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# FILTERS IN ORDERED Γ-SEMIGROUPS

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ABSTRACT. In this paper we characterize the principal filters on any ordered $\Gamma$-semigroup $M$ and their structure and properties are investigated by using the relation $\mathcal{N}$ which is the smallest complete semilattice congruence on $M$. In particular, we prove that every principal filter of any ordered $\Gamma$-semigroup $M$ can be uniquely determined by its $\mathcal{N}$-classes of $M$. Also, by using the relation $\mathcal{N}$, we will observe that $\mathcal{N}$ on any ordered $\Gamma$-semigroup $M$ is the equality relation on $M$ if and only if $M$ is a semilattice such that $a \leq a\gamma a$ for all $a \in M$, $\gamma \in \Gamma$, and $\mathcal{N}$ is the universal relation on $M$ if and only if $M$ is the only principal filter. We also investigate properties of the complete semilattice congruence classes of $M$.

**1. Introduction and preliminaries.** In 1987, Kehayopulu [**8**] introduced the concept of filter in poe-semigroups. Later Kehayopulu [**12**] defined the relation $\mathcal{N}$ on a po-semigroup and obtained some results. Various kinds of ordered semigroups have been widely studied by many authors [**1, 2, 8–14, 17, 20**] by using the notion of filter and the relation $\mathcal{N}$. In [**15**] Kwon introduced the concept of filter and the relation $\mathcal{N}$ in ordered $\Gamma$-semigroups and obtained some results extending those for ordered semigroups. Also, in [**3, 4**] we have used these notions to characterize some classes of ordered $\Gamma$-semigroups. In the present paper we give some new results extending those for ordered semigroups, dealing with the principal filters on any ordered $\Gamma$-semigroup $M$ and their structure and properties, which are investigated by using the relation $\mathcal{N}$ which is the smallest complete semilattice congruence on $M$. In particular, we prove that every principal filter of any ordered $\Gamma$-semigroup $M$ can be uniquely determined by its $\mathcal{N}$-classes of $M$. Also, we will consider a structure of principal filter on ordered $\Gamma$-semigroups and by using the relation $\mathcal{N}$, we will observe that $\mathcal{N}$ on any ordered $\Gamma$-semigroup $M$ is the equality relation on $M$ if and only if $M$ is a semilattice having the property $a \leq a\gamma a$ for all $a \in M$,

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$\gamma \in \Gamma$, and $\mathcal{N}$ is the universal relation on $M$ if and only if $M$ is the only principal filter. We also investigate properties of the complete semilattice congruence classes of $M$.

We introduce below necessary notions and present a few auxiliary results that will be used throughout the paper.

In 1986, Sen and Saha [**19**] defined $\Gamma$-semigroup as a generalization of semigroup as follows:

**Definition 1.1.** Let $M$ and $\Gamma$ be two nonempty sets. Denote by the letters of the English alphabet the elements of $M$ and with the letters of the Greek alphabet the elements of $\Gamma$. Then $M$ is called a $\Gamma$-semigroup if there exists a mapping $M \times \Gamma \times M \to M$, written as $(a, \gamma, b) \mapsto a\gamma b$ satisfying the following identity

$$(a\alpha b)\beta c = a\alpha(b\beta c) \text{ for all } a, b, c \in M \text{ and for all } \alpha, \beta \in \Gamma.$$

A $\Gamma$-semigroup $M$ is called *commutative* $\Gamma$*-semigroup* if for all $a, b \in M$ and $\gamma \in \Gamma$, $a\gamma b = b\gamma a$. A nonempty subset $K$ of a $\Gamma$-semigroup $M$ is called a *sub*-$\Gamma$*-semigroup* of $M$ if for all $a, b \in K$ and $\gamma \in \Gamma$, $a\gamma b \in K$.

Examples of $\Gamma$-semigroups can be found in [**18, 19**].

**Definition 1.2.** A *po*-$\Gamma$*-semigroup* (: *ordered* $\Gamma$*-semigroup*) is an ordered set $M$ at the same time a $\Gamma$-semigroup such that for all $a, b, c \in M$ and for all $\gamma \in \Gamma$

$$a \leq b \Longrightarrow a\gamma c \leq b\gamma c, c\gamma a \leq c\gamma b.$$

**Example 1.3.** Let $M$ be the set of all $2 \times 3$ matrices over the set of positive integers, and let $\Gamma$ be the set of all $3 \times 2$ matrices over the same set. Then $M$ is a $\Gamma$-semigroup with respect to the usual matrix multiplication. Also $M$ and $\Gamma$ are posets with respect to "$\leq$" defined by $(a_{ik}) \leq (b_{ik})$ if and only if $a_{ik} \leq b_{ik}$ for all $i, k$. Then $M$ is a $po - \Gamma$-semigroup.

**Example 1.4.** Let $G$ be a group, $I, \wedge$ two index sets and $\Gamma$ the collection of some $\wedge \times I$ matrices over $G^o = G \cup \{0\}$, the group with zero. Let $\mu^0$ be the set of all elements $(a)_{i\lambda}$ where $i \in I$, $\lambda \in \wedge$ and $(a)_{i\lambda}$ the $I \times \wedge$ matrix over $G^0$ having $a$ in the $i$th row and $\lambda$th column, its remaining entries being zero. The expression $(0)_{i\lambda}$ will be used to denote the zero matrix. For any $(a)_{i\lambda}, (b)_{j\mu}, (c)_{k\nu} \in \mu^0$ and $\alpha = (p_{\lambda i}), \beta = (q_{\lambda i}) \in \Gamma$ we define $(a)_{i\lambda}\alpha(b)_{j\mu} = (ap_{\lambda j}b)_{i\mu}$. Then it is easily verified that $[(a)_{i\lambda}\alpha(b)_{j\mu}]\beta(c)_{k\nu} = (a)_{i\lambda}\alpha[(b)_{j\mu}\beta(c)_{k\nu}]$. Thus $\mu^0$ is a $\Gamma$-semigroup. We call $\Gamma$ the sandwich matrix set and $\mu^0$ the Rees $I \times \wedge$ matrix $\Gamma$-semigroup over $G^0$ with sandwich matrix set $\Gamma$ and denote it by $\mu^0(G : I, \wedge, \Gamma)$. In [**5**] we deal with lattice-ordered Rees matrix $\Gamma$-semigroups.

**Example 1.5.** Let $M$ be the set of all integers of the form $4n + 1$ and $\Gamma$ the set of all integers of the form $4n+3$, where $n$ is a nonnegative integer. If $a\alpha b$ is $a + \alpha + b$ and $\alpha a \beta$ is $\alpha + a + \beta$ (usual sum of the integers) and $a \leq b$ means $a$ is less than or equal to $b$ for all $a, b \in M$ and $\alpha, \beta \in \Gamma$, then $M$ is a $po - \Gamma$-semigroup.

**Example 1.6.** Let $M$ be the set of all isotone mappings from a poset $P$ into another poset $Q$ and $\Gamma$ the set of all isotone mappings from the poset $Q$ into the poset $P$. Let $f, g \in M$ and $\alpha \in \Gamma$. Denote by $f\alpha g$ the usual mapping composition of $f, \alpha$ and $g$. Then $M$ is a $\Gamma$-semigroup. We define a relation $\leq$ on $M$ by $f \leq g$ if and only if $af \leq ag$, for all $a \in P$. This relation is a partial order on $M$ and as such $M$ is a poset. We also define a relation $\leq$ on $\Gamma$ by $\alpha \leq \beta$ if and only if $x\alpha \leq x\beta$, for all $x \in Q$. For this relation $\Gamma$ is a poset. Then $M$ is a $po-\Gamma$-semigroup.

**Example 1.7.** For $a, b \in [0, 1]$, let $M = [0, a]$ and $\Gamma = [0, b]$. Then $M$ is an ordered $\Gamma$-semigroup under usual multiplication and usual partial order relation.

**Example 1.8.** Fix $m \in Z$, and let $M$ be the set of all integers of the form $mn+1$ and $\Gamma$ denotes the set of all integers of the form $mn+m-1$ where $n$ is an integer. Then $M$ is an ordered $\Gamma$-semigroup under usual addition and usual partial order relation.

Throughout this paper, $M$ stands for an ordered $\Gamma$-semigroup. For nonempty subsets $A$ and $B$ of $M$ and a nonempty subset $\Gamma'$ of $\Gamma$, let

$A\Gamma' B = \{a\gamma b : a \in A, b \in B \text{ and } \gamma \in \Gamma'\}$. If $A = \{a\}$, then we also write $a\Gamma' B$ instead of $\{a\}\Gamma' B$, and similarly if $B = \{b\}$ or $\Gamma' = \{\gamma\}$. A nonempty subset $A$ of $M$ is called a *right* (respectively *left*) ideal of $M$ if

(1) $A\Gamma M \subseteq A$ (respectively $M\Gamma A \subseteq A$)

(2) $a \in A, b \leq a$ for $b \in M \Rightarrow b \in A$.

$A$ is called an *ideal* of $M$ if it is right and left ideal of $M$.

Let $T$ be a sub-$\Gamma$-semigroup of $M$. For $A \subseteq T$ we denote

$$(A]_T = \{t \in T \mid t \leq a, \text{ for some } a \in A\}$$
$$[A)_T = \{t \in T | t \geq a, \text{ for some } a \in A\}.$$

If $T = M$, then we always write $(A]$ (respectively $[A)$) instead of $(A]_M$ (respectively $[A)_M$). Clearly, $A \subseteq (A]_T \subseteq (A]$ and $A \subseteq B$ implies that $(A]_T \subseteq (B]_T$ for any nonempty subsets $A, B$ of $T$. For $A = \{a\}$, we write $(a]$ (respectively $[a)$) instead of $(\{a\}]$ (respectively $[\{a\})$).

An ideal $T$ of $M$ is said to be *prime* if $A\Gamma B \subseteq T$ implies that $A \subseteq T$ or $B \subseteq T$, where $A, B \subseteq M$ or equivalently, an ideal $T$ of $M$ is said to be *prime* if $a\Gamma b \subseteq T$ implies that $a \in T$ or $b \in T$ $(a, b \in M)$ [**9, 15**]. An ideal $T$ of $M$ is said to be *semiprime* if $A\Gamma A \subseteq T$ implies that $A \subseteq T$, where $A \subseteq M$ or equivalently, an ideal $T$ of $M$ is said to be *semiprime* if $a\Gamma a \subseteq T$ implies that $a \in T$ $(a \in M)$ [**11, 16**].

**Definition 1.9** [**8, 15**]. Let $M$ be a $po-\Gamma$-semigroup and $F$ a sub-$\Gamma$-semigroup. Then $F$ is called a filter of $M$ if

(1) $a, b \in M, a\gamma b \in F$ $(\gamma \in \Gamma) \Rightarrow a \in F$ and $b \in F$

(2) $a \in F, a \leq c$ $(c \in M) \Rightarrow c \in F$ or equivalently $[F) \subseteq F$.

**Lemma 1.10.** *Let $F_1, F_2$ be two filters of $M$. Then the intersection $F_1 \cap F_2$, if it is nonempty, is a filter of $M$.*

*Proof.* It can easily be verified from the definition above. □

Lemma 1.10 shows that the intersection of a finite number of filters of $M$, if it is nonempty, is a filter of $M$. It is clear that for every $a \in M$ there is a unique smallest filter of $M$ containing the element $a$, denoted by $N(a)$, which is called the principal filter generated by $a$.

The following example shows that, in general, the union of two filters is not a filter.

**Example 1.11.** Let $M = \{a, b, c\}$ and $\Gamma = \{\gamma\}$ with the multiplication defined by

$$x\gamma y = \begin{cases} b & \text{if } x = y = b \\ c & \text{if } x = y = c \\ a & \text{otherwise.} \end{cases}$$

First we show that $M$ is a $\Gamma$-semigroup, suppose not. Then there exist $x, y, z \in M$ such that $(x\gamma y)\gamma z \neq x\gamma(y\gamma z)$. If $(x\gamma y)\gamma z = b$, then $x = y = z = b$. Thus $x\gamma(y\gamma z) = b$, which is impossible. If $x\gamma(y\gamma z) = b$, then $x = y = z = b$. Thus $(x\gamma y)\gamma z = b$, which is impossible. If $(x\gamma y)\gamma z = c$, then $x = y = z = c$. Thus $x\gamma(y\gamma z) = c$, which is impossible. If $x\gamma(y\gamma z) = c$, then $x = y = z = c$. Thus $(x\gamma y)\gamma z = c$, which is impossible. Hence $(x\gamma y)\gamma z = x\gamma(y\gamma z)$ for all $x, y, z \in M$. Obviously, $x\gamma y = y\gamma x$ for all $x, y \in M$. Therefore $M$ is a commutative $\Gamma$-semigroup.

Define a relation $\leq$ on $M$ as follows:

$$\leq := 1_M \cup \{(a, b), (a, c)\}.$$

Then $(M, \leq)$ is a partially ordered set. Let $x, y \in M$ be such that $x \leq y$. Since $x\gamma z \leq y\gamma z$, $z\gamma x \leq z\gamma y$ for all $z \in M$ and $a \leq b, a \leq c$, then $M$ is an ordered $\Gamma$-semigroup.

It is easy to see that $N(a) = \{a, b, c\}$, $N(b) = \{b\}$, $N(c) = \{c\}$ are all the filters of the ordered $\Gamma$-semigroup $M$. But $N(b) \cup N(c)$ isn't a filter of $M$ because $b\gamma c = a$ is not in $N(b) \cup N(c)$.

An equivalence relation $\Re$ on $M$ is called *congruence* [**12, 15**] if for all $\gamma \in \Gamma$ and $c \in M$,

$$(a, b) \in \Re \Longrightarrow (a\gamma c, b\gamma c) \in \Re, (c\gamma a, c\gamma b) \in \Re.$$

A congruence $\Re$ on $M$ is called *semilattice congruence* [**12, 15**] if for all $\gamma \in \Gamma$ and $a, b \in M$,

$$(a\gamma a, a) \in \Re \quad \text{and} \quad (a\gamma b, b\gamma a) \in \Re.$$

A semilattice congruence $\Re$ on $M$ is called *complete* [**4, 14**] if for any $a, b \in M$, $\gamma \in \Gamma$, $a \leq b$ implies $(a, a\gamma b) \in \Re$.

We denote by "$\mathcal{N}$" the equivalence relation on $M$ defined by $\mathcal{N} = \{(a,b) \in M^2 \mid N(a) = N(b)\}$ [**9, 15**]. $\mathcal{N}$ is a semilattice congruence on $M$ [**15**, Theorem 2.7]. We have proved that $\mathcal{N}$ is a complete semilattice congruence on $M$ [**4**, Remark 2.17].

For any $a \in M$, the $\mathcal{N}$-class containing $a$ is denoted by $(a)_{\mathcal{N}}$ and it is clear that it is an ordered sub-$\Gamma$-semigroup of $M$ [**4**, Remark 2.3]. On the set $M/\mathcal{N} = \{(a)_{\mathcal{N}} | a \in M\}$ we define $(a)_{\mathcal{N}}\gamma(b)_{\mathcal{N}} = (a\gamma b)_{\mathcal{N}}$, for all $(a)_{\mathcal{N}}, (b)_{\mathcal{N}} \in M/\mathcal{N}$, $\gamma \in \Gamma$. It is clear that the set $M/\mathcal{N}$ is a $\Gamma$-semigroup. In this set we define $(a)_{\mathcal{N}} \preceq (b)_{\mathcal{N}}$ if and only if $(a)_{\mathcal{N}} = (a\gamma b)_{\mathcal{N}}$, for all $\gamma \in \Gamma$, then it can be easily seen that the set $M/\mathcal{N}$ is an ordered $\Gamma$-semigroup induced by the complete semilattice congruence $\mathcal{N}$ on $M$ [**3, 4**].

**2. On the structure of principal filters.** In [**15**] the following lemma is proved as an easy modification of the Lemma in [**12**]:

**Lemma 2.1.** *Let $M$ be an ordered $\Gamma$-semigroup and $F$ a nonempty set of $M$. Then the following are equivalent*:

(1) *$F$ is a filter of $M$.*

(2) *$M \backslash F = \varnothing$ or $M \backslash F$ is a prime ideal of $M$.*

A direct result of above lemma is the following lemma.

**Lemma 2.2.** it If $N(b) \subseteq N(a)$, then $N(a) \backslash N(b)$, if it is nonempty, is a prime ideal of $N(a)$.

The following lemma is an immediate result from the definition of the principal filter.

**Lemma 2.3.** *Let $a, b \in M$. Then $b \in N(a)$ implies that $N(b) \subseteq N(a)$.*

**Lemma 2.4.** *Let $a, b \in M$. Then $a \leq b$ implies that $N(b) \subseteq N(a)$.*

*Proof.* Since $b \geq a$ then it is clear that $b \in N(a)$. By using Lemma 2.3, we have that $N(b) \subseteq N(a)$. $\square$

**Theorem 2.5.** *Let $a \in M$. Then $(a)_{\mathcal{N}}$ is a semiprime ideal of $N(a)$.*

*Proof.* We first prove that $(a)_{\mathcal{N}}$ is a sub-Γ-semigroup of $N(a)$. For any $x \in (a)_{\mathcal{N}}$, it is clear that $x \in N(x) = N(a)$. Thus we have $(a)_{\mathcal{N}} \subseteq N(a)$. Since $N(a)$ is a filter of $M$, we have that $x\gamma y \in N(a)$ for any $x, y \in (a)_{\mathcal{N}} \subseteq N(a)$ and $\gamma \in \Gamma$. By using Lemma 2.3, we have $N(x\gamma y) \subseteq N(a)$ for all $\gamma \in \Gamma$. Since $x\gamma y \in N(x\gamma y)$ which is a filter of $M$, we deduce that $x \in N(x\gamma y)$. By Lemma 2.3, we have that $N(a) = N(x) \subseteq N(x\gamma y)$ and $N(x\gamma y) = N(a)$. By definition, we see that $x\gamma y \in (a)_{\mathcal{N}}$. Hence $(a)_{\mathcal{N}}$ is a sub-Γ-semigroup of $N(a)$.

We prove now that $(a)_{\mathcal{N}}$ is an ideal of $N(a)$. Assume that $x \in (a)_{\mathcal{N}} \subseteq N(a)$ and $y \in N(a)$. Then $x\gamma y \in N(a)$ since $N(a)$ is a principal filter. Hence $N(x\gamma y) \subseteq N(a)$ by Lemma 2.3. Since $x\gamma y \in N(x\gamma y)$ which is a principal filter of $M$, we have $x \in N(x\gamma y)$ and so $N(a) = N(x) \subseteq N(x\gamma y)$ by Lemma 2.3. Consequently, $N(a) = N(x\gamma y)$, that is, $x\gamma y \in (a)_{\mathcal{N}}$. Similarly, $y\gamma x \in (a)_{\mathcal{N}}$.

In order to prove that $(a)_{\mathcal{N}}$ is an ideal of $N(a)$, we need to show that $((a)_{\mathcal{N}}]_{N(a)} \subseteq (a)_{\mathcal{N}}$. Assume that $x \in ((a)_{\mathcal{N}}]_{N(a)}$. Then there exists $t \in (a)_{\mathcal{N}}$ such that $x \leq t$. By Lemma 2.4, it is clear that $N(a) = N(t) \subseteq N(x)$. On the other hand, $x \in N(a)$. By Lemma 2.3, we have $N(x) \subseteq N(a)$, and $N(x) = N(a)$. This implies that $x \in (a)_{\mathcal{N}}$. Hence $((a)_{\mathcal{N}}]_{N(a)} \subseteq (a)_{\mathcal{N}}$. Thus, we have shown that $(a)_{\mathcal{N}}$ is an ideal of $N(a)$.

Finally, we prove that $(a)_{\mathcal{N}}$ is a semiprime ideal of $N(a)$. Let $x \in N(a)$ and $x\gamma x \in (a)_{\mathcal{N}}$, for all $\gamma \in \Gamma$. Clearly, by Lemma 2.3, $N(x) \subseteq N(a)$. Also, since $x\gamma x \in N(x)$, for all $\gamma \in \Gamma$, we have by Lemma 2.3 that $N(a) = N(x\gamma x) \subseteq N(x)$. Hence, we obtain that

$N(x) = N(a)$ which implies that $x \in (a)_{\mathcal{N}}$. Thus, we have shown that $(a)_{\mathcal{N}}$ is a semiprime ideal of $N(a)$. □

**Theorem 2.6.** *Let $a, b \in M$. Then $(a)_{\mathcal{N}} \preceq (b)_{\mathcal{N}}$ if and only if $N(b) \subseteq N(a)$.*

*Proof.* Let $a, b \in M$. Since $(a)_{\mathcal{N}}, (b)_{\mathcal{N}}$ are respectively semilattice congruence classes of $M$, it is clear that $(a)_{\mathcal{N}} \preceq (b)_{\mathcal{N}}$ implies $(a)_{\mathcal{N}}\gamma(b)_{\mathcal{N}} \subseteq (a)_{\mathcal{N}}$, for all $\gamma \in \Gamma$. Hence, we have $x\gamma y \in (a)_{\mathcal{N}} \subseteq N(a)$ for any $x \in (a)_{\mathcal{N}}$ and $y \in (b)_{\mathcal{N}}$. Since $N(a)$ is a principal filter, $y \in N(a)$. By Lemma 2.3, we have $N(b) = N(y) \subseteq N(a)$.

Conversely, we only need to prove that $(a)_{\mathcal{N}}\gamma(b)_{\mathcal{N}} \subseteq (a)_{\mathcal{N}}$, $(b)_{\mathcal{N}}\gamma(a)_{\mathcal{N}} \subseteq (a)_{\mathcal{N}}$, for all $\gamma \in \Gamma$. Suppose that $x \in (a)_{\mathcal{N}}$ and $y \in (b)_{\mathcal{N}} \subseteq N(b) \subseteq N(a)$. By Theorem 2.5, $(a)_{\mathcal{N}}$ is an ideal of $N(a)$. Hence, we have $x\gamma y \in (a)_{\mathcal{N}}$ and $y\gamma x \in (a)_{\mathcal{N}}$. This shows that $(a)_{\mathcal{N}}\gamma(b)_{\mathcal{N}} \subseteq (a)_{\mathcal{N}}$, $(b)_{\mathcal{N}}\gamma(a)_{\mathcal{N}} \subseteq (a)_{\mathcal{N}}$. □

Let us now consider the subset of $M$ given by

$$K(a) = \{b \in M : (b)_{\mathcal{N}} \succ (a)_{\mathcal{N}}\}$$

for any $a \in M$.

**Theorem 2.7.** *Let $a \in M$. Then the following sets are equal:*

(1) $K(a) = \{b \in M : (b)_{\mathcal{N}} \succ (a)_{\mathcal{N}}\}$.

(2) $A = \{b \in N(a) : (b)_{\mathcal{N}} \succ (a)_{\mathcal{N}}\}$.

(3) $B = \cup\{(b)_{\mathcal{N}} : (b)_{\mathcal{N}} \succ (a)_{\mathcal{N}}\}$.

(4) $C = N(a) \backslash (a)_{\mathcal{N}}$.

*Proof.* We prove that $K(a) \subseteq A \subseteq B \subseteq C \subseteq K(a)$.

Let $b \in K(a)$. Since $(b)_{\mathcal{N}} \succ (a)_{\mathcal{N}}$, by Theorem 2.6, we have $N(b) \subsetneq N(a)$ and so $b \in N(a)$. This shows that $K(a) \subseteq A$. Clearly, $A \subseteq B$.

Let $x \in B$. Then there exists $b \in M$ such that $x \in (b)_{\mathcal{N}} \succ (a)_{\mathcal{N}}$. Hence, by Theorem 2.6, we have $x \in N(x) = N(b) \subsetneq N(a)$. If

$x \in (a)_{\mathcal{N}}$, then $N(b) = N(x) = N(a)$ which contradicts $N(b) \subsetneq N(a)$. Hence $x \in N(a) \backslash (a)_{\mathcal{N}}$, that is, $B \subseteq C$.

Let $x \in C$. Then we have $N(x) \subseteq N(a)$. Since $x \notin (a)_{\mathcal{N}}$, then $N(x) \neq N(a)$ and so $N(x) \subsetneq N(a)$. Hence, by Theorem 2.6, we have $(x)_{\mathcal{N}} \succ (a)_{\mathcal{N}}$, $x \in M$. This shows that $C \subseteq K(a)$ and the proof is completed. □

**Corollary 2.8.** *Let $a \in M$. Then $N(a) = \cup\{(b)_{\mathcal{N}} : (b)_{\mathcal{N}} \succeq (a)_{\mathcal{N}}\} = \{b \in M : (b)_{\mathcal{N}} \succeq (a)_{\mathcal{N}}\}$.*

*Proof.* By the definition of $K(a)$ and Theorem 2.7, we have $N(a) = (a)_{\mathcal{N}} \cup K(a) = (a)_{\mathcal{N}} \cup (\cup\{(b)_{\mathcal{N}} : (b)_{\mathcal{N}} \succ (a)_{\mathcal{N}}\}) = \cup\{(b)_{\mathcal{N}} : (b)_{\mathcal{N}} \succeq (a)_{\mathcal{N}}\} = \{b \in M : (b)_{\mathcal{N}} \succeq (a)_{\mathcal{N}}\}$.

Using Corollary 2.8, we can easily see that every principal filter of an ordered Γ-semigroup $M$ can be uniquely expressed by its $\mathcal{N}$-classes of $M$. □

**Example 2.9.** Let $M = \{a, b, c, d, e, f, g\}$ and $\Gamma = \{\gamma\}$ with the multiplication be defined by

$$x\gamma y = \begin{cases} a & \text{if } x = a, y \in \{a, c\} \text{ or } x = y = b \text{ or } x = c, y = a \\ b & \text{if } x = a, y = b \text{ or } x = b, y \in \{a, c\} \text{ or } x = c, y = b \\ c & \text{if } x = y = c \\ d & \text{if } x = d, y \in \{d, c\} \text{ or } x = c, y = d \\ e & \text{if } x = y = e \text{ or } x = y = f \\ f & \text{if } x = e, y = f \text{ or } x = f, y = e \\ g & \text{otherwise.} \end{cases}$$

If we define a relation $\leq$ on $M$ as follows:

$$\leq := 1_M \bigcup \{(g, a), (g, b), (g, c), (g, d), (g, e), (g, f), (d, c)\}$$

then it can be easily verified that $M$ is an ordered Γ-semigroup. All the $\mathcal{N}$-classes of $M$ are $(a)_{\mathcal{N}} = \{a, b\}$, $(c)_{\mathcal{N}} = \{c\}$, $(d)_{\mathcal{N}} = \{d\}$, $(e)_{\mathcal{N}} = \{e, f\}$ and $(g)_{\mathcal{N}} = \{g\}$.

We consider now the principal filter on $M$. By Corollary 2.8, we can easily deduce that $N(a) = N(b) = \{a, b, c\}$, $N(c) = \{c\}$, $N(d) = \{c, d\}$, $N(e) = N(f) = \{e, f\}$ and $N(g) = \{a, b, c, d, e, f, g\}$. Also, we have that $N(a) \cup N(d)$ isn't a filter because $a\gamma d = g \notin N(a) \bigcup N(d)$.

From Lemma 2.1 and Corollary 2.8, we also have

**Corollary 2.10.** *Let $a \in M$. Then $M \backslash \{b : (b)_{\mathcal{N}} \succeq (a)_{\mathcal{N}}\}$ is a prime ideal of $M$.*

To prove the following theorems we use some important notions and results proved in [**7**, 1.3.2] for ordered semigroups, the modification of which can easily be done for the ordered $\Gamma$-semigroups.

**Theorem 2.11.** *The following statement are equivalent:*

(1) *$M$ is a semilattice such that $a \leq a\gamma a$, for all $a \in M$, $\gamma \in \Gamma$.*

(2) *For every $a \in M, N(a) = [a)$.*

(3) *$\mathcal{N}$ is the equality relation on $M$.*

*Proof.* (1) $\Rightarrow$ (2). Let $M$ be a semilattice. For any $a \in M$ and $x, y \in [a)$, we have $x \geq a, y \geq a$. This implies that $x\gamma y \geq a\gamma a = a$ and $x\gamma y \in [a)$ for all $\gamma \in \Gamma$. Hence, $[a)$ is a sub-$\Gamma$-semigroup of $M$.

To prove that $[a)$ is a filter containing $a$, we suppose that $b, c \in M$ such that $b\gamma c \in [a)$ for all $\gamma \in \Gamma$. Then we have $b\gamma c \geq a$, for all $\gamma \in \Gamma$. Since $M$ is semilattice, there exist $\gamma_1, \gamma_2 \in \Gamma$, such that $b = b\gamma_1 b$ and $c = c\gamma_2 c$. Since $b\gamma c \geq a$, for all $\gamma \in \Gamma$, then we have $b\gamma_1 c \geq a$ and there exists $\gamma_2 \in \Gamma$ such that $a = a\gamma_2 b\gamma_1 c$. Hence,

$$a\gamma_1 b = b\gamma_1 a = a\gamma_2 b\gamma_1 c\gamma_1 b = a\gamma_2 b\gamma_1 b\gamma_1 c = a.$$

Also, there exists $\gamma_3 \in \Gamma$ such that $a = a\gamma_3 b\gamma_2 c$. Hence,

$$a\gamma_2 c = c\gamma_2 a = a\gamma_3 b\gamma_2 c\gamma_2 c = a\gamma_3 b\gamma_2 c = a,$$

and so $b \geq a$, $c \geq a$. This shows that $b \in [a)$, $c \in [a)$. Since $[a) \subseteq [a)$ always holds, $[a)$ is a filter containing $a$, as required.

Let $T$ be a filter containing $a$. By the definition of filters, we have $[T) \subseteq T$. Since $a \in T$, then $[a) \subseteq [T) \subseteq T$. Consequently, $[a)$ is the smallest filter containing $a$ and then $N(a) = [a)$.

(2) $\Rightarrow$ (3). Assume that $a\mathcal{N}b$ for $a, b \in M$. Then $[a) = N(a) = N(b) = [b)$. Since $a \in [a) = [b)$ and $b \in [b) = [a)$, we have that $a \geq b, b \geq a$ and so $a = b$. This implies that $\mathcal{N} = 1_M$.

(3) $\Rightarrow$ (1). For any $a, b \in M$, we have $(a)_\mathcal{N} = \{a\}, (b)_\mathcal{N} = \{b\}$. Since $(a)_\mathcal{N}$ and $(b)_\mathcal{N}$ are both semilattice congruence classes of $M$, it is easy to see that $(a)_\mathcal{N}\gamma(a)_\mathcal{N} \subseteq (a)_\mathcal{N}$ and $(a)_\mathcal{N}\gamma(b)_\mathcal{N} = (b)_\mathcal{N}\gamma(a)_\mathcal{N}$ for all $\gamma \in \Gamma$. Clearly, $a\gamma a = a$, $a\gamma b = b\gamma a$. This shows that $M$ is a semilattice as required.

Moreover, the partial order on $M$ is the natural order of semilattice. Indeed: $(a)_\mathcal{N} \preceq (b)_\mathcal{N}$ if and only if $(a)_\mathcal{N}\gamma(b)_\mathcal{N} = (b)_\mathcal{N}\gamma(a)_\mathcal{N} = (a)_\mathcal{N}$, for all $\gamma \in \Gamma$. Since $\mathcal{N} = 1_M$, we have that $a \leq b$ if and only if $a\gamma b = b\gamma a = a$. □

**Theorem 2.12.** *The following statements are equivalent*:

(1) $\mathcal{N}$ *is the universal relation on* $M$.

(2) $M$ *has only one filter and* $N(a) = M$, *for any* $a \in M$.

(3) $M$ *has only one complete semilattice congruence on* $M$.

*Proof*. Since $\mathcal{N}$ is the smallest complete semilattice congruence on $M$, it is trivial that (1) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (2). Since $\mathcal{N}$ is the universal relation on $M$ which means that for every $a \in M$, $(a)_\mathcal{N} = M$, we have that $(a)_\mathcal{N} \subseteq N(a) \subseteq M$ by Theorem 2.5. Hence, $N(a) = M$ as required.

(2) $\Rightarrow$ (1). For any $a, b \in M$, we have $N(a) = M = N(b)$. This shows that $a\mathcal{N}b$ and $\mathcal{N} = \omega_M$, as required. □

**Theorem 2.13.** *Let* $\sigma$ *be a complete semilattice congruence on an ordered* $\Gamma$*-semigroup* $M$ *and* $Y$ *the semilattice* $M/\sigma$. *Then for any* $x \in Y$, *we have*

(1) $M_x$ *is the union of some* $\mathcal{N}$*-classes.*

(2) *The set* $T = \cup\{M_y : y \succeq x, y \in Y\}$ *is a filter.*

(3) *For any* $a \in M_x$, $N(a) = T$ *if and only if* $\sigma$ *is the smallest complete semilattice congruence on* $M$.

*Proof.* (1) Since $\mathcal{N}$ is the smallest complete semilattice congruence on $M$, we have that $(a,b) \in \mathcal{N} \subseteq \sigma$ for every $a \in M_x$ and $b \in (a)_{\mathcal{N}}$. It is clear that $M_x$ is a semilattice congruence class of $M$ and so $b \in M_x$. We have proved that $(a)_{\mathcal{N}} \subseteq M_x$. Consequently, $\cup_{a \in M_x}(a)_{\mathcal{N}} \subseteq M_x$. Clearly, $M_x \subseteq \cup_{a \in M_x}(a)_{\mathcal{N}}$. Hence we have $M_x = \cup_{a \in M_x}(a)_{\mathcal{N}}$. This is exactly the union of some $\mathcal{N}$-classes.

(2) To see that $T$ is a filter, we first prove that $T$ is a sub-$\Gamma$-semigroup of $M$. Since $\varnothing \neq M_x \subseteq T$, $T$ is not empty. For any $a, b \in T$, there exist $y, z \in Y$ such that $a \in M_y$, $b \in M_z$, $y \succeq x$ and $z \succeq x$. This implies that $a\gamma b \in M_y \Gamma M_z \subseteq M_{y\Gamma z}$ and $y\gamma z \succeq x$, for all $\gamma \in \Gamma$. Hence, $a\gamma b \in T$ and $T$ is a sub-$\Gamma$-semigroup of $M$.

Assume that $a\gamma b \in T$ and $a, b \in M$. Let $y, z$ and $t \in Y$ such that $a \in M_z$, $b \in M_t$, $a\gamma b \in M_y$ and $y \succeq x$, for all $\gamma \in \Gamma$. This implies that $a\gamma b \in M_z \Gamma M_t \subseteq M_{z\Gamma t}$ and $z\gamma t = y \succeq x$, for all $\gamma \in \Gamma$. Since $Y$ is a semilattice, it is easy to see that $z \succeq x$ and $t \succeq x$. Thus, we have $a \in T$ and $b \in T$.

For any $a \in [T)$, there exists an element $y \in Y$ such that $a \in M_y$ and an element $b \in M_z$ such that $a \geq b$, where $z \in Y$ and $z \succeq x$. This shows that $a\gamma b \in M_y \Gamma M_z \subseteq M_{y\Gamma z}$, for all $\gamma \in \Gamma$. Since $\sigma$ is a complete semilattice congruence, we can see that $(a\gamma b, b) \in \sigma$. From $b \in M_z$, we immediately have $a\gamma b \in M_z$. Then we have $y\gamma z = z$ and so $y \succeq z \succeq x$ in $Y$. Hence, $a \in T$ and $[T) \subseteq T$ as required. We have shown that $T$ is a filter.

(3) If $\sigma$ is the smallest complete semilattice congruence on $M$, we have $\sigma = \mathcal{N}$ and $M_x$ is a $\mathcal{N}$-class for every $x \in Y$. Then we have $M_x = (a)_{\mathcal{N}}$ for any $a \in M_x$. $T$ is the union of all the $\mathcal{N}$-classes which are greater than $(a)_{\mathcal{N}}$. This is exactly the set $\cup\{(b)_{\mathcal{N}} : (b)_{\mathcal{N}} \succeq (a)_{\mathcal{N}}\}$. By Theorem 2.7, $N(a) = \cup\{M_y : y \succeq x, y \in Y\}$.

Conversely, suppose that $(a,b) \in \sigma$ and $a \in M_x$; then we have $b \in M_x$. Since $N(a) = \cup\{M_y : y \succeq x, y \in Y\}$ for any $a \in M_x$, we now have $N(a) = N(b)$ and $(a,b) \in \mathcal{N}$, then $\sigma \subseteq \mathcal{N}$. Since $\mathcal{N}$ is the smallest complete semilattice congruence on $M$, we have $\sigma = \mathcal{N}$; thus, $\sigma$ is the smallest complete semilattice congruence on $M$. □

The following proposition is an immediate corollary of the above theorem.

**Corollary 2.14.** *Let $\sigma$ be a complete semilattice congruence on an ordered $\Gamma$-semigroup $M$ and $Y$ the semilattice $M/\sigma$. For any $x \in Y$ and $a \in M_x, N(a) \subseteq \cup\{M_y : y \succeq x, y \in Y\}$.*

**Corollary 2.15.** *Let $\sigma$ be a complete semilattice congruence on an ordered $\Gamma$-semigroup $M$ and $Y$ the semilattice $M/\sigma$. If there exists a maximal element $x \in Y$ such that $M_x$ has no proper sub-$\Gamma$-semigroups, we have $N(a) = M_x$ for any $a \in M_x$.*

*Proof.* Assume that $x$ is a maximal element in $Y$. By Corollary 2.14, we have $N(a) \subseteq \cup\{M_y : y \succeq x, y \in Y\} = M_x$ for any $a \in M_x$. This shows that $N(a)$ is a sub-$\Gamma$-semigroup of $M_x$. Since $a \in M_x$, we have that $M_x$ is nonempty. Since $M_x$ has no proper sub-$\Gamma$-semigroups, we have $N(a) = M_x$. $\square$

**Example 2.16.** Let $M = \{a, b, c, d, e, f, g\}$ and $\Gamma = \{\gamma\}$ with the multiplication defined by

$$x\gamma y = \begin{cases} a & \text{if } x = a, y \in \{a, c, e, g\} \text{ or } x = c, y \in \{a, e, g\} \\ & \text{or } x = e, y \in \{a, c\} \\ & \text{or } x = g, y \in \{a, c\} \\ c & \text{if } x = y = c \\ d & \text{if } x = d, y \in \{c, d\} \text{ or } x = c, y = d \\ e & \text{if } x = e, y \in \{e, g\} \text{ or } x = g, y = e \\ f & \text{if } x = e, y = f \text{ or } x = f, y \in \{e, f, g\} \text{ or } x = g, y = f \\ g & \text{if } x = g, y = g \\ b & \text{otherwise.} \end{cases}$$

If we define a relation $\leq$ on $M$ as follows:

$$\begin{aligned} \leq := 1_M \bigcup \{&(a, c), (a, e), (a, g), (b, a), (b, c), (b, d), \\ &(b, e), (b, f), (b, g), (d, c), (e, g), (f, e), (f, g)\}, \end{aligned}$$

then it can be easily verified that $M$ is an ordered $\Gamma$-semigroup. We now define a complete semilattice congruence $\sigma$ on $M$ as follows:

$$\sigma := 1_M \bigcup \{(a, b), (b, a), (c, d), (d, c), (e, f), (f, e)\}.$$

Then $M/\sigma = \{\{a,b\},\{c,d\},\{e,f\},\{g\}\}$. If we denote $M_x = \{a,b\}$, $M_y = \{c,d\}$, $M_z = \{e,f\}$, $M_t = \{g\}$, the order on semilattice $Y = M/\sigma$ is shown below.

$$\begin{aligned}\preceq = \{&(x,x),(y,y),(z,z),(t,t)\\&(x,y),(x,z),(x,t),(z,t)\}\end{aligned}$$

It can be easily seen that $M$ is a semilattice. By Theorem 2.11, we can easily see that $N(a) = \{a,c,e,g\}$, $N(b) = \{a,b,c,d,e,f,g\}$, $N(c) = \{c\}$, $N(d) = \{c,d\}$, $N(e) = \{e,g\}$, $N(f) = \{e,f,g\}$, $N(g) = \{g\}$ and $\mathcal{N} = 1_M$. By Corollary 2.14, we can see that $N(a) \subseteq M_x \cup M_y \cup M_z \cup M_t$, $N(b) \subseteq M_x \cup M_y \cup M_z \cup M_t$, $N(c) \subseteq M_y$, $N(d) \subseteq M_y$, $N(e) \subseteq M_z \cup M_t$, $N(f) \subseteq M_z \cup M_t$ and $N(g) = M_t$ by Corollary 2.15.

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## REFERENCES

**1.** Y.L. Cao and X.Z. Xu, *Nil-extensions of simple po-semigroups*, Commun. Algebra **28** (2000), 2477–2496.

**2.** Z.L. Gao and K.P. Shum, *On cyclic commutative po-semigroups*, PU.M.A. **8** (1997), 261–273.

**3.** K. Hila, *On prime, weakly prime ideals and prime radical on ordered $\Gamma$-semigroups*, submitted.

**4.** K. Hila and E. Pisha, *On bi-ideals in ordered $\Gamma$-semigroups* II, submitted.

**5.** ———, *On lattice-ordered Rees matrix $\Gamma$-semigroups*, An. Stiint Univ. Al. I. Cuza Iasi, Mat., to appear.

**6.** J.M. Howie, *An introduction to semigroup theory*, Academic Press, London 1976.

**7.** ———, *Fundamentals of semigroup theory*, Oxford University Press, New York 1995.

**8.** N. Kehayopulu, *On weakly commutative poe-semigroups*, Semigroup Forum **34** (1987), 367–370.

**9.** ———, *On filtres generated in poe-semigroups*, Math. Japon. **35** (1990), 789–796.

**10.** ———, *On weakly prime ideals of ordered semigroups*, Math. Japon. **35** (1990), 1051–1056.

**11.** N. Kehayopulu, *On left regular ordered semigroups*, Math. Japon. **35** (1990), 1057–1060.

**12.** ———, *Remark on ordered semigroups*, Math. Japon. **36** (1990), 1061–1063.

**13.** N. Kehayopulu, P. Kiriakuli, S. Hanumantha Rao and P. Lakshmi, *On weakly commutative poe-semigroups*, Semigroup Forum **41** (1990), 373–376.

**14.** N. Kehayopulu and M. Tsingelis, *Remark on ordered semigroups*, in *Decompositions and homomorphic mappings of semigroups*, E.S. Ljapin, ed., Interuniversitary collection of scientific works, St. Petersburg: "Obrazovanie" (ISBN 5-233-00033-4), (1992), pages 50–55.

**15.** Y.I. Kwon, *The filters of ordered* $\Gamma$*-semigroups*, J. Korea Soc. Math. Education: The Pure and Applied Mathematics **4** (1997), 131–135.

**16.** ———, *The left regular ordered* $\Gamma$*-semigroups*, Far East J. Math. Sci. **6** (1998), 613–618.

**17.** X.M. Ren, J.Z. Yan and K.P. Shum, *On principal filters of po-semigroups*, PU.M.A. **16** (2005), 37–42.

**18.** N.K. Saha, *On* $\Gamma$*-semigroup*-II, Bull. Calif. Math. Soc. **79** (1987), 331–335.

**19.** M.K. Sen and N.K. Saha, *On* $\Gamma$*-semigroup*-I, Bull. Calif. Math. Soc. **78** (1986), 180–186.

**20.** J.Z. Yan, X.M. Ren and S. Ma, *The structure of principal filters on po-semigroups*, Scient. Magna **2** (2006), 50–54.

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