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ASYMPTOTIC DISTANCE AND ITS APPLICATION

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ABSTRACT. We define the concept of asymptotic distance of two disjoint sets of positive integers. Some properties of this concept are investigated and, as an application, a new criterion for an infinite series to be a Liouville number is derived.

1. Introduction. In this paper we introduce the concept of asymptotic distance between two disjoint infinite sets of positive integers. We will relate it to asymptotic density, and we will use this to obtain some results about Liouville numbers. First let us recall some basic definitions.

For a set $A \subset \mathbf{N}$ and $n \in \mathbf{N}$, denote $A(n)$ as the number of elements of the set $A \cap \{1, 2, \dots, n\}$. The *lower* and *upper asymptotic densities* of the set A are defined and denoted by

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{A(n)}{n} \quad \text{and} \quad \overline{d}(A) = \limsup_{n\to\infty} \frac{A(n)}{n},$$

respectively. In the case when these values are equal, the common value is called the *asymptotic density* of set A and denoted by $d(A)$.

For two infinite disjoint sets $X, Y \subset \mathbf{N}$, set

$$D_1(X, Y) = \liminf_{x\in X, y\in Y, x<y} \left(\frac{y}{x} - 1\right).$$

Definition 1.1. Let $A, B \subset \mathbf{N}$ be disjoint infinite sets. We call

$$D(A, B) = \min\{D_1(A, B), D_1(B, A)\}$$

the asymptotic distance of the sets A and B.

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Let α be a real number. If for every positive real number r there exist integers p and q such that $0 < |\alpha - p/q| < 1/q^r$, then the number α is called *Liouville*.

A survey of results concerning Liouville numbers can be found in the book by Nishioka [**4**]. Also the results of Petruska [**5**] establish several interesting results about *strong Liouville numbers*, first defined by Erdös in [**1**]. In 1975 Erdös [**2**] proved a criterion for Liouville numbers.

Theorem 1.1 (Erdős). *For a strictly increasing sequence $\{a_n\}_{n=1}^{\infty}$, suppose that*

$$\limsup_{n\to\infty} a_n^{1/t^n} = \infty$$

for every $t > 0$, and that there exist $\varepsilon_0 > 0$ and $n_0(\varepsilon_0) \in \mathbf{N}$ such that

$$a_n > n^{1+\varepsilon_0}$$

if $n > n_0(\varepsilon_0)$. Then

$$\alpha = \sum_{n=1}^{\infty} \frac{1}{a_n}$$

is a Liouville number.

Recently, Hančl [**3**] introduced the concept of Liouville sequences and improved Theorem 1.1. Other related results can be found in [**6**].

In the sequel, by $[x]$ and $\{x\}$ we will denote the integer and fractional parts of real number x, respectively.

2. Main results. We will start by proving some upper bounds of asymptotic distances of two sets.

Lemma 2.1. *Let $A \subset \mathbf{N}$, and let $B = \mathbf{N} - A = \cup_{n=1}^{\infty}(c_n, d_n] \cap \mathbf{N}$ with $0 < c_n < d_n < c_{n+1}$ for every $n \in \mathbf{N}$. Then for every set $X \subset \mathbf{N}$ disjoint from A we have*

$$D(A, X) \le \limsup_{n\to\infty} \sqrt{\frac{d_n}{c_n}} - 1. \tag{1}$$

Proof. Set $I = \{n \in \mathbf{N};\ (c_n, d_n] \cap X \neq \varnothing\}$. Then $X \subset \cup_{n\in I}(c_n, d_n] \cap \mathbf{N}$. Choose an arbitrary $x \in X$. Then there is a unique $n = n(x) \in \mathbf{N}$ such that $x \in (c_n, d_n]$. Consequently, for every $a_1, a_2 \in A$ such that $a_1 < x < a_2$ we have

$$\frac{x}{a_1} \geq \frac{x}{c_n} \quad \text{and} \quad \frac{a_2}{x} > \frac{d_n}{x}.$$

Set

$$m_n = \min\left\{\frac{x}{c_n}, \frac{d_n}{x};\ x \in (c_n, d_n] \cap B\right\}, \quad n \in I.$$

Suppose, for example, that $m_n = x_n/c_n$ for a particular $x_n \in (c_n, d_n] \cap B$. Then $m_n = x_n/c_n \leq d_n/x_n$, i.e., $x_n \leq \sqrt{c_n d_n}$ and, consequently,

$$m_n = \frac{x_n}{c_n} \leq \sqrt{\frac{d_n}{c_n}}. \tag{2}$$

Notice that the assumption $m_n = d_n/y_n$ for a particular $y_n \in (c_n, d_n] \cap B$ leads to the same inequality. Using (2) we have

$$\begin{aligned} D(A, X) &= \min\{D_1(A, X), D_1(X, A)\} \\ &= \liminf_{n\in I}(m_n - 1) \leq \limsup_{n\to\infty} \sqrt{\frac{d_n}{c_n}} - 1. \qquad \square \end{aligned}$$

The next example shows that the upper bound in the previous lemma is the best possible.

Example 2.1. Let $A \subset \mathbf{N}$ be arbitrary. Write $B = \mathbf{N} - A = \cup_{n=1}^{\infty}(c_n, d_n] \cap \mathbf{N}$ with $0 < c_n < d_n < c_{n+1}$. Also, let $\{n_k\}_{k=1}^{\infty}$ be some increasing sequence of positive integers such that

$$\lim_{k\to\infty} \frac{d_{n_k}}{c_{n_k}} = \limsup_{n\to\infty} \frac{d_n}{c_n},$$

and choose $X = \{[\sqrt{c_{n_k} d_{n_k}}];\ k \in \mathbf{N}\}$. Then, from the proof of the previous lemma, it can be seen that

$$D(A, X) = \lim_{k\to\infty} \sqrt{\frac{d_{n_k}}{c_{n_k}}} - 1 = \limsup_{n\to\infty} \sqrt{\frac{d_n}{c_n}} - 1.$$

Sometimes it is useful to estimate the asymptotic distance of two sets in terms of asymptotic densities. We need the following lemma.

Lemma 2.2. *Let* $B = \cup_{n=1}^{\infty}(c_n, d_n] \cap \mathbf{N}$ *with* $0 < c_n < d_n < c_{n+1}$ *for every* $n \in \mathbf{N}$. *If* $\overline{d}(B) < 1$, *then the inequality*

$$\limsup_{k\to\infty} \frac{d_k - c_k}{c_k} \leq \frac{\overline{d}(B) - \underline{d}(B)}{1 - \overline{d}(B)} \tag{3}$$

holds.

Proof. From the definition of asymptotic density we obtain that, for every $\varepsilon > 0$, there exists an N such that, for every $k > N$,

$$\underline{d}(B) - \varepsilon < \frac{B(c_k)}{c_k} \tag{4}$$

and

$$\frac{B(d_k)}{d_k} < \overline{d}(B) + \varepsilon. \tag{5}$$

We also have

$$B(d_k) = B(c_k) + d_k - c_k. \tag{6}$$

Inequality (5) and equation (6) yield

$$\frac{B(c_k) + d_k - c_k}{d_k - c_k + c_k} < \overline{d}(B) + \varepsilon.$$

Hence,

$$\frac{B(c_k)/c_k + (d_k - c_k)/c_k}{((d_k - c_k)/c_k) + 1} < \overline{d}(B) + \varepsilon,$$

and therefore,

$$\frac{B(c_k)}{c_k} + \frac{d_k - c_k}{c_k} < (\overline{d}(B) + \varepsilon)\left(\frac{d_k - c_k}{c_k} + 1\right).$$

From this and (4) we obtain

$$\underline{d}(B) - \varepsilon + \frac{d_k - c_k}{c_k} < (\overline{d}(B) + \varepsilon)\left(\frac{d_k - c_k}{c_k} + 1\right).$$

Thus,

$$\frac{d_k - c_k}{c_k}\big(1 - \overline{d}(B) - \varepsilon\big) < \overline{d}(B) - \underline{d}(B) + 2\varepsilon$$

and (3) follows. The proof of Lemma 2.2 is complete. □

The following theorem provides the upper estimate of the asymptotic distance in terms of asymptotic densities.

Theorem 2.1. *Let $A \subset \mathbf{N}$ be an infinite set with $\underline{d}(A) > 0$. Then, for every infinite $X \subset \mathbf{N}$ disjoint from A, we have*

$$D(A, X) \leq \frac{\overline{d}(A) - \underline{d}(A)}{2\underline{d}(A)}.$$

Proof. Set $B = \mathbf{N} - A$, and write B in the form $B = \cup_{n=1}^{\infty}(c_n, d_n] \cap \mathbf{N}$ with $0 < c_n < d_n < c_{n+1}$. Then $\underline{d}(B) = 1 - \overline{d}(A)$ and $\overline{d}(B) = 1 - \underline{d}(A) < 1$ and, using Lemmas 2.1 and 2.2, we have

$$\begin{aligned} D(A, X) &\leq \limsup_{n\to\infty} \frac{\sqrt{d_n} - \sqrt{c_n}}{\sqrt{c_n}} \leq \limsup_{n\to\infty} \frac{\sqrt{d_n} - \sqrt{c_n}}{\sqrt{c_n}} \frac{\sqrt{d_n} + \sqrt{c_n}}{2\sqrt{c_n}} \\ &= \limsup_{n\to\infty} \frac{d_n - c_n}{2c_n} \leq \frac{\overline{d}(A) - \underline{d}(A)}{2\underline{d}(A)}. \qquad \square \end{aligned}$$

The next corollary is a straightforward consequence of the previous theorem.

Corollary 2.1. *Let $A \subset \mathbf{N}$ be a set of positive asymptotic density. Then for every $X \subset \mathbf{N}$ disjoint from A, we have $D(A, X) = 0$.*

The next example shows that $D(A, X)$ can be any nonnegative value, including ∞, in the case when $\underline{d}(A) = 0$.

Example 2.2. Define $A = \{2^{2^{2n}};\ n \in \mathbf{N}\}$ and $X_\infty = \{2^{2^{2n+1}};\ n \in \mathbf{N}\}$. Also define

$$X_q = \left\{ \left[2^{2^{2n}}(1+q)\right] + 1;\ n \in \mathbf{N} \right\} \text{ for every } q \in [0,\infty).$$

Then, using ideas similar to those in Lemma 2.1 and Example 2.1, one can show that $D(A, X_q) = q$ holds for every $q \in [0,\infty]$.

Now we are going to apply the concept of asymptotic distance to prove that the sum of a certain series are Liouville numbers.

Theorem 2.2. *Let $\{a_n\}_{n=1}^\infty$ be a nondecreasing sequence of positive integers with*

$$\gamma = \liminf_{n\to\infty} \frac{1}{n} \log_2 \log_2 a_n > 0. \tag{7}$$

Assume that there are numbers $\alpha > \beta \geq \gamma$ such that the sets

$$\begin{aligned} A &= \left\{ n \in \mathbf{N};\ \frac{1}{n} \log_2 \log_2 a_n > \alpha \right\}, \\ B &= \left\{ n \in \mathbf{N};\ \frac{1}{n} \log_2 \log_2 a_n < \beta \right\} \end{aligned} \tag{8}$$

are infinite and $D(A,B) = 0$ holds. Then the sum of the series $\sum_{n=1}^\infty 1/a_n$ is a Liouville number.

Proof. Set $x = \sum_{n=1}^\infty 1/a_n$. We prove that, for every positive integer K, there are positive integers p and q such that $|x - (p/q)| < 1/q^K$. To prove this we find a positive integer M such that

$$\sum_{n=M+1}^\infty \frac{1}{a_n} < \frac{1}{(\prod_{j=1}^M a_j)^K}. \tag{9}$$

Let N be a sufficiently large positive integer. First of all, we estimate $\sum_{n=N+1}^\infty 1/a_n$. From (7) we obtain that $a_k > 2^k$ for every $k > N$. This fact and the fact that $\{a_n\}_{n=1}^\infty$ is a nondecreasing sequence of positive integers imply that

(10)
$$\begin{aligned}\sum_{n=N+1}^{\infty}\frac{1}{a_n} &= \sum_{N<n\le\log_2 a_{N+1}}\frac{1}{a_n}+\sum_{\log_2 a_{N+1}<n}\frac{1}{a_n}\\ &\le \frac{\log_2 a_{N+1}}{a_{N+1}}+\sum_{\log_2 a_{N+1}<n}\frac{1}{a_n}\le\frac{\log_2 a_{N+1}}{a_{N+1}}+\sum_{\log_2 a_{N+1}<n}\frac{1}{2^n}\\ &\le \frac{\log_2 a_{N+1}}{a_{N+1}}+\frac{2}{2^{\log_2 a_{N+1}}}\le\frac{2\log_2 a_{N+1}}{a_{N+1}}.\end{aligned}$$

Set $x_n = 1/n\log_2\log_2 a_n$ for $n \in \mathbf{N}$ and notice that, as $\{a_n\}_{n=1}^{\infty}$ is nondecreasing, the inequality

$$\frac{x_n}{x_{n+1}} \le \frac{n+1}{n} \tag{11}$$

holds for all $n \in \mathbf{N}$. Then, for every pair $m \in A$, $n \in B$, with $m < n$ we have
$$\frac{\alpha}{\beta} < \frac{x_m}{x_n} \le \frac{n}{m}$$
which yields
$$D_1(A,B) = \liminf_{x\in A, y\in B, x<y}\left(\frac{y}{x}-1\right) \ge \frac{\alpha}{\beta}-1>0.$$
That is why $D(A,B) = D_1(B,A) = 0$. Consequently, for every $\varepsilon > 0$ and positive integer N_0, there are integers $N_2 > N_1 > N_0$ with $N_2 \in A$, $N_1 \in B$ such that

$$\frac{N_2 - N_1}{N_1} < \varepsilon. \tag{12}$$

Let us define the finite sequence $\{b_n\}_{n=N_1}^{N_2}$ as follows.
$$b_t = \begin{cases} a_t^t & \text{if } t = N_1\\ a_t^{1/((K+2)^{t-N_1})} & \text{if } t = N_1+1, N_1+2, \dots, N_2.\end{cases}$$
Let T denote the integer for which $b_T = \max_{t=N_1,N_1+1,\dots,N_2} b_t$. Let us note that $b_T \ne b_{N_1}$, otherwise from (8) and (12) we obtain that
$$\begin{aligned}2^{N_1 2^{\beta N_1}} &> a_{N_1}^{N_1} = b_{N_1} \ge b_{N_2} = a_{N_2}^{1/((K+2)^{N_2-N_1})}\\ &> 2^{2^{\alpha N_2}/((K+2)^{N_2-N_1})}\\ &> 2^{2^{\alpha N_1}/((K+2)^{N_1\varepsilon})}.\end{aligned}$$

Hence, applying $\log_2$ twice to the above inequality, we get

$$\beta N_1 + \log_2 N_1 > \alpha N_1 - N_1 \varepsilon \log_2 (K+2),$$

and this is a contradiction for a sufficiently large N_1 and a sufficiently small ε. Thus, $b_T \geq b_t$ for every $t = N_1, N_1+1, \ldots, T-1$. From this and from the fact that the sequence $\{a_n\}_{n=1}^{\infty}$ is nondecreasing, we obtain that

$$\begin{aligned} a_T &\geq \left(\max_{t=N_1, N_1+1, \ldots, T-1} b_t \right)^{(K+2)^{T-N_1}} \\ &\geq \left(\max_{t=N_1, N_1+1, \ldots, T-1} b_t \right)^{(K+1)\sum_{j=0}^{T-N_1-1}(K+2)^j} \\ &\geq \left(\prod_{t=N_1}^{T-1} b_t^{(K+2)^{t-N_1}} \right)^{K+1} \geq \left(\prod_{t=1}^{T-1} a_t \right)^{K+1}. \end{aligned}$$

This implies that

$$a_T = a_T^{K/(K+1)+1/(K+1)} \geq a_T^{1/(K+1)} \left(\prod_{t=1}^{T-1} a_t \right)^K. \tag{13}$$

From (10) and (13) we obtain that

$$\sum_{n=T}^{\infty} \frac{1}{a_n} \leq \frac{2\log_2 a_T}{a_T} \leq \frac{2\log_2 a_T}{a_T^{1/(K+1)} (\prod_{t=1}^{T-1} a_t)^K} < \frac{1}{(\prod_{t=1}^{T-1} a_t)^K}$$

for sufficiently large T. Thus, (9) follows for $M = T-1$. The proof of Theorem 2.2 is complete. □

Example 2.3. As an immediate consequence of Theorem 2.2, we obtain that the number

$$H = \sum_{n=1}^{\infty} \left[2^{2^{(1/9(3+\sum_{j=3}^{n}(([\log j]-[\log(j-[\log j])])/[\log j]-(1/j))))n}} \right]^{-1}$$

is Liouville.

Proof. Let us put

$$a_n = 2^{2^{(1/9(3+\sum_{j=3}^{n}(([\log j]-[\log(j-[\log j])])/[\log j]-(1/j))))n}}.$$

Then we have

$$\gamma_1 = \liminf_{n\to\infty} \frac{1}{n}\log_2\log_2 a_n < \limsup_{n\to\infty} \frac{1}{n}\log_2\log_2 a_n = \alpha_1.$$

Set $\beta = 1/4(3\gamma_1 + \alpha_1)$ and $\alpha = 1/4(\gamma_1 + 3\alpha_1)$. Now we have $0 < \gamma_1 < \beta < \alpha < \alpha_1$. If we define A and B as in Theorem 2.2, then we obtain that

$$D(A,B) \le \lim_{n\to\infty} \frac{\log n}{n} = 0.$$

So all the assumptions of Theorem 2.2 are fulfilled. Thus, the number H is Liouville. □

Open problem. Is the number

$$G = \sum_{n=1}^{\infty}\left[2^{2^{(1/9(3+\log n-\sum_{j=1}^{n}(1/j)))n}}\right]^{-1}$$

Liouville?

Corollary 2.2. *Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} c_n = \infty$ and $0 < c_n < 1/n$ holds for every $n \in \mathbf{N}$. Then the number*

$$\sum_{n=1}^{\infty}\left[2^{2^{1/3(2-\{\sum_{j=1}^{n} c_j\})n}}\right]^{-1}$$

is Liouville.

Proof. One can easily show that

$$\liminf_{n\to\infty} \frac{1}{n}\log_2\log_2 a_n = \frac{1}{3} \quad \text{and} \quad \limsup_{n\to\infty} \frac{1}{n}\log_2\log_2 a_n = \frac{2}{3}.$$

Choose $\beta < \alpha$ any numbers in the interval $(1/3, 2/3)$ to fulfill the assumptions of Theorem 2.2. $\square$

The next two examples follow from Corollary 2.2.

Example 2.4. The number

$$\sum_{n=1}^{\infty} \left[2^{2^{1/3(2-\left\{\sum_{j=1}^{n}(1/p_j)\right\})n}} \right]^{-1},$$

where p_j is the jth prime, is Liouville.

Example 2.5. The number

$$\sum_{n=1}^{\infty} \left[2^{2^{1/3(2-\{\log n\})n}} \right]^{-1}$$

is Liouville.

Example 2.6. Another Liouville number can be defined as follows. Let $x_1 > 0$ be arbitrary. Let us define the sequence $\{x_n\}_{n=1}^{\infty}$ as follows

$$x_2 = \frac{x_1}{2},\ x_3 = x_1,\ x_4 = \frac{3}{4}x_1,\ x_5 = \frac{3}{5}x_1,\ x_6 = \frac{1}{2}x_1,\ x_7 = x_1, \dots$$

such that $x_k = x_1$ for all $k = k_m = 2^m - 1$, $m = 1, 2, 3, \dots$ and

$$x_{k_m+l} = \frac{k_m}{k_m + l} x_1 \quad l = 0, 1, 2, \dots k_m.$$

Define $a_n = [2^{2^{x_n n}}]$. Then the number $\sum_{n=1}^{\infty} 1/a_n$ is Liouville.

Proof. Notice that $a_n = 2^{2^{x_n n}} = 2^{2^{x_1 k_m}}$ is constant for all $n \in [k_m, k_{m+1}) \cap \mathbf{N}$ and

$$\gamma = \liminf_{n\to\infty} \frac{1}{n} \log_2 \log_2 a_n = \liminf_{n\to\infty} x_n = \frac{x_1}{2}.$$

One can easily prove that all the assumptions of Theorem 2.2 hold for any $1/2x_1 < \beta < \alpha < x_1$. Thus, the sum of the series $\sum_{n=1}^{\infty} 1/a_n$ is a Liouville number. $\square$

Remark 2.1. Denote

$$H(x) = \sum_{n=1}^{\infty} \left[2^{2^{(x(3+\sum_{j=3}^{n}(([\log j]-[\log(j-[\log j])])/[\log j]-(1/j))))n}} \right]^{-1}.$$

Then, for all $x \in \mathbf{R}^+$, we obtain that the number $H(x)$ is Liouville. To prove this fact, follow the proof of Example 2.3. So the function $H(x)$ maps $\mathbf{R}^+$ to the set with the zero Lebesgue measure. We find similar phenomena in Corollary 2.2 and Example 2.6.

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