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TRANSFORMATION OF SPECTRA OF GRAPH LAPLACIANS

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ABSTRACT. We study how the spectrum of a graph Laplacian is transformed under two types of graph transformations: 1) replacing a graph G by its edge graph G_E; 2) edge substitution, where each edge of G is replaced by a specified graph H, yielding a graph denoted G_H. Since we allow a rather broad definition of what constitutes a Laplacian on a graph, part of the problem is to define a Laplacian on the new graphs G_E and G_H that is naturally related to the original Laplacian and such that the spectra are closely related. Our work extends results of Shirai [**11**] on specific Laplacians on regular graphs.

1. Introduction. How does the spectrum of a graph Laplacian transform when you transform the graph? This is a natural question that we investigate for two types of graph transformations: 1) the passage from a graph to its edge graph; and 2) edge substitution, replacing each edge in a graph by a specified graph. We adopt the point of view, promoted by Colin de Verdière [**2**], that there are many different Laplacians associated to a single graph. Suppose G is a graph with vertices V and edges E. A *weight* on G is an assignment of positive values to the elements of V and E. We write μ_x for the weight of $x \in V$ and view μ_x as a measure on V. We write $c(x,y)$ for the weight of the edge $e(x,y) \in E$ joining vertices x and y (we also write $x \sim y$ to indicate that the vertices are joined by an edge), and regard $c(x,y)$ as a conductance whose reciprocal $r(x,y) = 1/c(x,y)$ is a resistance. Thus, the edge weights allow us to imagine the graph as an electric network where the edges are resistors joining the vertices. The edge weights give rise to a bilinear form called *energy*:

$$\mathcal{E}(u,v) = \sum_{x \sim y} c(x,y)(u(x)-u(y))(v(x)-v(y)) \tag{1.1}$$

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for u and v functions on V, and we write $\mathcal{E}(u) = \mathcal{E}(u, u)$ for the associated quadratic form. (In case V is infinite, $\mathcal{E}(u)$ is well-defined for all functions, but may take on the value $+\infty$, while $\mathcal{E}(u, v)$ is well-defined and finite if both u and v have finite energy.) The Laplacian associated to the weight is defined by

$$\mathcal{E}(u, v) = -\sum_V \Delta u(x) v(x) \mu_x, \tag{1.2}$$

which easily yields the pointwise formula

$$\Delta u(x) = \frac{1}{\mu_x} \sum_{y \sim x} c(x, y)(u(y) - u(x)). \tag{1.3}$$

Note that the Laplacian is unchanged if we multiply all weights by a fixed constant. We say that the weight (and the associated Laplacian) is *consistent* if

$$\mu_x = \sum_{y \sim x} c(x, y). \tag{1.4}$$

In that case the formula for the Laplacian simplifies to

$$\Delta u(x) = \left(\sum_{y \sim x} \frac{c(x, y)}{\mu_x} u(y) \right) - u(x). \tag{1.5}$$

In this work we always begin with a consistent Laplacian for our original graph, but we do not always end up with a consistent Laplacian on the edge graph. The reader should keep in mind that there are situations in which one needs to work with Laplacians that do not satisfy the consistency condition. Our work is a continuation of work of Shirai [**11**] that studies these problems for a single Laplacian on a regular graph, and only studies one kind of edge substitution.

Let us assume for the moment that the graph is both finite and connected. Then Δ is a self-adjoint operator on the finite-dimensional inner product space $L^2(G) = L^2(V, \mu)$ and so has a complete set of real-valued eigenfunctions $\{u_j\}$ with eigenvalues $\{\lambda_j\}$, by convention

$$-\Delta u_j = \lambda_j u_j, \quad j = 1, 2, \dots, \# V, \tag{1.6}$$

so that λ_j are nonnegative. We may arrange the eigenvalues in nondecreasing order, in which case $\lambda_1 = 0$ (with u_1 constant), and all the other eigenvalues are strictly positive. We call $\{\lambda_j\}$ the *spectrum* of the Laplacian, but we note that information about the eigenfunctions should also be considered as part of the spectral data. In case of multiplicity, the individual eigenvalue is repeated in the spectrum $\{\lambda_j\}$, and there is no canonical choice of eigenfunctions in an eigenspace. Without multiplicity, it is often convenient to normalize the eigenfunctions to have $\|u_j\|_{L^2} = 1$, but this still only determines the eigenfunction up to a multiple of ± 1. In the case of a consistent Laplacian it is easy to see that the eigenvalues satisfy the inequality

$$0 \le \lambda_j \le 2 \tag{1.7}$$

(this follows by examining what happens at a point where $u_j(x)$ assumes its maximum or minimum value), and $\lambda_j = 2$ is possible if and only if G is bipartite, in which case the multiplicity of 2 is one, with u_j assuming values $+1$ on one set of vertices and -1 on the other set in a 2-coloring of the graph.

In Section 2 we consider the edge graph G_E, whose vertices V_E are the edges E of G, and the edge relation $e \sim e'$ holds exactly when e and e' have a vertex in common, so $e(x,y) \sim e(x,z)$. Given a weight on G, how do we associate a weight on G_E? Since the weight on G assigns weights $c(x,y)$ to the edge $e(x,y)$, it is natural to take a weight on G_E with

$$\mu_{e(x,y)} = c(x,y). \tag{1.8}$$

It is less clear how to assign a conductance $c(e(x,y), e(x,z))$ to edges in G_E. We will find it convenient to take

$$c(e(x,y), e(x,z)) = \frac{a c(x,y) c(x,z)}{\mu_x} \tag{1.9}$$

for some positive constant a. With this choice of weight we have a Laplacian Δ_E on G_E whose spectrum is easy to relate to the spectrum of Δ, at least in the case that the original weight is consistent. We will show that the ordered spectrum of Δ_E is

$$\{a\lambda_1, a\lambda_2, \dots, a\lambda_{\#V}, 2a, \dots, 2a\} \tag{1.10}$$

if G is not bipartite, and

$$\{a\lambda_1, a\lambda_2, \ldots, a\lambda_{\#V-1}, 2a, \ldots, 2a\} \tag{1.11}$$

if G is bipartite. The multiplicity of $2a$ is $\#E - \#V$ in the first case and $\#E - \#V + 1$ in the second case, so the total number of eigenvalues is $\#E$. Moreover, the eigenfunction associated with $a\lambda_j$ with $\lambda_j \neq 2$ on G_E is Su_j, for the sum operator

$$Su(e(x,y)) = u(x) + u(y) \tag{1.12}$$

mapping functions on V to functions on E, and the eigenspace associated to $2a$ is the orthogonal complement of the image of S. It is not always the case that the Laplacian Δ_E is consistent, although in some cases, for example if G is k-regular with all conductances equal, this can be achieved by the appropriate choice of the constant a. We do not mean to suggest that the weight on G_E given by (1.8) and (1.9) is the only, or even the best, choice.

In Section 3 we consider the construction of a new graph G_H via edge substitution, where H is a fixed graph with $N+2$ vertices V_H, two of which, denoted q_0 and q_1, are considered as boundary points. We will assume that there is a graph isometry τ of H that interchanges q_0 and q_1. We will fix a consistent weight for H that is invariant under τ. We write $c_H(g,h)$ for the conductance of an edge $e(g,h)$ in E_H, and $\nu_h = \sum_{g \sim h} c_H(g,h)$ for the measure on V_H. Edge substitution replaces each edge $e(x,y)$ in the original graph with a copy of H, identifying x with q_0 and y with q_1. Because of the symmetry, it doesn't matter if we interchange x and y. The new graph G_H has two types of vertices, the *old vertices* V_{old} being just V, and the *new vertices* V_{new}, indexed $v(x,y,h)$ where $e(x,y)$ is an edge of G and $h \in V_H \setminus \{q_0, q_1\}$. The edges of G_H are exactly the edges E_H in each copy of H, so

$$\begin{cases} v(x,y,h) \sim v(x,y,g) & \text{if } g \sim h \text{ in } H, \\ v(x,y,h) \sim x & \text{if } h \sim q_0 \text{ and} \\ v(x,y,h) \sim y & \text{if } h \sim q_1. \end{cases} \tag{1.13}$$

We create a consistent weight on G_H by assigning conductances to edges multiplicatively, so

$$\begin{cases} c(v(x,y,h), v(x,y,g)) = c(x,y)c_H(g,h) \\ c(v(x,y,h), x) = c(x,y)c_H(q_0,h) \\ c(v(x,y,h), y) = c(x,y)c_H(q_1,h). \end{cases} \tag{1.14}$$

The consistency condition implies

$$\mu_{v(x,y,h)} = c(x,y)\nu_h \tag{1.15}$$

for the new vertices. In order to have the weight of the old vertices unchanged, we will assume

$$\nu_{q_0} = 1. \tag{1.16}$$

This condition can be achieved by multiplying all weights on H by a constant, and so it does not affect the Laplacian Δ_{G_H}. Note that at new vertices the Laplacian Δ_{G_H} agrees with the Laplacian Δ_H on the inserted copy of H. The simplest example of edge substitution is to choose H to be the 3-element graph with two edges (symmetry requires the conductance to be the same on both edges); here G_H inserts a vertex in the middle of each edge. This is discussed in Example 3.4.

It is clear that every eigenfunction on G_H with eigenvalue λ' restricts to a λ'-eigenfunction on the interior of each copy of H. The problem is to understand the λ'-eigenvalue equation at the old vertices. We will show that there are two distinct types of solutions. There are *new eigenvalues* corresponding to λ' in the Dirichlet spectrum of H (eigenfunctions vanishing on the boundary $\{q_0, q_1\}$). There are exactly N Dirichlet eigenvalues λ' (counting multiplicity), and these give rise to λ'-eigenfunctions on G_H that vanish on V_{old}. The number of such eigenfunctions depends on the nature of the Dirichlet eigenfunctions on H. If the eigenfunction on H is a joint Dirichlet-Neumann eigenfunction, then the eigenfunctions on each copy of H may be chosen independently, and the dimension of the λ'-eigenspace on G_H is $\#E$. Otherwise, there are $\#V$ linear constraints arising from the λ'-eigenvalue equation at the old vertices, so the dimension is at least $\#E - \#V$.

The other type of solution we will call a *bifurcated eigenvalue*. In this case the eigenfunction restricted to V_{old} is a λ-eigenfunction G, and the values λ and λ' are related by

$$\lambda = R(\lambda') \tag{1.17}$$

where R is a rational function. Thus, there are a finite number of solutions λ' for each λ. In the generic case (λ' is not a Dirichlet eigenvalue of H) there is a unique extension from V_{old} to V_{new} of a λ-eigenfunction on G to a λ'-eigenfunction on G_H. Between the two types of eigenvalues, we obtain a complete description of the spectrum of G_H.

In this paper we also consider the case of infinite graphs. Here the question of interest is to relate the spectral resolutions of the two Laplacians. In the case of the edge graph we give a complete description, while for edge substitution we are only able to offer a reasonable conjecture.

Graph Laplacians and their spectra have been studied extensively; see the books [**1, 2, 3**]. Aside from its intrinsic interest, this subject has applications to the study of Laplacians on fractals, as developed in the work of Kigami [**5**–**8**]. See [**12, 14**] for expository accounts. Other works that develop this connection include [**4, 9, 10, 13, 15**]. Indeed, [**13**] explicitly uses some of the result of [**11**]. We hope that some of the results of this paper will have applications to the study of Laplacians on fractals.

2. Edge graph. Let G be a finite connected graph with a consistent weight. We denote the associated Laplacian by Δ_G. Let G_E be the edge graph of G, with weight given by (1.8) and (1.9), and denote its Laplacian by Δ_E. To make the equations clearer, we use lower case letters for functions on G and upper case letters for functions on G_E. We denote the *sum operator* S from functions on G to functions on G_E by

$$Sf(e(x,y)) = f(x) + f(y). \tag{2.1}$$

We compute the adjoint operator S^* from the definition

$$\sum_V S^*F(x)g(x)\mu_x = \sum_E F(e(x,y)Sg(e(x,y))c(x,y). \tag{2.2}$$

Using (2.1), the right side of (2.2) becomes

$$\sum_x g(x) \sum_{y\sim x} F(e(x,y))c(x,y);$$

hence

$$S^*F(x) = \sum_{y\sim x} \frac{c(x,y)}{\mu_x} F(e(x,y)). \tag{2.3}$$

These operators intertwine $a\Delta_G$ and Δ_E.

Lemma 2.1. (a) $-\Delta_E S = -aS\Delta_G$,
(b) $-S^*\Delta_E = -a\Delta_G S^*$,
(c) $SS^* = (1/a)\Delta_E + 2I$,
(d) $f \in \ker S \Leftrightarrow -\Delta_G f = 2f$.

Proof. (a) From the definition of Δ_E and (1.8) and (1.9) we find

$$\begin{aligned} -\Delta_E F(e(x,y)) &= a \sum_{\substack{z\sim x\\ z\neq y}} \frac{c(x,z)}{\mu_x}(F(e(x,y) - F(e(x,z)) \\ &\quad + a \sum_{\substack{z'\sim y\\ z'\neq x}} \frac{c(y,z')}{\mu_y}(F(e(x,y) - F(e(y,z'))) \end{aligned} \tag{2.4}$$

for any function F on G_E. Note that we can drop the conditions $z \neq y$ and $z' \neq x$ in each sum because the last factor vanishes for the deleted value. When $F = Sf$, this yields

$$\begin{aligned} -\Delta_E Sf(e(x,y)) &= a \sum_{z\sim x} \frac{c(x,z)}{\mu_x}(f(y) - f(z)) \\ &\quad + a \sum_{z'\sim y} \frac{c(y,z')}{\mu_y}(f(x) - f(z')). \end{aligned} \tag{2.5}$$

Because the weight on G is consistent, the right side of (2.5) is equal to

$$af(y) - a\sum_{z\sim x} \frac{c(x,z)}{\mu_x} f(z) + af(x) - a\sum_{z'\sim y}\frac{c(y,z')}{\mu_y} f(z').$$

Rearranging terms, this is seen to be equal to

$$a(-\Delta_G f(x) - \Delta_G f(y)) = -aS\Delta_G f(e(x,y)).$$

(b) This follows from (a) by taking adjoints.

(c) By (2.1) and (2.3) we find

$$\begin{aligned} SS^*F(e(x,y)) &= S^*F(x) + S^*F(y) \\ &= \sum_{z\sim x} \frac{c(x,z)}{\mu_x} F(e(x,z)) \\ &\quad + \sum_{z'\sim y} \frac{c(y,z')}{\mu_y} F(e(y,z')) \end{aligned} \tag{2.6}$$

and the result follows from (2.4).

(d) Clearly $f \in \ker S$ if and only if $f(x) = -f(y)$ whenever $x \sim y$. This implies $-\Delta_G f = 2f$. Conversely, if f satisfies this 2-eigenvalue equation, then G must be bipartite and $f(x) = -f(y)$ if $x \sim y$. □

Theorem 2.2. *Let $\{\lambda_j\}$ denote the eigenvalues of Δ_G with eigenfunctions $\{u_j\}$. Then the spectrum of Δ_E consists of eigenvalues $a\lambda_j$ for all $\lambda_j < 2$, with the same multiplicity as for Δ_G, and eigenfunction Su_j, and the eigenvalue $2a$ with multiplicity $\#E - \#V + 1$ if G is bipartite and $\#E - \#V$ if G is not bipartite.*

Proof. From part (a) of the lemma

$$-\Delta_E Su_j = -aS\Delta_G u_j = a\lambda_j Su_j,$$

and by part (d) Su_j is not zero if $\lambda_j \neq 2$. Thus, Su_j is an $a\lambda_j$-eigenfunction of Δ_E if $\lambda_j \neq 2$. Conversely, suppose U is a λ-eigenfunction of Δ_E with $\lambda \neq 2a$. Then, by part (b) of the lemma

$$-\Delta_G S^*U = -a^{-1}S^*\Delta_E U = a^{-1}\lambda S^*U,$$

and by part (c) S^*U is not zero. Thus, S^*U is an $a^{-1}\lambda$-eigenfunction of Δ_G, so it must be a multiple of u_j for some j with $\lambda_j \neq 2$. Then U is a multiple of Su_j. This shows that the multiplicities are the same, and there are no eigenvalues other than $\{a\lambda_j\}$ and $2a$. A dimension count gives the claimed multiplicity for the eigenvalue $2a$. □

Note that the multiplicity of $2a$ might be zero, in which case $2a$ is not an eigenvalue of Δ_E. This is always the case if G is a tree, for then G is bipartite and $\#E = \#V - 1$.

It follows from the proof that a function is in the image of S if and only if it is orthogonal to the $2a$-eigenspace.

Next we consider the normalization of the eigenfunctions. Suppose u_j is normalized so

$$\|u_j\|^2_{L^2(G)} = \sum u_j(x)^2\mu_x = 1. \tag{2.7}$$

Then

$$\begin{aligned}\|Su_j\|^2_{L^2(G_E)} &= \sum_E (u_j(x)+u_j(y))^2 c(x,y)\\ &= \sum_{x\in V} u_j(x)^2 \sum_{y\sim x} c(x,y) + \sum_{x\in V}\sum_{y\sim x} c(x,y)u_j(x)u_j(y).\end{aligned} \tag{2.8}$$

Of course $\sum_{y\sim x} c(x,y) = \mu_x$ so the first term on the right side of (2.8) is 1. Also

$$\lambda_j u_j(x) = -\Delta_G u_j(x) = u_j(x) - \sum_{y\sim x}\frac{c(x,y)}{\mu_x}u_j(y). \tag{2.9}$$

Multiplying (2.9) by $\mu_x u_j(x)$ and summing over $x\in V$, we obtain

$$\sum_{x\in V}\sum_{y\sim x} c(x,y)u_j(x)u_j(y) = (1-\lambda_j)\sum u_j(x)^2\mu_x = 1-\lambda_j,$$

so (2.8) becomes simply

$$\|Su_j\|_{L^2(G_E)} = 2-\lambda_j. \tag{2.10}$$

Thus, to obtain a normalized eigenfunction, we should take $U_j = (2-\lambda_j)^{-1/2}Su_j$.

The weight on G_E that we are using is not necessarily consistent. However, it is easy to give a necessary and sufficient condition such that there exists a choice of the constant a that will make the weight consistent. Indeed, the consistency condition is

$$c(x,y) = \frac{a}{\mu_x}\sum_{\substack{z\sim x\\ z\neq y}} c(x,y)c(x,z) + \frac{a}{\mu_y}\sum_{\substack{z'\sim y\\ z'\neq x}} c(x,y)c(y,z'). \tag{2.11}$$

We may cancel $c(x,y)$ from (2.11) and use the consistency of the weight on G to obtain the equivalent condition

$$2 - \frac{1}{a} = c(x,y)\left(\frac{1}{\mu_x} + \frac{1}{\mu_j}\right), \tag{2.12}$$

which we can write as

$$c(x,y) = b\left(\frac{\mu_x \mu_y}{\mu_x + \mu_y}\right) \tag{2.13}$$

for $b = 2 - 1/a$. In other words, the measure determines the conductances. However, we cannot choose any measure, because the consistency condition requires

$$\sum_{y \sim x} \frac{\mu_y}{\mu_x + \mu_y} = \frac{1}{b}. \tag{2.14}$$

Thus, we require that the left side of (2.14) be independent of x. (This value must be greater than $1/2$ in order that a be positive, but it is easy to see that this condition is automatic by considering the vertex x where μ_x attains its minimum.) Then we use (2.14) to define b and (2.13) to define the conductances. The consistency condition then holds for the weight on G, and then the choice $a = (2-b)^{-1}$ implies the consistency condition for the weight on G_E.

For example, suppose G is k-regular (for $k \geq 3$) and we choose all conductances equal, say $c(x,y) = 1$, so $\mu_x = k$ for all vertices. Then (2.14) holds with $b = 2/k$, hence $a = k/(2(k-1))$. Of course, G_E is $2(k-1)$-regular, and the weight on G_E gives equal conductance to all edges. If G is not bipartite, all eigenvalues of Δ_G are multiplied by $k/(2(k-1))$ to obtain eigenvalues of Δ_E, and the eigenvalue $k/(k-1)$ appears with multiplicity $((k/2)-1)\#V$. This result is essentially in [**11**].

Finally, we consider the case of an infinite connected graph. Sometimes it is convenient to assume that each vertex has finite order, but in fact we can obtain the same results assuming only that the weight is chosen so that the possibly infinite sum in (1.4) converges.

Lemma 2.3. *Under the above assumption, the operator $-\Delta_G$ is a bounded operator on $L^2(G)$ with bound at most* 2.

Proof. If u and v have finite support,

$$\langle \Delta_G u, v\rangle = \sum_E c(x,y)u(y)v(x) - \langle u, v\rangle.$$

By Cauchy-Schwarz (twice) and consistency,

$$\begin{aligned}
&\Big|\sum_{x\in V}\sum_{y\sim x} c(x,y)u(y)v(x)\Big| \\
&\qquad\le \sum_{x\in V}\bigg(\sum_{y\sim x} c(x,y)u(y)^2\bigg)^{1/2}\bigg(\sum_{y\sim x} c(x,y)\bigg)^{1/2}|v(x)| \\
&\qquad= \sum_{x\in V}\bigg(\sum_{y\sim x} c(x,y)u(y)^2\bigg)^{1/2}|v(x)|\mu_x^{1/2} \\
&\qquad\le \bigg(\sum_{x\in V}\sum_{y\sim x} c(x,y)u(y)^2\bigg)^{1/2}\bigg(\sum_{x\in V} v(x)^2\mu_x\bigg)^{1/2} \\
&\qquad= \bigg(\sum_{y\in V} u(y)^2\mu_y\bigg)^{1/2}\bigg(\sum_{x\in V} v(x)^2\mu_x\bigg)^{1/2} \\
&\qquad= \|u\|_{L^2(G)}\|v\||_{L^2(G)}.
\end{aligned}$$

Since functions of finite support are dense in $L^2(G)$, the same estimate holds for $u, v \in L^2(G)$; hence, Δ_G is bounded with bound at most 2. □

Lemma 2.4. *The operator $-\Delta_E$ is bounded on $L^2(G_E)$ with bound at most $4a$.*

Proof. Since the weight is not necessarily consistent, we need to modify the proof of Lemma 2.3. The key observation is that

$$\sum_{e'\sim e} c(e,e') \le 2a\mu_e, \tag{2.15}$$

which then allows us to essentially repeat the proof. By (1.8) and (1.9), we have

$$\sum_{e'\sim e}\frac{c(e,e')}{\mu(e)} = \sum_{\substack{z\sim x\\ z\ne y}}\frac{ac(x,z)}{\mu_x} + \sum_{\substack{z'\sim y\\ z'\ne x}}\frac{ac(y,z')}{\mu_y} \tag{2.16}$$

for $e = e(x, y)$. By adding the omitted values $z = y$ and $z' = x$ to the right side of (2.16) we obtain $2a$, which yields (2.15). $\square$

Since the Laplacians are self-adjoint, the spectral theorem implies the existence of a spectral resolution. One way to describe this is by spectral projection operators. For Δ_G, there is a spectrum Λ, a closed subset of $[0, 2]$, and a measure $dm(\lambda)$ supported on Λ, and for each $\lambda \in \Lambda$ an operator $\mathcal{P}_\lambda$ with kernel $P_\lambda(x, y)$,

$$\mathcal{P}_\lambda f(x) = \sum_y P_\lambda(x, y) f(y) \mu_y, \tag{2.17}$$

satisfying

$$-\Delta_G \mathcal{P}_\lambda f = \lambda \mathcal{P}_\lambda f \tag{2.18}$$

and

$$f = \int_\Lambda \mathcal{P}_\lambda f \, dm(\lambda). \tag{2.19}$$

Note that we are not claiming that $\mathcal{P}_\lambda$ is a bounded operator on L^2, but (2.17) makes sense for any finitely supported function f, and these functions are dense in L^2. For such functions (2.19) makes sense pointwise. Also (2.18) is equivalent to

$$-\Delta_G P(\cdot, y) = \lambda P(\cdot, y) \text{ for all } y, \tag{2.20}$$

and $P_\lambda(x, y) = P_\lambda(y, x)$ because $-\Delta_G$ is self-adjoint. Again, we do not claim that either $\mathcal{P}_\lambda$ or the measure $dm(\lambda)$ are unique, since we can always multiply $\mathcal{P}_\lambda$ by a function $\varphi(\lambda)$ and simultaneously multiply $dm(\lambda)$ by $1/\varphi(\lambda)$ without changing (2.18) and (2.19). It is true that the product $\mathcal{P}_\lambda dm(\lambda)$ is unique.

It is convenient to break up the measure $dm(\lambda)$ into a discrete $dm_d(\lambda)$ and continuous $dm_c(\lambda)$ part. The discrete part can be written

$$dm_d(\lambda) = \sum b_j \delta(\lambda - \lambda_j), \tag{2.21}$$

and, without loss of generality, we can take all the constants b_j equal to 1. Then $\{\lambda_j\}$ are the L^2-eigenvalues of $-\Delta_G$, and the corresponding eigenspaces may have finite or infinite multiplicity. If $\{u_{jk}\}$ is an orthonormal basis for the λ_j-eigenspace, then

$$P_{\lambda_j}(x,y) = \sum_k u_{jk}(x)u_{jk}(y). \tag{2.22}$$

We call $\Lambda_d = \{\lambda_j\}$ the *discrete spectrum.* We do not claim that it is a closed set. The continuous spectrum Λ_c is the support of $dm_c(\lambda)$, and by definition it is a closed set. Strictly speaking, the spectrum of $-\Delta_G$ also contains the limit points of Λ_d, which may or may not belong to Λ_c, but these points do not play an essential role in the spectral resolution (2.19).

The discrete eigenvalue $\lambda = 2$ plays a special role. Such eigenvalues can only occur if G is bipartite, and the associated eigenfunctions may or may not be in L^2. If the total measure of G is finite, then the function that takes values ± 1 on the two parts of G is an L^2 eigenfunction. It is not clear whether or not the converse statement is true. In any case, we define Λ' to be Λ with the discrete value $\lambda = 2$ removed from Λ_d, and $dm'(\lambda) = dm_c(\lambda) + dm'_d(\lambda)$, where dm'_d is equal to dm_d with the atom $\delta(x-2)$ removed.

The value $2a$ may or may not be in the discrete spectrum of $-\Delta_E$. Let Λ'_E denote the spectrum Λ_E of $-\Delta_E$ with the discrete eigenvalue $2a$ removed. Then every function $F \in L^2(G_E)$ can be written uniquely $F = F' + F_{2a}$ where F_{2a} is an L^2 $2a$-eigenfunction and F' is orthogonal to all L^2 $2a$-eigenfunctions.

Theorem 2.5. $\Lambda'_E = a\Lambda'$, *and we may take*

$$\mathcal{P}^E_{a\lambda} = \frac{1}{2-\lambda} S\mathcal{P}_\lambda S^* \text{ for } \lambda \neq 2 \tag{2.23}$$

for the corresponding spectral projections. Then

$$F' = \int_{\Lambda'} \frac{1}{2-\lambda} S\mathcal{P}_\lambda S^* F \, dm'(\lambda) \tag{2.24}$$

with

$$-\Delta_E \frac{1}{2-\lambda} S\mathcal{P}_\lambda S^* F = a\lambda \frac{1}{2-\lambda} S\mathcal{P}_\lambda S^* F. \tag{2.25}$$

Moreover, we can write

$$\mathcal{P}^E_{a\lambda}F(e(x,y)) = \sum_{e(z,w)\in E} P^E_{a\lambda}(e(x,y),e(z,w))F(e(z,w))c(z,w) \tag{2.26}$$

for

$$\begin{aligned} P^E_{a\lambda}&(e(x,y),e(z,w)) \\ &= \frac{1}{2(2-\lambda)}(P_\lambda(x,z)+P_\lambda(x,w)+P_\lambda(y,z)+P_\lambda(y,w)). \end{aligned} \tag{2.27}$$

Proof. It is easy to see that S is a bounded operator from $L^2(G)$ to $L^2(G_E)$, and hence S^* is bounded from $L^2(G_E)$ to $L^2(G)$. We also observe that Lemma 2.1 continues to hold for infinite graphs.

We have

$$(\Delta_E + 2I)\int_{\Lambda'} \frac{1}{2-\lambda}\mathcal{P}_\lambda S^* F' dm\,(\lambda) = \int_{\Lambda'} \mathcal{P}_\lambda S^* F' dm\,(\lambda) = S^*F'$$

by (2.18) and (2.19). Composing on the left with S and using parts (a) and (c) of Lemma 2.1, we obtain

$$\left(\frac{1}{a}\Delta_E + 2I\right)\int_{\Lambda'} \frac{1}{2-\lambda} S\mathcal{P}_\lambda S^* F' dm\,(\lambda) = \left(\frac{1}{a}\Delta_E + 2I\right)F'.$$

Since F' is orthogonal to the kernel of $(1/a)\Delta_E + 2I$, we obtain

$$\int_{\Lambda'} \frac{1}{2-\lambda} S\mathcal{P}_\lambda S^* F' dm\,(\lambda) = F'. \tag{2.28}$$

However, S^*F_{2a} is a 2-eigenfunction by part (b) of Lemma 2.1, so $\mathcal{P}_\lambda S^* F_{2a} = 0$ for $\lambda \neq 2$; hence, $1/(2-\lambda)S\mathcal{P}_\lambda S^* F' = 1/(2-\lambda)S\mathcal{P}_\lambda S^* F$. Thus, (2.28) is the same as (2.24). Then (2.25) follows from (2.18) and parts (a) and (b) of Lemma 2.1.

To establish (2.26) and (2.27), we compute from (2.17) and (2.3) that

$$\begin{aligned} S\mathcal{P}_\lambda S^* F(e(x,y)) &= \sum_z (P_\lambda(x,z)+P_\lambda(y,z))S^*F(z)\mu_z \\ &= \sum_z (P_\lambda(x,z)+P_\lambda(y,z))\sum_{w\sim z} F(e(z,w))c(z,w). \end{aligned}$$

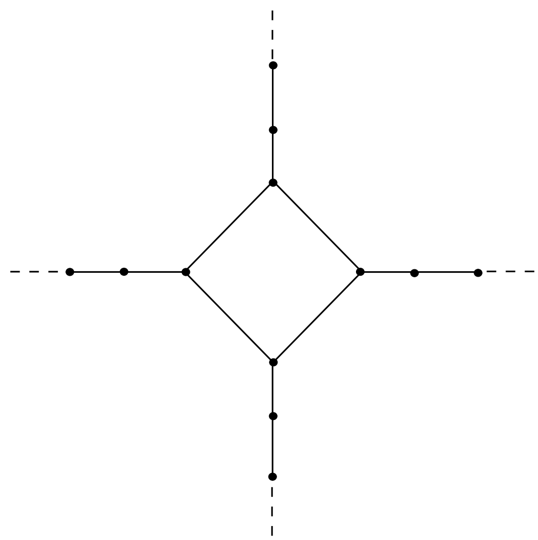

FIGURE 2.1.

We interchange z and w and then average to obtain (2.26) from (2.23) with the kernel given by (2.27). □

What is the multiplicity of the discrete eigenvalue $2a$ for $-\Delta_E$? Shirai [**9**] shows that for any infinite k-regular graph with $k \geq 3$ and all conductances equal, the multiplicity is always infinite. On the other hand, when $k = 2$, the multiplicity is zero, i.e., $2a$ is not an eigenvalue. It is not difficult to construct examples with any given finite multiplicity. For example, to get multiplicity one, just take a square with radiating half-lines, as shown in Figure 2.1, with all conductances equal (say 1). Take F to be the function that alternates ± 1 on the edges around the square and vanishes on all other edges. It is obvious that $-\Delta_E F(e) = 0$ on all the edges where $F(e) = 0$. If e is one of the edges around the square, we have $c(e, e') = a/3$ for each of its four adjacent edges e' by (1.9). A simple computation then shows that $-\Delta_E F(e) = 2aF(e)$. On the other hand, any $2a$-eigenfunction that does not vanish identically on the edges of one of the half-lines cannot lie in $L^2(G_E)$, so the multiplicity is exactly one. To get multiplicity N we need to weave together N squares in a similar fashion.

3. Edge substitution graph. Let G_H be an edge substitution graph with weight as described in the introduction. We always assume that H is finite and connected. We begin by discussing the case when G is finite.

Suppose that λ' is not a Dirichlet eigenvalue of H. Then every λ'-eigenfunction on the interior of H is uniquely determined by its values on the boundary $\{q_0, q_1\}$:

$$u(h) = a_0(\lambda', h)u(q_0) + a_1(\lambda', h)u(q_1) \text{ for } h \in V_H \tag{3.1}$$

for certain functions $a_i(\lambda', h)$. Since the λ'-eigenvalue equation is

$$(1-\lambda')u(h) = \sum_{g\sim h} \frac{c_H(g,h)}{\nu_h} u(g), \tag{3.2}$$

it follows by Cramer's rule that $a_i(\lambda', h)$ are rational functions of λ' with a common denominator being a polynomial of degree N and the numerators being polynomials of degree $N-1$. Of course, the Dirichlet eigenvalues are exactly the zeroes of the denominator. Define

$$A_i(\lambda') = \sum_{h\sim q_0} c_H(q_0, h)a_i(\lambda', h), \quad i = 0, 1, \tag{3.3}$$

so $A_i(\lambda')$ is a rational function of the same type as $a_i(\lambda', h)$. If desired we could compute these functions explicitly from the data (the structure of H and the weight).

Now we observe that, for u a λ'-eigenfunction,

$$\begin{aligned}
&\sum_{h\sim q_0} c(v(x,y,h),x)u(v(x,y,h)) \\
&\qquad = c(x,y)\sum_{h\sim q_0} c_H(q_0,h)(a_0(\lambda',h)u(x) + a_1(\lambda',h)u(y)) \\
&\qquad = c(x,y)(A_0(\lambda')u(x) + A_1(\lambda')u(y)).
\end{aligned}$$

It follows that

$$\begin{aligned}
\Delta_{G_H} u(x) &= \sum_{y\sim x} \frac{1}{\mu_x} \sum_{h\sim q_0} c(v(x,y,h),x)u(v(x,y,h)) - u(x) \\
&= (A_0(\lambda') - 1)u(x) + A_1(\lambda') \sum_{y\sim x} \frac{c(x,y)}{\mu_x} u(y) \\
&= (A_0(\lambda') - 1)u(x) + A_1(\lambda')(\Delta_G u(x) + u(x)).
\end{aligned}$$

In other words,

$$-\Delta_{G_H} u(x) = \lambda' u(x) \text{ if and only if } -\Delta_G u(x) = \lambda u(x), \tag{3.4}$$

where λ and λ' are related by

$$A_1(\lambda')(1-\lambda) = 1 - \lambda' - A_0(\lambda'). \tag{3.5}$$

We summarize this computation as follows.

Theorem 3.1. *Let λ' be an eigenvalue of Δ_{G_H} with eigenfunction u. Then either λ' is a Dirichlet eigenvalue of Δ_H, or $u|_{V_{\text{old}}}$ is a λ-eigenfunction of Δ_G where λ and λ' are related by* (3.5).

We observe that (3.5) is a polynomial equation of degree $N+1$ in λ' for each fixed λ. Thus, each of the $\#V$ eigenvalues λ can give rise to at most $N+1$ values of λ', for a total upper bound of $(N+1)(\#V)$ eigenvalues of Δ_{G_H}. There are two reasons why the true count could be lower. One is that the polynomial equation (3.5) might have multiple roots. The other is that one of the roots might be a Dirichlet eigenvalue of H. On the other hand, there are two other potential problems that do not arise. One is that the same value of λ' might arise for different choices of λ; this can't happen because it is clear from (3.5) that λ' determines λ. The other is that some of the values of λ' might not lie in the interval $[0,2]$; this can't happen because our argument shows that every λ' that solves (3.5) gives an eigenvalue of Δ_{G_H}, hence must lie in $[0,2]$.

If λ' is a Dirichlet eigenvalue of H, we would like to count its multiplicity as an eigenvalue of G_H. Assume first that λ' does not arise as a solution of (3.5). If the associated eigenfunction is a joint Dirichlet-Neumann of H, which in this case means $u(q_i) = 0$ and

$$\sum_{h \sim q_i} c_H(q_i, h) u(h) = 0, \quad i = 0, 1, \tag{3.6}$$

then we can place a copy of this eigenfunction on any of the $\#E$ copies of H in G_H, so the multiplicity is $\#E$. On the other hand, if (3.6) does not hold, we obtain linear constraints at each old vertex. Specifically, for each choice of coefficients $b(x, y)$ for $x \sim y$, consider the function $\widetilde{u}$ vanishing at V_{old} with

$$\widetilde{u}(v(x, y, h)) = b(x, y)u(h). \tag{3.7}$$

Clearly $\widetilde{u}$ satisfies the λ'-eigenvalue equation at each new vertex. Without loss of generality, we may take every eigenfunction on H to be either symmetric or skew-symmetric; in the symmetric case, we require $b(x, y) = b(y, x)$, while in the skew-symmetric case $b(x, y) = -b(y, x)$. The λ'-eigenvalue equation at the point $x \in V_{\text{old}}$ takes the form

$$\sum_{y \sim x} c(x, y)b(x, y) = 0. \tag{3.8}$$

Lemma 3.2. *The multiplicity of the λ'-eigenfunctions of $-\Delta_{G_H}$ vanishing on V_{old} for λ' a Dirichlet eigenfunction of $-\Delta_H$ that is not a joint Dirichlet-Neumann eigenfunction is given as follows*:

(a) $\#E - \#V + 1$ *if the Dirichlet eigenfunction is skew-symmetric*;

(b) $\#E - \#V$ *if the Dirichlet eigenfunction is symmetric and G is not bipartite*;

(c) $\#E - \#V + 1$ *if the Dirichlet eigenfunction is symmetric and G is bipartite.*

Proof. The multiplicity is equal to the dimension of solutions of (3.8). Since these are $\#V$ linear equations in $\#E$ unknowns, the issue is the possible redundancy in the system. In other words, can we find coefficients $a(x)$ for $x \in V$ such that the equation

$$\sum_{x \in V} a(x) \sum_{y \sim x} c(x, y)b(x, y) = 0$$

is trivial? Note that the coefficient of $b(x, y)$ is $(a(x) + a(y))c(x, y)$ in the symmetric case and $(a(x) - a(y))c(x, y)$ in the skew-symmetric case. Clearly $a(x) =$ constant gives the unique solution in the skew-symmetric case, while $a(x) = \pm$ constant alternating on different colored vertices gives the unique solution for bipartite G in the symmetric case. □

Note that $\#E \geq \#V - 1$, and equality holds only when G is bipartite. Thus, the multiplicities in the lemma are always nonnegative.

It is also possible that a Dirichlet eigenvalue λ' of $-\Delta_H$ may correspond to an eigenfunction of $-\Delta_{G_H}$ that does not vanish on V_{old}. If the Dirichlet eigenfunction is symmetric, then we have an obstruction to (3.1) when $u(q_0) = u(q_1)$, but not if $u(q_0) = -u(q_1)$, while if the Dirichlet eigenfunction is skew-symmetric, then the reverse is true. There are two ways this can arise:

(i) if $\lambda = 0$ and u is the constant function on V_{old}, if λ' is related to λ by (3.5) and λ' is a Dirichlet eigenvalue of $-\Delta_H$ corresponding to a skew-symmetric eigenfunction;

(ii) if $\lambda = 2$, G is bipartite and $u = \pm 1$ on V_{old}, if λ' is related to λ by (3.5) and λ' is a Dirichlet eigenvalue of $-\Delta_H$ corresponding to a symmetric eigenfunction.

Next we discuss the correct normalization of eigenfunctions. Suppose we start with u a λ-eigenfunction on G normalized so that

$$\|u\|^2_{L^2(G)} = \sum_{x \in V} |u(x)|^2 \mu_x = 1, \tag{3.9}$$

and let u also denote its extension to a λ'-eigenfunction on G_H, so

$$u(v(x, y, h)) = a_0(\lambda', h)u(x) + a_1(\lambda', h)u(y) \tag{3.10}$$

on new vertices, where we have chosen one value of λ' satisfying (3.5) where λ' is not a Dirichlet eigenvalue of $-\Delta_H$. Define

$$B_0(\lambda') = \sum_{h \in V_H \setminus \{q_0, q_1\}} \nu_h a_0(\lambda', h)^2 \tag{3.11}$$

and

$$B_1(\lambda') = 2 \sum_{h \in V_H \setminus \{q_0, q_1\}} \nu_h a_0(\lambda', h) a_1(\lambda', h). \tag{3.12}$$

Then we have

$$\begin{aligned}\sum_{h \in V_H \setminus \{q_0, q_1\}} & u(v(x, y, h))^2 c(x, y) \nu_h \\ &= B_0(\lambda') c(x, y) u(x)^2 \\ &\quad + B_0(\lambda') c(x, y) u(y)^2 + B_1(\lambda') c(x, y) u(x) u(y).\end{aligned}$$

It follows that

$$
\begin{aligned}
\|u\|^2_{L^2(G_H)} &= \sum_{x\in V} u(x)^2\mu_x + \sum_{x\in V}\sum_{y\sim x} B_0(\lambda')c(x,y)u(x)^2 \\
&\quad + \sum_{x\in V}\sum_{y\sim x} B_1(\lambda')c(x,y)u(x)u(y)) \\
&= 1 + B_0(\lambda') + B_1(\lambda')\sum_{x\in V}\sum_{y\sim x} c(x,y)u(x)u(y)
\end{aligned} \tag{3.13}
$$

by the consistency of the weight on G. Now

$$\Delta u(x) = \sum_{y\sim x} \frac{c(x,y)}{\mu_x}\ u(y) - u(x),$$

so multiplying by $\mu_x u(x)$, we obtain

$$\sum_{y\sim x} c(x,y)u(x)u(y) = \mu_x u(x)\Delta u(x) + \mu_x u(x)^2.$$

If we substitute this into (3.13) and use the λ-eigenvalue equation, we obtain

$$\|u\|^2_{L^2(G_H)} = 1 + B_0(\lambda') + (1-\lambda')B_1(\lambda'). \tag{3.14}$$

Thus, to obtain a normalized eigenfunction, we should take

$$\widetilde{u} = (1 + B_0(\lambda') + (1-\lambda')B_1(\lambda'))^{-1/2}u. \tag{3.15}$$

In particular, if the λ-eigenspace of $-\Delta_G$ has an orthonormal basis $\{u_k\}$, then

$$P_\lambda(x,y) = \sum_k u_k(x)u_k(y) \tag{3.16}$$

is the kernel of the orthogonal projection operator

$$\mathcal{P}_\lambda f(x) = \sum_{y\in V} P_\lambda(x,y)f(y)\mu_y \tag{3.17}$$

onto the λ-eigenspace of $-\Delta_G$. Then

$$P_{\lambda'}^{G_H}(x,y) = (1 + B_0(\lambda') + (1-\lambda')B_1(\lambda'))^{-1}\widetilde{P}_\lambda(x,y) \tag{3.18}$$

is the kernel of the orthogonal projection operator $\mathcal{P}_{\lambda'}^{G_H}$ onto the λ'-eigenspace of $-\Delta_{G_H}$, where $\widetilde{P}_\lambda(x,y)$ is obtained from $P_\lambda(x,y)$ by extending the values on $V_{\text{old}} \times V_{\text{old}}$ by solving the λ'-eigenvalue equation in each variable on $V_{\text{old}} \times V_{\text{new}}$, $V_{\text{new}} \times V_{\text{old}}$ and $V_{\text{new}} \times V_{\text{new}}$.

Next, we consider the case when G is infinite. (As before, H is finite.) As in Section 2, we assume the weight on G is chosen so that (1.4) converges. Then Lemma 2.3 applies to both $-\Delta_G$ and $-\Delta_{G_H}$. Given a spectral resolution (2.17)–(2.19) for $-\Delta_G$, split into a discrete Λ_d and a continuous Λ_c spectrum, denote by Λ' the solutions to (3.5) corresponding to λ in Λ, but with Dirichlet eigenfunctions of $-\Delta_H$ deleted. Note that the set of $\lambda \in \Lambda_c$ corresponding under (3.5) to a Dirichlet eigenvalue of $-\Delta_H$ is a finite set, hence has measure zero for $dm_c(\lambda)$.

Conjecture 3.3. *The spectrum of $-\Delta_{G_H}$ is $\Lambda' \cup \{\lambda'_j\}$, where $\{\lambda'_j\}$ are the Dirichlet eigenvalues of $-\Delta_H$ and form part of the discrete spectrum of $-\Delta_{G_H}$. The spectral resolution*

$$f = \int_{\Lambda'} \mathcal{P}_{\lambda'}^{G_H} f\, dm'(\lambda') + \sum_j \mathcal{P}_{\lambda'_j}^{G_H} f \tag{3.19}$$

has spectral operators $\mathcal{P}_{\lambda'}^{G_H}$ for $\lambda' \in \Lambda'$ with kernels given by (3.18), *and*

$$dm'(\lambda') = dm\,(\lambda) \tag{3.20}$$

where λ is determined from λ' by (3.5).

The evidence for the conjecture is that it is clearly valid for the discrete part of the spectrum by the previous discussion. The multiplicity of each of the Dirichlet eigenvalues λ'_j in the spectrum of $-\Delta_{G_H}$ is difficult to determine. If λ'_j is a joint Dirichlet-Neumann eigenvalue, then the multiplicity is infinite. Otherwise, it is equal to the dimension of L^2 solutions of (3.8).

We conclude with some examples.

Example 3.4. Let H be the 3-element graph $q_0 \sim h_1 \sim g_1$ with $c_H(q_0, h_1) = c_H(h_1, q_1) = 1$. Then $a_0(\lambda', h_1) = a_1(\lambda', h_1) = 1/(2(1-\lambda'))$. It follows that

$$A_0(\lambda') = A_1(\lambda') = \frac{1}{2(1-\lambda')}. \tag{3.21}$$

Note that $\lambda' = 1$ is the only Dirichlet eigenvalue of $-\Delta_H$, and it is not a joint Dirichlet-Neumann eigenvalue. The corresponding eigenfunction is symmetric. Then the solutions of (3.5) are given by

$$\lambda' = 1 \pm \sqrt{1 - \frac{\lambda}{2}}. \tag{3.22}$$

Note that for every λ satisfying $0 \leq \lambda < 2$ there are two distinct solutions satisfying $0 \leq \lambda' \leq 2$, but for $\lambda = 2$ there is just one solution, and it happens to be the Dirichlet eigenvalue. So if G is not bipartite we have $2\#V$ bifurcated eigenvalues, and the eigenvalue 1 has multiplicity $\#E - \#V$, while if G is bipartite we have $2(\#V - 1)$ bifurcated eigenvalues, and the eigenvalue 1 has multiplicity $\#E - \#V + 2$, made up of the $\#E - \#V + 1$-dimensional space vanishing on V_{old} given by Lemma 3.2, and a one-dimensional space generated by the function $u(x) = \pm 1$ on V_{old} and extended to be zero on V_{new}.

In this example $B_0(\lambda') = 1/(2(1-\lambda')^2)$ and $B_1(\lambda') = 1/((1-\lambda')^2)$, so

$$1 + B_0(\lambda') + (1-\lambda')B_1(\lambda) = \frac{2\lambda'^2 - 6\lambda' + 5}{2(1-\lambda')^2}. \tag{3.23}$$

Example 3.5. Let H be the 4-element square graph with vertices q_0, q_1, h_1, h_2 and edges $q_0 \sim h_1 \sim q_1$ and $q_0 \sim h_2 \sim q_1$, with all conductances equal to $1/2$, so (1.16) holds. In this example $a_0(\lambda', h_i) = a_1(\lambda', h_i) = 1/(2(1-\lambda'))$ for $i = 1, 2$, so $A_0(\lambda')$ and $A_1(\lambda')$ are again given by (3.21) as in the previous example, and (3.22) again describes all solutions to (3.5). Note that, although $N = 2$, we only have generically two solutions. However, the Dirichlet eigenvalue $\lambda' = 1$ of $-\Delta_H$ has multiplicity 2. One eigenfunction has $u(h_1) = u(h_2) = 1$, and the other has $u(h_1) = -u(h_2) = 1$. The second one is a joint Dirichlet-Neumann eigenfunction, so it adds $\#E$ dimensions to the $\lambda' = 1$ eigenspace of $-\Delta_{G_H}$, bringing the number of eigenvalues up to $\#V + 2\#E$.

Example 3.6. Let H be a complete-3 graph with $c(q_0, h_1) = c(q_1, h_1) = t$ and $c(q_0, q_1) = 1 - t$, where t is a parameter satisfying $0 < t < 1$ (note that $t = 1$ just gives Example 3.4). Again $a_i(\lambda', h_1) = 1/(2(1-\lambda'))$, $i = 0, 1$, but $a_i(\lambda', q_i) = 1$, $i = 0, 1$. Thus,

$$A_0(\lambda') = \frac{t}{2(1-\lambda')}, \qquad A_1(\lambda') = \frac{t}{2(1-\lambda')} + (1-t). \tag{3.24}$$

We solve (3.5) to obtain

$$1 - \lambda' = \frac{(1-t)(1-\lambda) \pm \sqrt{(1-t)^2(1-\lambda)^2 + 4t(1-(\lambda/2))}}{2}. \tag{3.25}$$

It is straightforward to show that this yields two distinct solutions in $[0, 2]$ for each λ in $[0, 2]$. The only Dirichlet eigenvalue of $-\Delta_H$ is $\lambda' = 1$, which corresponds to $\lambda = 2$. The eigenfunction is symmetric and not a joint Dirichlet-Neumann eigenfunction. Thus, if G is not bipartite, then there are $2\#V$ bifurcated eigenvalues, and the eigenvalue $\lambda' = 1$ has multiplicity $\#E - \#V$, with all eigenfunctions vanishing on V_{old}. If G is bipartite, then there are $2\#V - 1$ bifurcated eigenvalues, and the eigenvalue $\lambda' = 1$ has multiplicity $\#E - \#V + 1$, with all eigenfunctions vanishing on V_{old}. (The difference between this example and Example 3.4 is that the function satisfying $u(x) = \pm 1$ on V_{old} and vanishing on V_{new} is an eigenfunction with eigenvalue $2 - t$.)

Example 3.7. Let H be the linear graph $q_0 \sim h_1 \sim h_2 \sim \cdots \sim h_N \sim q_1$ with all conductances equal to 1. In this example it is convenient to identify h_j with the point $j/(N+1)$ on the unit interval, q_0 with 0 and q_1 with 1. We introduce a parameter s with $0 \le s \le 1$, related to λ by

$$1 - \lambda = \cos \pi s. \tag{3.26}$$

Then (3.1) can be written

$$u(x) = \frac{\sin \pi s(1-x)}{\sin \pi s} u(q_0) + \frac{\sin \pi s x}{\sin \pi s} u(q_1). \tag{3.27}$$

We have

$$A_0(\lambda') = \frac{\sin \pi s N/(N+1)}{\sin \pi s}$$

and

$$A_1(\lambda') = \frac{\sin \pi s 1/(N+1)}{\sin \pi s},$$

and (3.5) simplifies to

$$1 - \lambda' = \cos \frac{\pi s}{N+1}. \tag{3.28}$$

The Dirichlet eigenfunctions of $-\Delta_H$ are

$$u_j(x) = \sin \pi j x, \quad j = 1, \dots, N, \tag{3.29}$$

with eigenvalues given by (3.28) with $s = j$. These are never joint Dirichlet-Neumann eigenfunctions, and they are symmetric when j is odd and skew-symmetric when j is even.

Now if $0 < s < 1$, then there are $N + 1$ distinct solutions to (3.5), given by

$$1 - \lambda' = \cos\left(\frac{\pi(s+2j)}{N+1}\right), \quad j = 0, 1, \dots, N, \tag{3.30}$$

and none of these are Dirichlet eigenvalues of $-\Delta_H$. This breaks down when $\lambda = 0$ (corresponding to $s = 0$), when the solutions corresponding to j and $N + 1 - j$ are equal, and also when $\lambda = 2$ (corresponding to $s = 1$), when the solutions corresponding to j and $N - j$ are equal. Note that these exceptional cases yield Dirichlet eigenvalues, except $\lambda' = 0$ for $\lambda = 0$ ($j = 0$ in (3.30)), $\lambda' = 2$ for $\lambda = 2$ when N is even ($j = N/2$ in (3.30)), and $\lambda' = 2$ for $\lambda = 0$ when N is odd ($j = (N+1)/2$ in (3.30)). Of course $\lambda = 0$ always occurs with multiplicity one in the spectrum of $-\Delta_G$, while $\lambda = 2$ occurs with multiplicity one if and only if G is bipartite.

If G is not bipartite and N is even there are $(N+1)(\#V-1)+1$ bifurcated eigenvalues of $-\Delta_{G_H}$, namely, $(N+1)(\#V-1)$ corresponding to eigenvalues $\lambda \neq 0$, and the constant function corresponding to $\lambda = 0$ and $\lambda' = 0$. There are $N/2$ symmetric Dirichlet eigenfunctions on H that give rise to $(N/2)(\#E - \#V)$ eigenfunctions on G_H that vanish on V_{old}, and $N/2$ skew-symmetric Dirichlet eigenfunctions on H that give rise to $(N/2)(\#E - \#V + 1)$ eigenfunctions on G_H that vanish on V_{old}, according to Lemma 3.2. Finally, the $N/2$ skew-symmetric Dirichlet eigenfunctions on H each give rise to a single eigenfunction on G_H that is constant on V_{old}. The total count is

$$(N+1)(\#V-1)+1+\frac{N}{2}(\#E-\#V)+\frac{N}{2}(\#E-\#V+1)+\frac{N}{2} = \#V+N\#E.$$

Similarly, if N is odd, there are $(N+1)(\#V-1)+2$ bifurcated eigenvalues of $-\Delta_{G_H}$, and there are $(N+1)/2$ symmetric and $(N-1)/2$ skew-symmetric Dirichlet eigenfunctions of $-\Delta_H$, reducing the count of multiplicities of eigenspaces corresponding to Dirichlet eigenvalues by one.

Similarly, if G is bipartite, then there are $(N+1)(\#V-2)+2$ bifurcated eigenvalues of $-\Delta_{G_H}$, there are N Dirichlet eigenvalues of $-\Delta_H$, each corresponding to $\#E - \#V + 1$ eigenfunctions of $-\Delta_{G_H}$ vanishing on V_{old}, and one either constant or ± 1 on V_{old}.

REFERENCES

1. Fan R.K. Chung, *Specral graph theory*, CBMS **92**, American Mathematical Society, 1997.

2. Y. Colin de Verdière, *Spectres de Graphes*, Soc. Math. France, 1998.

3. D.M. Cvetković, M. Doob and H. Sachs, *Spectra of graphs–Theory and application*, Deutscher Verlag Wissen., Berlin, and Academic Press, New York, 1980.

4. M. Fukushima and T. Shima, *On a spectral analysis for the Sierpinski gasket*, Potential Anal. **1** (1992), 1–35.

5. J. Kigami, *A harmonic calculus on the Sierpinski spaces*, Japan J. Appl. Math. **8** (1989), 259–290.

6. ———, *Harmonic calculus on p.d.f. self-similar sets*, Trans. Amer. Math. Soc. **335** (1993), 721–755.

7. ———, *Analysis on fractals*, Cambridge University Press, New York, 2001.

8. ———, *Harmonic analysis for resistance forms*, J. Functional Anal. **204** (2003), 399–444.

9. T. Shima, *On eigenvalue problems for the random walks on the Sierpinski pre-gaskets*, Japan J. Indust. Appl. Math. **8** (1991), 127–141.

10. ———, *On eigenvalue problems for Laplacians on p.c.f. self-similar sets*, Japan J. Indust. Appl. Math. **13** (1996), 1–23.

11. T. Shirai, *The spectrum of infinite regular line graphs*, Trans. Amer. Math. Soc. **352** (2000), 115–132.

12. R. Strichartz, *Analysis on fractals*, Amer. Math. Soc. **46** (1999), 1199–1208.

13. ———, *Fractafolds based on the Sierpinski gasket and their spectra*, Trans. Amer. Math. Soc. **355** (2003), 4019–4043.

14. ———, *Differential equations on fractals: A tutorial*, Princeton University Press, Princeton, 2006.

15. A. Teplyaev, *Spectral analysis on infinite Sierpinski gaskets*, J. Functional Anal. **159** (1989), 537–567.

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