,j=1,\dots,n}$$

exist on E and

$$\|\psi'(t, x) - \psi'(t, \bar{x})\| \le Q\,\|x - \bar{x}\| \quad \text{on } E.$$

Suppose that Assumption $H[\partial_q f, \psi]$ is satisfied and $\varphi \in \mathbf{K}$, $z \in C_{\varphi.c}[d, \lambda]$, $v \in C_{\partial\varphi_i.c}[s, \gamma]$ and $(t, x) \in [a_i, c] \times \mathbf{R}^n$. Write $\partial_q f_i = (\partial_{q_1} f_i, \dots, \partial_{q_n} f_i$, $1 \le i \le k$, and consider the Cauchy problem

$$\eta'(\tau) = -\partial_q f_i(\tau, \eta(\tau), z_{\psi(\tau,\eta(\tau))}, v(\tau, \eta(\tau))), \quad \eta(t) = x, \tag{6}$$

and denote by $g_i[z, v](\cdot, t, x)$ its Carathéodory solution. The function $g_i[z, v](\cdot, t, x)$ is the i-th bicharacteristic of (1) corresponding to (z, v). We put $i = 1, \dots, k$ in the above definitions. We prove a lemma on the existence and uniqueness and on the regularity of bicharacteristics.

Lemma 2.1. *Suppose that Assumption $H[\partial_q f, \psi]$ is satisfied and*

$$\varphi,\ \overline{\varphi} \in \mathbf{K},\ \ z \in C_{\varphi.c}[d,\lambda],\ \ \bar{z} \in C_{\bar{\varphi}.c}[d,\lambda],$$
$$v \in C_{\partial\varphi_i,c}[s,\gamma], \quad \bar{v} \in C_{\partial\overline{\varphi}_i.c}[s,\gamma],$$

where $\widetilde{a} < c \le a$ and $1 \le i \le k$. Then the bicharacteristics $g_i[z,v](\cdot,t,x)$ and $g_i[\bar{z},\bar{v}](\cdot,t,x)$ exist on $[a_i,c]$. Solutions of (6) *are unique and we have the estimates*

$$(7)\ \ \|g_i[z,v](\tau,t,x) - g_i[z,v](\tau,\bar{t},\bar{x})\| \le \Theta(c)\left[\left|\int_t^{\bar{t}} \alpha(\xi)\,d\xi\right| + \|x-\bar{x}\|\right]$$

where $(\tau,t,x), (\tau,\bar{t},\bar{x}) \in [a_i,c]\times[a_i,c]\times\mathbf{R}^n$ and

$$(8)\quad \|g_i[z,v](\tau,t,x) - g_i[\bar{z},\bar{v}](\tau,t,x)\| \\ \le \Theta(c)\left|\int_\tau^t L(\xi)\left[\|z-\bar{z}\|_{(\xi,\mathbf{R}^k)} + \|v-\bar{v}\|_{(\xi,\mathbf{R}^n)}\right]d\xi\right|$$

where $(\tau,t,x) \in [a_i,c]\times[a_i,c]\times\mathbf{R}^n$ and

$$\Theta(\tau) = \exp\left\{\left(1+dQ+s_2\right)\int_0^\tau L(\xi)\,d\xi\right\}.$$

Proof. We begin with the observation that

$$(9)\qquad \|z_{\psi(\tau,y)} - z_{\psi(\tau,\bar{y})}\|_B \le dQ\|y-\bar{y}\|$$

where (τ,y), $(\tau,\bar{y}) \in [a_i,c]\times\mathbf{R}^n$. We conclude from Assumption $H[\partial_q f,\psi]$ that the following Lipschitz condition is satisfied

$$(10)\quad \|\partial_q f_i(\tau,y,z_{\psi(\tau,y)},v(\tau,y)) - \partial_q f_i(\tau,\bar{y},z_{\psi(\tau,\bar{y})},v(\tau,\bar{y}))\| \\ \le L(\tau)(1+dQ+s_2)\|y-\bar{y}\|$$

where $(\tau,y),(\tau,\bar{y}) \in [a_i,c]\times\mathbf{R}^n$. Now, the existence and uniqueness of the solution of (6) follows from classical theorems on Carathéodory solutions of initial problems.

It is easily seen that the integral inequality

$$\|g_i[z,v](\tau,t,x) - g_i[z,v](\tau,\bar t,\bar x)\| \le \|x-\bar x\| + \left| \int_t^{\bar t} \alpha(\xi)\, d\xi \right| + (1+dQ+s_2)\left| \int_\tau^t L(\xi)\, \|g_i[z,v](\xi,t,x) - g_i[z,v](\xi,\bar t,\bar x)\|\, d\xi \right|$$

is satisfied for $(\tau,t,x),(\tau,\bar t,\bar x) \in [a_i,c]\times[a_i,c]\times\mathbf{R}^n$. Then we obtain (7) by the Gronwall inequality.

Now we prove (8). For $z \in C_{\varphi.c}[d,\lambda]$, $\bar z \in C_{\bar\varphi.c}[d,\lambda]$, $v \in C_{\partial\varphi_i,c}[s,\gamma]$, $\bar v \in C_{\partial\varphi_i.c}[s,\gamma]$, we have the integral inequality

$$\begin{aligned}&\|g_i[z,v](\tau,t,x) - g_i[\bar z,\bar v](\tau,t,x)\|\\ &\quad\le \left| \int_\tau^t L(\xi)\left[\|z-\bar z\|_{(\xi,\mathbf{R}^k)} + \|v-\bar v\|_{(\xi,\mathbf{R}^n)}\right] d\xi \right|\\ &\qquad + (1+dQ+s_2)\left| \int_\tau^t L(\xi)\, \|g_i[z,v](\xi,t,x) - g_i[\bar z,\bar v](\xi,t,x)\|\, d\xi \right|\end{aligned}$$

where $(\tau,t,x) \in [a_i,c]\times[a_i,c]\times\mathbf{R}^n$. From the Gronwall inequality we deduce (8). This proves Lemma 2.1.

6. Integral functional equations. We denote by $CL(B,\mathbf{R})$ the set of all linear and continuous real functions defined on $C(B,\mathbf{R})$. Let $\|\cdot\|_\star$ be the norm in $CL(B,\mathbf{R})$ generated by the maximum norm in $C(B,\mathbf{R})$. For

$$W = \left[w_{ij}\right]_{i,j=1,\dots,k} \text{ where } w_{ij} \in CL(B,\mathbf{R})$$

we put

$$\|W\|_\star = \max\left\{ \sum_{j=1}^k \|w_{ij}\|_\star :\ 1 \le i \le k \right\}.$$

Assumption $H[f,\psi]$. Assumption $H[\partial_q f,\psi]$ is satisfied and

1) the derivatives

$$\partial_x f = \left[\partial_{x_j} f_i\right]_{\substack{i=1,\dots,k,\\ j=1,\dots,n}}$$

exist on Ω and $\partial_x f(\cdot, x, w, q) : [0, a] \to M_{k\times n}$ is measurable,

2) there exist on Ω the Fréchet derivatives

$$\partial_w f = \left[\partial_{w_j} f_i\right]_{i,j=1,\dots,k}$$

and $\partial_{w_j} f_i(t, x, w, q) \in CL(B, \mathbf{R})$ for $i, j = 1, \dots, k$,

3) the function

$$\partial_w f(\,\cdot\,, x, w, q)\,\chi : [0, a] \longrightarrow M_{k\times k},$$
$$\partial_w f(\,\cdot\,, x, w, q)\,\chi = \left[\partial_{w_j} f_i(\cdot, x, w, q)\,\chi\right]_{i,j=1,\dots,k},$$

is measurable for each $\chi \in C(B, \mathbf{R})$, $(x, w, q) \in \mathbf{R}^n \times C(B, \mathbf{R}^k) \times \mathbf{R}^n$,

4) there are α_0, $\beta \in L([0, a], \mathbf{R}_+)$ such that

$$\|f(t, x, w, q)\|_\infty \le \alpha_0(t),$$
$$\|\partial_x f(t, x, w, q)\| \le \beta(t),\ \ \|\partial_w f(t, x, w, q)\|_\star \le \beta(t)$$

and

$$\begin{aligned}\|\partial_x f(t, x, w, q) - \partial_x f(t, \bar{x}, \overline{w}, \bar{q})\| &\le L(t)\big[\|x - \bar{x}\| \\ &\quad + \|w - \overline{w}\|_B + \|q - \bar{q}\|\,\big],\\ \|\partial_w f(t, x, w, q) - \partial_w f(t, \bar{x}, \overline{w}, \bar{q})\|_\star &\le L(t)\big[\|x - \bar{x}\| + \|w - \bar{w}\|_B \\ &\quad + \|q - \bar{q}\|\big]\end{aligned}$$

for (x, w, q), $(\bar{x}, \overline{w}, \bar{q}) \in \mathbf{R}^n \times C(B, \mathbf{R}^k) \times \mathbf{R}^n$ and for almost all $t \in [0, a]$,

5) there is a $\widetilde{Q} \in \mathbf{R}_+$ such that

$$\|\partial_x \psi'(t, x) - \partial_x \psi'(t, \bar{x})\| \le \widetilde{Q}\|x - \overline{x}\| \text{ on } E.$$

Let us denote by z and u the unknown functions of the variables (t, x) where

$$z = (z_1, \dots, z_k)^T,\ \ u = \left[u_{ij}\right]_{\substack{i=1,\dots,k,\\ j=1,\dots,n}}.$$

Write

$$u_{(t,x)} = \left[(u_{ij})_{(t,x)}\right]_{\substack{i=1,\dots,k,\\ j=1,\dots,n}},\ \ u_{[i]} = (u_{i1}, \dots, u_{in}),$$
$$\partial_x u_{[i]} = \left[\partial_{x_\mu} u_{i\nu}\right]_{\nu,\,\mu=1,\dots,n},\ \ \partial_t u_{[i]} = \big(\partial_t u_{i1}, \dots, \partial_t u_{in}\big)^T$$

where $i = 1, \dots, k$. Set

$$
\begin{aligned}
W[z,u] &= \big[W_{ij}[z,u]\big]_{\substack{i=1,\dots,k,\\ j=1,\dots,n}},\\
W_{[i]}[z,u] &= \big(W_{i1}[z,u], \dots, W_{in}[z,u]\big), \quad i = 1, \dots, k,
\end{aligned}
$$

and

$$
\begin{aligned}
&W_{ij}[z,u](t,x)\\
&\quad= \sum_{\mu=1}^{k} \sum_{\nu=1}^{n} \partial_{x_j} \psi_\nu(t,x)\, \partial_{w_\mu} f_i\big(t, x, z_{\psi(t,x)}, u_{[i]}(t,x)\big)\, (u_{\mu\nu})_{\psi(t,x)}
\end{aligned}
$$

for $i = 1, \dots, k$, $j = 1, \dots, n$. Set

$$
\begin{aligned}
&P[z, u_{[i]}](\tau, t, x)\\
&= \big(\tau, g_i[z, u_{[i]}](\tau, t, x), z_{\psi(\tau, g_i[z, u_{[i]}](\tau,t,x))}, u_{[i]}(\tau, g_i[z, u_{[i]}](\tau, t, x))\big).
\end{aligned}
$$

Let us denote by

$$
\begin{aligned}
F[z,u] &= \big(F_1[z,u], \dots, F_k[z,u]\big),\\
G[z,u] &= \big[G_{ij}[z,u]\big]_{\substack{i=1,\dots,k,\\ j=1,\dots,n}},\\
G_{[i]}[z,u] &= \big(G_{i1}[z,u], \dots, G_{in}[z,u]\big), \quad i = 1, \dots, k,
\end{aligned}
$$

the functions given by

$$
\begin{aligned}
F_i[z,u](t,x) = \varphi_i(a_i, g_i[z, u_{[i]}](a_i, t, x))&\\
+ \int_{a_i}^{t} \big[f_i(P[z, u_{[i]}](\xi, t, x)) - \partial_q f_i(P[z, u_{[i]}](\xi, t, x))&\\
\circ\, u_{[i]}(\xi, g_i[z, u_{[i]}](\xi, t, x))\big]\, d\xi&
\end{aligned}
$$

and

$$
\begin{aligned}
G_{[i]}[z,u](t,x) = \partial_x \varphi_i(a_i, g_i[z, u_{[i]}](a_i, t, x))&\\
+ \int_{a_i}^{t} \Big\{\partial_x f_i(P[z, u_{[i]}](\xi, t, x))&\\
+ W_{[i]}[z,u](\xi, g_i[z, u_{[i]}](\xi, t, x))\Big\}\, d\xi,&
\end{aligned}
$$

where $i = 1, \dots, k$. We shall consider the following system of functional integral equations

$$z_i(t,x) = F_i[z,u](t,x), \quad u_{[i]}(t,x) = G_{[i]}[z,u](t,x), \tag{11}$$

$$g_i[z,u_{[i]}](\tau,t,x) = x + \int_\tau^t \partial_q f_i(P[z,u_{[i]}](\xi,t,x))\,d\xi, \tag{12}$$

where $i = 1, \dots, k$ and
(13)
$z_i(t,x) = \varphi_i(t,x),\ u_{[i]}(t,x) = \partial_x \varphi_i(t,x)$ on $[-d_0, a_i] \times \mathbf{R}^n$ for $1 \le i \le k$.

The proof of the existence of a solution of (1), (2) will be based on the following method of successive approximations. Suppose that $\varphi \in \mathbf{K}$ and Assumption $H[f,\psi]$ is satisfied. We define sequences $\{z^{(m)}\}$, $\{u^{(m)}\}$ where

$$z^{(m)} = (z_1^{(m)}, \dots, z_k^{(m)})^T, \quad u^{(m)} = \left[u_{ij}^{(m)}\right]_{\substack{i=1,\dots,k,\\ j=1,\dots,n}},$$
$$u_{[i]}^{(m)} = \left(u_{i1}^{(m)}, \dots, u_{in}^{(m)}\right), \qquad i = 1, \dots, k,$$

in the following way. We put first

$$\begin{aligned} z_i^{(0)}(t,x) &= \varphi_i(t,x) \quad \text{on } E_{0.i},\\ z_i^{(0)}(t,x) &= \varphi_i(a_i,x) \quad \text{on } [a_i,c] \times \mathbf{R}^n, \end{aligned} \tag{14}$$

and

$$\begin{aligned} u_{[i]}^{(0)}(t,x) &= \partial_x \varphi_i(t,x) \quad \text{on } E_{0.i},\\ u_{[i]}^{(0}(t,x) &= \partial_x \varphi_i(a_i,x) \quad \text{on } [a_i,c] \times \mathbf{R}^n, \end{aligned} \tag{15}$$

and we take $i = 1, \dots, k$ in the above definitions. Suppose that $(z^{(m)}, u^{(m)})$ where $z^{(m)} : [-d_0, c] \times \mathbf{R}^n \to \mathbf{R}^k$, $u^{(m)} : [-d_0, c] \times \mathbf{R}^n \to M_{k\times n}$, are known functions. Then $u_{[i]}^{(m+1)}$ is a solution of the functional integral problem

$$u_{[i]}(t,x) = G_{[i]}^{(m)}[\,u_{[i]}\,](t,x),\ (t,x) \in E_{0.i} \cup ([a_i,c] \times \mathbf{R}^n), \tag{16}$$

where

$$G^{(m)}_{[i]}[\,u_{[i]}\,](t,x) = \partial_x\varphi_i(a_i, g_i[z^{(m)}, u_{[i]}](a_i,t,x)) + \int_{a_i}^{t}\Big\{\partial_x f_i(P[z^{(m)}, u_{[i]}](\xi,t,x)) + W^{(m)}_{[i]}[z^{(m)}, u_{[i]}](\xi, g_i[z^{(m)}, u_{[i]}](\xi,t,x))\Big\}\,d\xi,$$

on $[a_i, c] \times \mathbf{R}^n$ and

$$G^{(m)}_{[i]}[\,u_{[i]}](t,x) = \partial_x\varphi_i(t,x) \quad \text{on } E_{0.i}$$

and

$$W^{(m)}_{[i]}[z^{(m)}, u_{[i]}] = \big(W^{(m)}_{i1}[z^{(m)}, u_{[i]}], \dots, W^{(m)}_{in}[z^{m)}, u_{[i]}]\big),$$

where

$$W^{(m)}_{ij}[z^{(m)}, u_{[i]}](t,x) = \sum_{\mu=1}^{k}\sum_{\nu=1}^{n}\partial_{x_j}\psi_\nu(t,x)\,\partial_{w_\mu} f_i\big(t,x,(z^{(m)})_{\psi(t,x)}, u_{[i]}(t,x)\big)\,(u^{(m)}_{\mu\nu})_{\psi(t,x)}$$

for $i = 1,\dots,k$, $j = 1,\dots,n$. The function $z^{(m+1)}$ is given by

$$z^{(m+1)}_i(t,x) = F_i[z^{(m)}, u^{(m+1)}](t,x), \quad (t,x) \in [a_i, c] \times \mathbf{R}^n, \tag{17}$$

$$z^{(m+1)}_i(t,x) = \varphi_i(t,x) \quad \text{on } E_{0.i} \tag{18}$$

where $i = 1,\dots,k$.

Remark 3.1. The sequences $\{z^{(m)}\}$ and $\{u^{(m)}\}$ are obtained in the following way. Suppose that $z^{(m)} : [-d_0, c] \times \mathbf{R}^n \to \mathbf{R}^k$ and $u^{(m)} : [-d_0, c] \times \mathbf{R}^n \to M_{k\times n}$ are known functions. Let us consider the classical Cauchy problem

$$\partial_t z_i(t,x) = f_i(t,x,(z^{(m)})_{\psi(t,x)}, \partial_x z_i(t,x)), \tag{19}$$

$$z(a_i, x) = \varphi_i(a_i, x) \quad \text{for } x \in \mathbf{R}^n \tag{20}$$

where $1 \le i \le k$. We adopt a method of quasilinearization for (19), (20). We introduce first additional unknown functions $u_{[i]} = \partial_x z_i$ in (19). Then we consider the linearization of (19) with respect to $u_{[i]}$ and we obtain the differential equations

$$\begin{aligned}(21)\quad &\partial_t z_i(t,x)\\ &= f_i\big(Q[z^{(m)},u_{[i]}](t,x)\big)+\partial_q f_i\big(Q[z^{(m)},u_{[i]}](t,x)\big)\circ\big(\partial_x z_i(t,x)-u_{[i]}(t,x)\big)\end{aligned}$$

where

$$Q[z^{(m)},u_{[i]}](t,x) = (t,x,(z^{(m)})_{\psi(t,x)},u_{[i]}(t,x)).$$

By virtue of (19) we get the following equations for $u_{[i]}$:

$$\begin{aligned}\partial_t u_{[i]}(t,x) = {}& \partial_x f_i\big(Q[z^{(m)},u_{[i]}](t,x)\big)\\ &+ \widetilde{W}^{(m)}_{[i]}[z^{(m)},u_{[i]}](t,x)\\ &+ \partial_q f_i\big(Q[z^{(m)},u_{[i]}](t,x)\big)\ \big[\partial_x u_{[i]}(t,x)\big]^T\end{aligned}\tag{22}$$

where

$$\widetilde{W}^{(m)}_{[i]} = \big(\widetilde{W}^{(m)}_{i1},\dots,\widetilde{W}^{(m)}_{in}\big),$$

$$\begin{aligned}&\widetilde{W}^{(m)}_{ij}[z^{(m)},u_{[i]}](t,x)\\ &\quad= \sum_{\mu=1}^{k}\sum_{\nu=1}^{n}\partial_{x_j}\psi_\nu(t,x)\,\partial_{w_\mu} f_i\big(Q[z^{(m)},u_{[i]}](t,x)\big)\big(\partial_{x_\nu} z^{(m)}_\mu\big)_{\psi(t,x)}\end{aligned}$$

and

$$\big[\partial_x u_{[i]}\big] = \big[\partial_{x_j} u_{i\nu}\big]_{\nu,\,j=1,\dots,n}.$$

It is natural to consider the following initial condition for (22):

$$u_{[i]}(a_i,x) = \partial_x\varphi_i(a_i,x), \quad x\in\mathbf{R}^n,$$

where $1 \le i \le k$. System (21), (22) has the following property: differential equations of bicharacteristics for (21) and for (22) are the same and they have the form

$$\eta'(\tau) = -\partial_q f_i\big(Q[z^{(m)},u_{[i]}](\tau,\eta(\tau))\big).$$

We put $u^{(m)}_{\mu\nu} = \partial_{x_\nu} z^{(m)}_\mu$ in (22). Then we integrate (21), (22) along the bicharacteristics $g_i[z^{(m)}, u_{[i]}](\,\cdot\,, t, x)$ and we obtain the system equations

$$z_i(t,x) = F_i[z^{(m)}, u_{[i]}](t,x), \quad u_{[i]}(t,x) = G^{(m)}_{[i]}[u_{[i]}](t,x),$$

with initial conditions (13) where $i = 1, \dots, k$.

We prove that there exists a solution $u^{(m+1)}_{[i]}$ of the second equation and we define $z^{(m+1)}_i(t,x) = F_i[z^{(m)}, u^{(m+1)}_{[i]}](t,x)$ where $i = 1, \dots, k$.

4. Successive approximations for integral functional equations. The main difficulty in carrying out our construction of a solution of (1), (2) is the existence of the sequences $\{z^{(m)}\}$ and $\{u^{(m)}\}$. We introduce some technical notations and assumptions. We define the functions Γ, $\widetilde{\Gamma} : [0,a] \to \mathbf{R}_+$ by

$$\begin{aligned}\Gamma(\tau) = \Theta(\tau)\Big[c_2 + (1 + s_1 Q)(1 + dQ + s_2)\int_0^\tau L(\xi)\,d\xi \\ + (s_2 Q^2 + s_1 \widetilde{Q})\int_0^\tau \beta(\xi)\,d\xi\Big],\\ \widetilde{\Gamma}(\tau) = \Theta(\tau)\Big[c_1 + (1 + dQ)\int_0^\tau \beta(\xi)\,d\xi + 2s_2\int_0^\tau \alpha(\xi)\,d\xi \\ + s_1(1 + dQ + s_2)\int_0^\tau L(\xi)\,d\xi\Big].\end{aligned}$$

Assumption $H_\star$. The constants $d \in \mathbf{R}_+$, $s \in \mathbf{R}^2_+$, $0 < c \le a$, satisfy the conditions

$$d \ge \widetilde{\Gamma}(c), \quad s_1 \ge c_1 + (1 + ns_1 Q)\int_0^c \beta(\xi)\,d\xi, \quad s_2 \ge \Gamma(c)$$

and

$$\gamma(\tau) = \Gamma(c)\alpha(\tau) + (1 + ns_1 Q)\beta(\tau), \quad \lambda(\tau) = (s_1 + \widetilde{\Gamma}(c))\alpha(\tau) + \alpha_0(\tau),$$

where $\tau \in [0, c]$.

Theorem 4.1. *If Assumptions $H[f,\psi]$, $H_\star$ are satisfied and $\varphi \in \mathbf{K}$, then for any $m \geq 0$ we have*

(I_m) $z^{(m)}$ *and* $u^{(m)}$ *are defined on* $[-d_0, c] \times \mathbf{R}^n$ *and*

$$z^{(m)} \in C_{\varphi.c}[d,\lambda], \quad u^{(m)}_{[i]} \in C_{\partial\varphi_i.c}[s,\gamma] \text{ for } i = 1,\dots,k.$$

(II_m) $\partial_x z_i^{(m)} = u^{(m)}_{[i]}$ *on* $[a_i, c] \times \mathbf{R}^n$ *for* $i = 1,\dots,k$.

Proof. We prove (I_m) and (II_m) by induction. It follows from (14), (15) that conditions (I_0) and (II_0) are satisfied. Suppose now that (I_m) and (II_m) hold for a given $m \geq 0$. We will prove that there exists a $u^{(m+1)}_{[i]} : [-d_0, c] \times \mathbf{R}^n \to \mathbf{R}^n$ and $u^{(m+1)}_{[i]} \in C_{\partial\varphi_i.c}[s,\gamma]$ for $i = 1,\dots,k$.

Suppose that $1 \leq i \leq k$ is fixed. We claim that

$$G^{(m)}_{[i]} : C_{\partial\varphi_i.c}[s,\gamma] \longrightarrow C_{\partial\varphi_i.c}[s,\gamma]. \tag{23}$$

Indeed, it follows from Assumption $H[f,\psi]$ and from Lemma 2.1 that

$$\|G^{(m)}_{[i]}[u_{[i]}](t,x)\| \leq c_1 + (1 + ns_1 Q) \int_0^c \beta(\xi)\, d\xi$$

and

$$\begin{aligned}&\big\|G^{(m)}_{[i]}[u_{[i]}](t,x) - G^{(m)}_{[i]}[u_{[i]}](\bar t,\overline{x})\big\| \\ &\qquad \leq \Gamma(c)\left[\left|\int_t^{\bar t} \alpha(\xi\, d\xi)\right| + \|x - \overline{x}\|\right] + (1 + ns_1 Q)\left|\int_t^{\bar t} \beta(\xi)\, d\xi\right|\end{aligned}$$

where (t,x), $(\bar t, \overline{x}) \in [a_i, c] \times \mathbf{R}^n$. By Assumption $H_\star$ we obtain (23).

It follows that there is a $\widetilde\beta \in \mathbf{L}([a_i,c], \mathbf{R}_+)$ such that

$$\big\|G^{(m)}_{[i]}[u_{[i]}](t,x) - G^{(m)}_{[i]}[\widetilde u_{[i]}](t,x)\big\| \leq \int_{a_i}^t \widetilde\beta(\xi)\, \|u_{[i]} - \widetilde u_{[i]}\|_{(\xi,\mathbf{R}^n)}\, d\xi$$

where $u_{[i]}$, $\widetilde u_{[i]} \in C_{\partial\varphi_i.c}[s,\gamma]$. For the above $u_{[i]}$, $\widetilde u_{[i]}$ we put

$$\big[|u_{]i]} - \widetilde u_{[i]}|\big] = \sup\left\{\|u_{[i]} - \widetilde u_{[i]}\|_{(t,\mathbf{R}^n)} \exp\left[-2\int_{a_i}^t \widetilde\beta(\xi)\, d\xi\right] t \in [a_i, c]\right\}.$$

Then we have

$$\begin{aligned}\big\| G^{(m)}_{[i]}[u_{[i]}](t,x) - G^{(m)}_{[i]}[\widetilde{u}_{[i]}](t,x)\big\| \\ &\le \big[|u_{]i]} - \widetilde{u}_{[i]}|\big] \int_{a_i}^{t} \widetilde{\beta}(\xi)\ \exp\big[2\int_{a_i}^{\xi} \widetilde{\beta}(\tau)\,d\tau\big]\,d\xi \\ &\le \frac{1}{2}\big[|u_{]i]} - \widetilde{u}_{[i]}|\big]\ \exp\big[2\int_{a_i}^{t} \gamma(\xi)\,d\xi\big],\end{aligned}$$

and consequently,

$$\big[|G^{(m)}_{[i]}[u_{[i]}] - G^{(m)}_{[i]}[\tilde{u}_{[i]}]|\big] \le \frac{1}{2}\big[|u_{[i]} - \widetilde{u}_{[i]}|\big].$$

From the Banach fixed point theorem it follows that there exists exactly one $u^{(m+1)}_{[i]} \in C_{\partial\varphi_i.c}[s,\gamma]$ satisfying (16). Then we have proved that there exists exactly one $u^{(m+1)} : [-d_0, c] \times \mathbf{R}^n \to M_{k\times n}$.

It is easily seen that $z^{(m+1)} \in C_{\varphi.c}[d,\lambda]$ where $z^{(m+1)}$ is given by (17), (18). Now we prove that $z^{(m+1)}$ satisfies the condition: $\partial_x z^{(m+1)}_i = u^{(m+1)}_{[i]}$ on $[a_i, c] \times \mathbf{R}^n$ for $1 \le i \le k$.

Suppose that i, $1 \le i \le k$, is fixed and

$$U(t,x,\overline{x}) = z^{(m+1)}_i(t,\overline{x}) - z^{(m+1)}_i(t,x) - u^{(m+1)}_{[i]}(t,x) \circ (\overline{x} - x)$$

where $(t,x), (t,\overline{x}) \in [a_i, c] \times \mathbf{R}^n$. We prove that there is a $C \in \mathbf{R}_+$ such that

$$\big|U(t,x,\overline{x})\big| \le C\|x - \overline{x}\|^2. \tag{24}$$

Write

$$g^{(m)}_i(\tau,t,x) = g_i[z^{(m)}, u^{(m+1)}_{[i]}](\tau,t,x),$$

$$\begin{aligned}P^{(m)}&(\xi,t,x) \\ &= \big(\xi, g^{(m)}_i(\xi,t,x), (z^{(m)})_{\psi(\xi, g^{(m)}_i(\xi,t,x))}, u^{(m+1)}_{[i]}(\xi, g^{(m)}_i(\xi,t,x))\big),\end{aligned}$$

$$Q^{(m)}(\xi,t,x,\bar{x},\tau) = \tau\, P^{(m)}(\xi,t,\overline{x}) + (1-\tau)\, P^{(m)}(\xi,t,x),\ 0 \le \tau \le 1,$$

and

$$\partial_w f_i(P^{(m)}(\xi,t,x)) \star (z^{(m)})_{(\xi,y))} = \sum_{j=1}^{k} \partial_{w_j} f_i(P^{(m)}(\xi,t,x)) \star (z_j^{(m)})_{(\xi,y))}$$

where $1 \le i \le k$. It follows from (16), (17) that
(25)

$$\begin{aligned}
U(t,x,\overline{x}) &= \varphi_i(a_i, g^{(m)}(a_i,t,\overline{x})) - \varphi_i(a_i, g^{(m)}(a_i,t,x)) \\
&+ \int_{a_i}^{t} \big[f_i(P^{(m)}(\xi,t,\overline{x})) - f_i(P^{(m)}(\xi,t,x))\big]\, d\xi \\
&- \int_{a_i}^{t} \partial_q f_i(P^{(m)}(\xi,t,\overline{x})) \circ u_{[i]}^{(m+1)}(\xi, g_i^{(m)}(\xi,t,\overline{x}))\, d\xi \\
&+ \int_{a_i}^{t} \partial_q f_i(P^{(m)}(\xi,t,x)) \circ u_{[i]}^{(m+1)}(\xi, g_i^{(m)}(\xi,t,x))\, d\xi \\
&- \partial_x \varphi_i(a_i, g_i^{(m)}(a_i,t,x)) \circ (\overline{x} - x) \\
&- \int_{a_i}^{t} \partial_x f_i(P^{(m)}(\xi,t,x))\, d\xi \circ (\overline{x} - x) \\
&- \int_{a_i}^{t} W_{[i]}^{(m)}[z^{(m)}, u_{[i]}^{(m+1)}]\big(\xi, g_i^{(m)}(\xi,t,x)\big)\, d\xi \circ (\overline{x} - x).
\end{aligned}$$

For simplicity of formulation of the next properties of the function U we define

$$\begin{aligned}
U_\varphi(t,x,\overline{x}) &= \varphi(a_i, g_i^{(m)}(a_i,t,\overline{x})) - \varphi(a_i, g_i^{(m)}(a_i,t,x)) \\
&- \partial_x \varphi(a_i, g_i^{(m)}(a_i,t,x)) \\
&\circ \big[g_i^{(m)}(a_i,t,\overline{x}) - g_i^{(m)}(a_i,t,x)\big], \widetilde{U}(t,x,\overline{x}) \\
&= \int_{a_i}^{t} \int_0^1 \big[\partial_x f_i(Q^{(m)}(\xi,t,x,\overline{x},\tau)) - \partial_x f_i(P^{(m)}(\xi,t,x))\big]\, d\tau \\
&\qquad\qquad \circ \big[g_i^{(m)}(\xi,t,\overline{x}) - g_i^{(m)}(\xi,t,x)\big]\, d\xi \\
&+ \int_{a_i}^{t} \int_0^1 \big[\partial_w f_i(Q^{(m)}(\xi,t,x,\overline{x},\tau)) - \partial_x f_i(P^{(m)}(\xi,t,x))\big]\, d\tau \\
&\qquad \star \big[(z^{(m)})_{\psi(\xi, g_i^{(m)}(\xi,t,\overline{x}))} - (z^{(m)})_{\psi(\xi, g_i^{(m)}(\xi,t,x))}\big]\, d\xi \\
&+ \int_{a_i}^{t} \int_0^1 \big[\partial_q f_i(Q^{(m)}(\xi,t,x,\overline{x},\tau)) - \partial_q f_i(P^{(m)}(\xi,t,x))\big]\, d\tau
\end{aligned}$$

$$\circ \big[u_{[i]}^{(m+1)}(\xi, g_i^{(m)}(\xi,t,\overline{x})) - u_{[i]}^{(m+1)}(\xi, g_i^{(m)}(\xi,t,x))\big]\, d\xi,$$

and

$$\begin{aligned} U_\star(t,x,\overline{x}) = {} & \partial_x \varphi_i(a_i, g_i^{(m)}(a_i,t,x)) \\ & \circ \big[g_i^{(m)}(a_i,t,\overline{x}) - g_i^{(m)}(a_i,t,x) - (\overline{x}-x)\big] \\ & + \int_{a_i}^t \partial_x f_i(P^{(m)}(\xi,t,x)) \\ & \qquad \circ \big[g_i^{(m)}(\xi,t,\overline{x}) - g_i^{(m)}(\xi,t,x) - (\overline{x}-x)\big]\, d\xi \\ & + \int_{a_i}^t W_{[i]}^{(m)}[z^{(m)}, u_{[i]}^{(m+1)}]\big(\xi, g_i^{(m)}(\xi,t,z)\big) \\ & \qquad \circ \big[g_i^{(m)}(\xi,t,\overline{x}) - g_i^{(m)}(\xi,t,x) - (\overline{x}-x)\big]\, d\xi, \end{aligned}$$

$$\begin{aligned} U_w(t,x,\overline{x}) = {} & \int_{a_i}^t \partial_w f_i(P^{(m)}(\xi,t,x)) \\ & \quad \star \big[(z^{(m)})_{\psi(\xi, g_i^{(m)}(\xi,t,\overline{x}))} - (z^{(m)})_{\psi(\xi, g_i^{(m)}(\xi,t,x))}\big]\, d\xi \\ & - \int_{a_i}^t W_{[i]}^{(m)}[z^{(m)}, u_{[i]}^{(m+1)}](\xi, g_i^{(m)}(\xi,t,x)) \\ & \qquad \circ \big[g_i^{(m)}(\xi,t,\overline{x}) - g_i^{(m)}(\xi,t,x)\big]\, d\xi \end{aligned}$$

and

$$\begin{aligned} U_q(t,x,\overline{x}) = {} & \int_{a_i}^t \partial_q f_i(P^{(m)}(\xi,t,x)) \circ u_{[i]}^{(m+1)}(\xi, g_i^{(m)}(\xi,t,x))\, d\xi \\ & + \int_{a_i}^t \partial_q f_i(P^{(m)}(\xi,t,x)) \\ & \circ \big[u_{[i]}^{(m+1)}(\xi, g_i^{(m)}(\xi,t,\overline{x})) - u_{[i]}^{(m+1)}(\xi, g_i^{(m)}(\xi,t,x))\big]\, d\xi \\ & - \int_{a_i}^t \partial_q f_i(P^{(m)}(\xi,t,\overline{x})) \circ u_{[i]}^{(m+1)}(\xi, g_i^{(m)}(\xi,t,\bar{x}))\, d\xi. \end{aligned}$$

By using the Hadamard mean value theorem to the difference

$$f_i(P^{(m)}(\xi,t,\overline{x})) - f_i(P^{(m)}(\xi,t,x))$$

and adding and subtracting $g_i^{(m)}(\xi,t,\overline{x}) - g_i^{(m)}(\xi,t,x)$ in the last three terms in (25) we can assert that

$$U(t,x,\overline{x}) = U_\varphi(t,x,\overline{x}) + \widetilde{U}(t,x,\overline{x}) + U_\star(t,x,\overline{x}) + U_w(t,x,\overline{x}) + U_q(t,x,\overline{x}).$$

It follows from Assumption $H[f,\psi]$ and from Lemma 2.1 that there is a $C_1 \in \mathbf{R}_+$ such that

$$|U_\varphi(t,x,\overline{x})| \le C_1\|x-\overline{x}\|^2, \quad |\widetilde{U}(t,x,\overline{x})| \le C_1\|x-\overline{x}\|^2.$$

According to (II_m) and Lemma 2.1 we have

$$|U_w(t,x,\overline{x}| \le C_2\|x-\overline{x}\|^2$$

with $C_2 \in \mathbf{R}_+$. It follows from Lemma 2.1 that the bicharacteristics satisfy the condition

$$g_i^{(m)}(\xi,\tau,g_i^{(m)}(\tau,t,x)) = g_i^{(m)}(\xi,t,x)$$

where $\xi,\tau \in [a_i,c]$, $(t,x) \in [a_i,c]\times\mathbf{R}^n$. The above relations and (16) imply

(26)
$$u_{[i]}^{(m+1)}(\tau,g_i^{(m)}(\tau,t,x)) = \partial_x\varphi_i(a_i,g_i^{(m)}(a_i,t,x)) + \int_{a_i}^{\tau}\big[\partial_x f_i(P^{(m)}(\xi,t,x)) + W_{[i]}^{(m)}[z^{(m)},u_{[i]}^{(m+1)}](\xi,g_i^{(m)}(\xi,t,x)\big]\,d\xi.$$

We conclude from (12) and (26) that

$$\begin{aligned}
U_\star(t,x,\overline{x}) = {} & \partial_x\varphi_i(a_i,g_i^{(m)}(a_i,t,x)) \\
& \circ \int_{a_i}^{t}\big[\partial_q f_i(P^{(m)}(\tau,t,\bar{x})) - \partial_q f_i(P^{(m)}(\tau,t,x))\big]\,d\tau \\
& + \int_{a_i}^{t}\partial_x f_i(P^{(m)}(\xi,t,x)) \\
& \qquad \circ \int_{\xi}^{t}\big[\partial_q f_i(P^{(m)}(\tau,t,\overline{x})) - \partial_q f_i(P^{(m)}(\tau,t,x))\big]\,d\tau\,d\xi \\
& + \int_{a_i}^{t} W_{[i]}^{(m)}[z^{(m)},u_{[i]}^{(m+1)}](\xi,g_i^{(m)}(\xi,t,x)) \\
& \qquad \circ \int_{\xi}^{t}\big[\partial_q f_i(P^{(m)}(\tau,t,\overline{x})) - \partial_q f_i(P^{(m)}(\tau,t,x))\big]\,d\tau\,d\xi \\
= {} & \int_{a_i}^{t}\big[\partial_q f_i(P^{(m)}(\tau,t,\overline{x})) - \partial_q f_i(P^{(m)}(\tau,t,x))\big]
\end{aligned}$$

$$\circ \Big\{ \partial_x \varphi_i(a_i, g_i^{(m)}(a_i,t,x)) + \int_{a_i}^{\tau} \partial_x f_i(P^{(m)}(\xi,t,x))\, d\xi$$
$$+ \int_{a_i}^{\tau} W_{[i]}^{(m)}[z^{(m)}, u_{[i]}^{(m+1)}](\xi, g_i^{(m)}(\xi,t,x))\, d\xi \Big\}\, d\tau$$
$$= \int_{a_i}^{t} \big[\partial_q f_i(P^{(m)}(\tau,t,\overline{x})) - \partial_q f_i(P^{(m)}(\tau,t,x))\big]$$
$$\circ\, u_{[i]}^{(m+1)}(\tau, g_i^{(m)}(\tau,t,x))\, d\tau.$$

The result is

$$U_q(t,x,\overline{x}) + U_\star(t,x,\overline{x})$$
$$= \int_{a_i}^{t} \big[\partial_q f_i(P^{(m)}(\tau,t,\overline{x})) - \partial_q f_i(P^{(m)}(\tau,t,x))\big]$$
$$\circ \big[u_{[i]}^{(m+1)}(\tau, g_i^{(m)}(\tau,t,\overline{x})) - u_{[i]}^{(m+1)}(\tau, g_i^{(m)}(\tau,t,x))\big]\, d\tau.$$

It follows that there is a $\widetilde{C} \in \mathbf{R}_+$ such that

$$\big|U_q(t,x,\bar{x}) + U_\star(t,x,\overline{x})\big| \le \widetilde{C}\|x - \overline{x}\|^2.$$

This establishes the formula (24). From (24) we conclude (II_{m+1}). This completes the proof of the lemma. □

Now we prove that the sequences $\{z^{(m)}\}$ and $\{u^{(m)}\}$ are convergent.

Lemma 4.2. *If Assumptions $H[f,\psi]$ and $H_\star$ are satisfied and $\varphi \in \mathbf{K}$, then the sequences $\{z^{(m)}\}$ and $\{u^{(m)}\}$ are uniformly convergent on $[-d_0, c] \times \mathbf{R}^n$.*

Proof. Write

$$Z_i^{(m)}(t) = \|z_i^{(m)} - z_i^{(m-1)}\|_t, \quad U_i^{(m)}(t) = \|u_{[i]}^{(m)} - u_{[i]}^{(m-1)}\|_{(t,\mathbf{R}^n)},$$

where $t \in [-d_0, c]$, $i = 1, \dots, k$, $m \ge 1$ and

$$Z^{(m)}(t) = \max\{Z_i^{(m)}(t) :\ 1 \le i \le k\},$$
$$U^{(m)}(t) = \max\{U_i^{(m)}(t) :\ 1 \le i \le k\}.$$

It follows from (16), (18) that $Z_i^{(m)}(t) = 0$, $U_i^{(m)}(t) = 0$ for $t \in [-d_0, a_i]$, $1 \le i \le k$, and there are $\gamma_0, \gamma_1, \gamma_2, \tilde{\gamma}_0, \tilde{\gamma}_1, \tilde{\gamma}_2 \in \mathbf{L}([0,c], \mathbf{R}_+)$ such that

$$\begin{aligned} Z_i^{(m+1)}(t) \le \int_{a_i}^t \gamma_0(\xi) Z^{(m)}(\xi)\, d\xi + \int_{a_i}^t \gamma_1(\xi)\, U_i^{(m)}(\xi)\, d\xi \\ + \int_{a_i}^t \gamma_2(\xi) U_i^{(m+1)}(\xi)\, d\xi \end{aligned} \tag{27}$$

and

$$\begin{aligned} U_i^{(m+1)}(t) \le \int_{a_i}^t \tilde{\gamma}_0(\xi) Z^{(m)}(\xi)\, d\xi + \int_{a_i}^t \tilde{\gamma}_1(\xi)\, U_i^{(m)}(\xi)\, d\xi \\ + \int_{a_i}^t \tilde{\gamma}_2(\xi) U_i^{(m+1)}(\xi)\, d\xi \end{aligned} \tag{28}$$

where $t \in [a_i, c]$, $1 \le i \le k$, $m \ge 1$. We conclude from (28) and from the Gronwall inequality that

$$U_i^{(m+1)}(t) \le \left\{ \int_{a_i}^t \tilde{\gamma}_0(\xi) Z^{(m)}(\xi)\, d\xi + \int_{a_i}^t \tilde{\gamma}_1(\xi) U_i^{(m)}(\xi)\, d\xi \right\} \exp\left[\int_{a_i}^t \tilde{\gamma}_2(\xi) \right], \tag{29}$$

where $t \in [a_i, c]$, $1 \le i \le k$. It follows from (27), (29) that there is a $\tilde{\gamma} \in \mathbf{L}([0,c], \mathbf{R}_+)$ such that

$$\begin{aligned} & Z^{(m+1)}(t) + U^{(m+1)}(t) \\ & \quad \le \int_0^t \tilde{\gamma}(\xi) \big[Z^{(m)}(\xi) + U^{(m)}(\xi) \big]\, d\xi, \quad t \in [0,c],\ m \ge 1. \end{aligned} \tag{30}$$

Write

$$V^{(m)}(t) = Z^{(m)}(t) + U^{(m)}(t), \quad t \in [0,c],$$

and

$$\big[|V^{(m)}|\big] = \sup \left\{ V^{(m)}(t)\ \exp\left[-2 \int_0^t \tilde{\gamma}(\tau)\, d\tau \right] : t \in [0,c] \right\}.$$

We conclude from (30) that

$$\begin{aligned} V^{(m+1)}(t) &\le \big[|V^{(m)}|\big] \int_0^t \tilde{\gamma}(\xi) \exp\left[2 \int_0^\xi \tilde{\gamma}(\tau)\, d\tau \right] d\xi \\ &\le \frac{1}{2} \big[|V^{(m)}|\big] \exp\left[\int_0^t \tilde{\gamma}(\tau)\, d\tau \right], \quad t \in [0,c], \end{aligned}$$

and consequently

$$\big[|V^{(m+1)}|\big] \le \frac{1}{2}\big[|V^{(m)}|\big], \quad m \ge 1. \tag{31}$$

There is a $\widetilde{C} \in \mathbf{R}_+$ such that $\big[|V^{(1)}|\big] \le \widetilde{C}$. From (31) we conclude that

$$\lim_{m\to\infty} \big[|V^{(m)}|\big] = 0,$$

and the lemma follows.

5. The main theorem.

Theorem 5.1. *If Assumptions $H[f,\psi]$, $H_\star$ are satisfied and $\varphi \in \mathbf{K}$ then there is a solution $\bar z = (\bar z_1, \dots, \bar z_k) : [-d_0, c] \times \mathbf{R}^n \to \mathbf{R}^k$ of problem (1), (2). Moreover, $\bar z \in C_{\varphi.c}[d,\lambda]$ and $\partial_x \bar z_i \in C_{\partial\varphi_i,c}[s,\gamma]$ for $1 \le i \le k$.*

Proof. It follows from Lemmas 4.1 and 4.2 that there are

$$\bar z \in C_{\varphi.c}[d,\lambda],\ \bar z = (\bar z_1, \dots, \bar z_k), \quad \text{and} \quad \bar u = \big[\,\bar u_{ij}\,\big]_{i=1,\dots,k\ j=1,\dots,n},$$
$$\bar u_{[i]} = (\bar u_{i1}, \dots, \bar u_{in}) \in C_{\partial\varphi_i.c}[s,\gamma] \quad \text{for } i = 1, \dots, k$$

such that

$$\lim_{m\to\infty} z_i^{(m)}(t,x) = \bar z_i(t,x), \quad \lim_{m\to\infty} u_{[i]}^{(m)}(t,x) = \bar u_{[i]}(t,x), \quad i = 1, \dots, k,$$

uniformly on $[a_i, c] \times \mathbf{R}^n$. Furthermore, $\partial_x \bar z_i$ exists on $[a_i, c] \times \mathbf{R}^n$ and $\partial_x \bar z_i = \bar u_{[i]}$ for $i = 1, \dots, k$. Thus we get from (17) that

$$\begin{aligned} \bar z_i(t,x) &= \varphi_i(a_i, \bar g_i(a_i,t,x)) + \int_{a_i}^t f_i\big(P[\bar z, \partial_x \bar z_i](\xi,t,x)\big)\,d\xi \\ &\quad - \int_{a_i}^t \partial_q f_i\big(P[\bar z, \partial_z \bar z_i](\xi,t,x)\big) \circ \partial_x \bar z_i(\xi, \bar g(\xi,t,x))\,d\xi, \\ &\qquad i = 1, \dots, k, \end{aligned} \tag{32}$$

where $\bar g_i(\xi,t,x) = g_i[\bar z, \partial_x \bar z_i](\xi,t,x)$. Now we prove that $\bar z$ is a solution of (1). Suppose that i, $1 \le i \le k$, is fixed. For a given $x \in \mathbf{R}^n$ we put

$y = \bar{g}_i(a_i, t, x)$. It follows from Lemma 2.1 that $\bar{g}_i(\tau, t, x) = \bar{g}_i(\tau, a_i, y)$ for $\tau \in [a_i, c]$ and (32) is equivalent to

$$
\begin{aligned}
(33)\quad & \bar{z}_i(t, \bar{g}_i(t, a_i, y)) \\
&= \varphi(a_i, y) \\
&\quad + \int_{a_i}^{t} f_i(\tau, \bar{g}_i(\tau, a_i, y), \bar{z}_{\psi(\tau, \bar{g}_i(\tau, a_i, y))}, \partial_x \bar{z}_i(\tau, \bar{g}_i(\tau, a_i, y)))\, d\tau \\
&\quad - \int_{a_i}^{t} \partial_q f_i(\tau, \bar{g}_i(\tau, a_i, y), \bar{z}_{\psi(\tau, \bar{g}_i(\tau, a_i, y))}, \partial_x \bar{z}_i(\tau, \bar{g}_i(\tau, a_i, y))) \\
&\qquad\qquad \circ\, \partial_x \bar{z}_i(\tau, \bar{g}_i(\tau, a_i, y))\, d\tau.
\end{aligned}
$$

The relations $y = \bar{g}_i(a_i, t, x)$ and $x = \bar{g}_i(t, a_i, y)$ are equivalent for $x, y \in \mathbf{R}^n$, $t \in [a_i, c]$. By differentiating (33) with respect to t and by putting again $x = \bar{g}_i(t, a_i, y)$ we obtain that $\bar{z}$ satisfies (1) for almost all $t \in [a_i, c]$ with fixed $x \in \mathbf{R}^n$. It is easily seen that $\bar{z}$ satisfies (2).

This completes the proof. □

Remark 5.2. Suppose that all the assumptions of Theorem 5.1 are satisfied and that the functions

$$f(\,\cdot\,, x, w, q),\ \partial_x f(\,\cdot\,, x, w, q),\ \partial_w f(\,\cdot\,, x, w, q),\ \partial_q f(\,\cdot\,, x, w, q)$$

are continuous on $[0, a]$ for $(x, w, q) \in \mathbf{R}^n \times C(B, \mathbf{R}^k) \times \mathbf{R}^n$. Then there is a classical solution $\bar{z} : [-d_0, c] \times \mathbf{R}^n \to \mathbf{R}^k$ of (1), (2).

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