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THE BOOLEAN SPACE OF R-PLACES

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ABSTRACT. We prove that every Boolean space is realized as a space of real places of some formally real field. This gives a partial answer to the problem posed in [**1, 7**].

1. Introduction. Let $\mathcal{X}(K)$ be the space of orders of a formally real field K, endowed with the Harrison topology introduced by subbasic sets of the form

$$H_K(a) := \{P \in \mathcal{X}(K) : a \in P\}, \quad a \in \dot{K} = K \setminus \{0\}.$$

It is known that $\mathcal{X}(K)$ is a Boolean space, i.e., compact, Hausdorff and totally disconnected. In [**3**] Craven presented a construction of a field K, whose space of orders $\mathcal{X}(K)$ is homeomorphic to a given Boolean space X. Spaces of orders are closely related to the spaces of $\mathbf{R}$-places, and some main results on this relationship can be found in [**8**]. We shall recall a part of this theory in the next section. In particular, spaces of $\mathbf{R}$-places are known to be compact and Hausdorff. An open problem posed in [**1, 7**] is:

Which compact and Hausdorff spaces occur as a spaces of real places?

It was pointed out in [**1**, Remark 2.16] that if K is a totally Archimedean field then the space of $\mathbf{R}$-places and the space of orders are homeomorphic and consequently the space of $\mathbf{R}$-places is Boolean. Thus, every finite discrete space is realized as a space of $\mathbf{R}$-places, since totally Archimedean fields exist with any finite number of orders. Our main theorem, presented in Section 4, states that every Boolean space is realized as a space of $\mathbf{R}$–places of some formally real field. Before we can get to this, we need to develop some new methods in the theory of extensions of $\mathbf{R}$-places; Section 3 includes these results.

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2. Basic properties of $M(K)$. Let K be a formally real field with the space of orders $\mathcal{X}(K)$. Denote by $\mathbf{Q}^+$ the set of positive rational numbers. For an order P of K, the set

$$A(P) := \{a \in K : \exists_{q \in \mathbf{Q}^+}\ q \pm a \in P\}$$

is a valuation ring of K with the maximal ideal

$$I(P) := \{a \in K : \forall_{q \in \mathbf{Q}^+}\ q \pm a \in P\}.$$

Moreover, P induces an Archimedean order on the residue field $k(P) = A(P)/I(P)$. Therefore, $k(P)$ can be considered as a subfield of $\mathbf{R}$, and the map

$$\xi_P : K \longrightarrow \mathbf{R} \cup \{\infty\},$$
$$\xi_P(a) = \begin{cases} a + I(P) & \text{if } a \in A(P) \\ \infty & \text{if } a \notin A(P) \end{cases}$$

is the $\mathbf{R}$-*place* associated to P. Conversely, if ξ is any $\mathbf{R}$-place of K, then there exists an order P of K such that $\xi_P = \xi$. By [**8**, Corollary 2.13, Proposition 9.1], two orders P_1 and P_2 determine the same $\mathbf{R}$-place if and only if $A(P_1) = A(P_2)$ and the Archimedean orders induced by P_1 and P_2 coincide. We let $M(K)$ denote the set of all $\mathbf{R}$-places of the field K. Therefore, the map:

$$\lambda_K : \mathcal{X}(K) \longrightarrow M(K), \qquad \lambda_K(P) = \xi_P$$

is onto and we can equip $M(K)$ with the quotient topology inherited from $\mathcal{X}(K)$. Since $\mathcal{X}(K)$ is compact, $M(K)$ is also compact. The ring

$$\mathcal{H}(K) = \{a \in K : \forall_{\xi \in M(K)}\ \xi(a) \neq \infty\}$$

is called *the real holomorphy ring of* K. We denote by $\mathbf{E}(K)$ the set of units of $\mathcal{H}(K)$. In fact [**8**, Theorem 9.11], the quotient topology on $M(K)$ coincides with the coarsest topology, such that the evaluation maps

$$e_a : M(K) \longrightarrow \mathbf{R}, \quad a \in \mathcal{H}(K),$$

defined by

$$e_a(\xi) = \xi(a)$$

are continuous. Moreover, this topology is the same as the topology introduced by subbasic sets of the form

$$U_K(a) = \{\xi \in M(K) : \xi(a) > 0\}, \text{ for } a \in \mathcal{H}(K).$$

Note that the complement $U_K^c(a) = \{\xi \in M(K) : \xi(a) \leqslant 0\}$ does not need to be an open set. However, if a is a unit in $\mathcal{H}(K)$, then $U_K^c(a) = U_K(-a)$ (since $\xi(a) \neq 0$, for every $\xi \in M(K)$) and thus $U_K(a)$ is a clopen set. By [**8**, Lemma 9.8], the evaluation maps e_a separate points of $M(K)$, thus $M(K)$ is a Hausdorff space.

A signature of a field K is a character $\chi : \dot{K} \to \{1, -1\}$ with additively closed kernel. It is known that the sets $\mathcal{X}(K)$ and $\{\ker \chi : \chi$ is a signature of $K\}$ are, in fact, equal. By [**2**, pages 60–61], if P_1, $P_2 \in \mathcal{X}(K)$ with $\lambda_K(P_1) = \lambda_K(P_2)$ and χ_1, χ_2 are their signatures then we have a relationship

$$\chi_2 = \chi_1 \cdot \tau \circ v,$$

where v is the valuation corresponding to the valuation ring $A(P_1) = A(P_2)$ with value group Γ and τ is a character of Γ with values in $\{1, -1\}$.

If K is totally Archimedean, then the value group Γ_P of the valuation associated to $A(P)$ is trivial for every $P \in \mathcal{X}(K)$. Then the map λ_K is injective, and hence a homeomorphism. In this case $M(K)$ is Boolean. It is well known that, for a given positive integer k, there exists a subfield of $\mathbf{R}$ with k Archimedean orders (see [**6**, page 582]). Thus every finite discrete space is realized as a space of $\mathbf{R}$-places of some formally real field K.

3. A Cantor cube as a space of R-places. One can find the theory of extensions of orders and signatures in [**2**]. Suppose that L is a field with an order P^L, and K is a subfield of L. Then $P^K = P^L \cap K$ is an order of K. We call P^L *an extension of* P^K. The map

$$\rho_{L/K} : \mathcal{X}(L) \longrightarrow \mathcal{X}(K), \qquad \rho_{L/K}(P^L) = P^L \cap K$$

is continuous, since $\rho_{L/K}^{-1}(H_K(a)) = H_L(a)$, for $a \in \dot{K}$.

If ξ^L is an $\mathbf{R}$-place of L, then the restriction $\xi^L|_K$ is an $\mathbf{R}$-place of K. Therefore, we have a map

$$\omega_{L/K} : M(L) \longrightarrow M(K), \qquad \omega_{L/K}(\xi^L) = \xi^L\,|_K\,.$$

By [**5**], the diagram

$$\begin{array}{ccc} \mathcal{X}(L) & \xrightarrow{\lambda_L} & M(L) \\ {\scriptstyle \rho_{L/K}}\downarrow & & \downarrow{\scriptstyle \omega_{L/K}} \\ \mathcal{X}(K) & \xrightarrow{\lambda_K} & M(K) \end{array}$$

commutes and all the maps are continuous.

Let $D_\mathfrak{m}$ be a Cantor cube with weight $\mathfrak{m}$, i.e. the set $\{1,-1\}^\mathfrak{m}$ with the product topology. We are going to show that there exists a field with space of $\mathbf{R}$-places homeomorphic to $D_\mathfrak{m}$. We need the following lemma.

Lemma 3.1. *Let P be an order of the field F, and let*

$$K = F(\{\sqrt{a},\, a \in \mathcal{A}\}),$$

where $\mathcal{A} \subset \{a \in F : 0 < \lambda_F(P)(a) < \infty\}$. Then the restriction of λ_K to the set $\rho_{K/F}^{-1}(P)$ is injective.

Proof. By induction we shall first show that this lemma is true if $\mathcal{A}$ is finite.

If $0 < \lambda_F(P)(a) < \infty$, then $a \in P$. Thus P has two extensions to orders of $F(\sqrt{a})$, call them P_1 and P_2. We can assume that $\sqrt{a} \in P_1$ and $-\sqrt{a} \in P_2$. Moreover, $0 \neq \lambda_K(P_i)(\sqrt{a}) < \infty$ since $0 < \lambda_K(P_i)(a) < \infty$, for $i = 1, 2$. Therefore, $\lambda_K(P_1)(\sqrt{a}) > 0$ and $\lambda_K(P_2)(\sqrt{a}) < 0$. Thus, $\lambda_K(P_1) \neq \lambda_K(P_2)$.

Now take a set $\{a_1, \dots, a_n\}$ of elements of F such that $0 < \lambda_F(P)(a_i) < \infty$, for $i = 1, \dots, n$. Let $K := F(\sqrt{a_1}, \dots \sqrt{a_n})$, and suppose that the lemma is true for the field $K' := F(\sqrt{a_1}, \dots \sqrt{a_{n-1}})$. Let Q_1 and Q_2 be two different orders of K that extend P. If $Q_1 \cap K' \neq Q_2 \cap K'$, then by the inductive hypothesis, $\lambda_{K'}(Q_1 \cap K') \neq \lambda_{K'}(Q_2 \cap K')$, and therefore $\lambda_K(Q_1) \neq \lambda_K(Q_2)$. If $Q_1 \cap K' = Q_2 \cap K' =: Q$, then we can repeat the argument from the case $n = 1$ with $F = K'$, $P = Q$ and $a = a_n$.

Now, suppose that

$$K = F(\{\sqrt{a},\, a \in \mathcal{A}\}),$$

where $\mathcal{A} \subset \{a \in F : O < \lambda_F(P)(a) < \infty\}$, and suppose that Q_1 and Q_2 are two different extensions of P in K. Then there exists an $\alpha \in K$ such that $\alpha \in Q_1$ and $-\alpha \in Q_2$. But α is in some $K' := F(\sqrt{a_1}, \dots \sqrt{a_n})$, where $a_1, \dots, a_n \in \mathcal{A}$. Therefore, $Q_1 \cap K' \neq Q_2 \cap K'$, and then $\lambda_{K'}(Q_1 \cap K') \neq \lambda_{K'}(Q_2 \cap K')$, which implies that $\lambda_K(Q_1) \neq \lambda_K(Q_2)$. □

Theorem 3.2. *For every infinite cardinal number* $\mathfrak{m}$, *the Cantor cube* $D_{\mathfrak{m}}$ *of weight* $\mathfrak{m}$ *is homeomorphic to the space* $M(K)$, *for some formally real field* K.

Proof. Let F be a real closed field of cardinality $\mathfrak{m}$. Consider the field $F(X)$ and two of its orders:

$$P_+ = \left\{ \frac{f}{g} : \frac{\mathrm{lc}\,(f)}{\mathrm{lc}\,(g)} \in \dot{F}^2 \right\},$$
$$P_- = \left\{ \frac{f}{g} : (-1)^{\deg(f) - \deg(g)} \frac{\mathrm{lc}\,(f)}{\mathrm{lc}\,(g)} \in \dot{F}^2 \right\},$$

where $\mathrm{lc}\,(f)$ and $\mathrm{lc}\,(g)$ denote the leading coefficients of the polynomials f and g, respectively. Easy computations show that the valuation rings $A(P_+)$ and $A(P_-)$ coincide and both orders induce the same order on the residue field F (see [**9**, pages 79–80]). Therefore, $\lambda_{F(X)}(P_+) = \lambda_{F(X)}(P_-)$. Let

$$K = F(X)(\{\sqrt{\frac{X-a}{X}} : a \in \dot{F}\}).$$

In [**9**] it was shown that $\rho^{-1}_{K/F(X)}(P_+) = H_K(X) \cong D_{\mathfrak{m}} \cong H_K(-X) = \rho^{-1}_{K/F(X)}(P_-)$. Moreover, for every $P_+^K \in \rho^{-1}_{K/F(X)}(P_+)$, there exists exactly one order $P_-^K \in \rho^{-1}_{K/F(X)}(P_-)$ such that $\lambda_K(P_+^K) = \lambda_K(P_-^K)$. Since $\mathcal{X}(K) = H_K(X) \dot{\cup} H_K(-X)$, the map $\lambda_K|_{H_K(X)}$ is surjective.

Finally, observe that $(X-a)/X = 1-(a/X) \in 1+I(P_+)$ and therefore $\lambda_{F(X)}(P_+)((X-a)/X) = 1$. By the previous lemma, $\lambda_K|_{H_K(X)}$ is injective. Thus, $\lambda_K|_{H_K(X)}$ is a continuous bijection of a compact space onto a Hausdorff space, so it is a homeomorphism. □

4. The main theorem. Every Boolean space is a closed subspace of some Cantor cube. In this section we shall show how we can eliminate **R**-places by field extensions. Of course, if we eliminate an **R**-place, then we eliminate all orders which determine this **R**-place.

We recall the following result by Craven [**3**] which allows us to eliminate orders:

Proposition 4.1 [**3**, Proposition 2]. *Let K be a formally real field, and let $Y \subset \mathcal{X}(K)$ be such that $Y = \cap_{\alpha \in \mathcal{A}} H_K(\alpha)$, where $\mathcal{A} \subset K$. Then there exists an algebraic extension L of K such that the map $\rho_{L/K} : \mathcal{X}(L) \to \mathcal{X}(K)$ is a homeomorphism onto Y.*

We note that the field L constructed in the proof of the proposition above is of the following form:

$$L = K(\{\sqrt[2^n]{\alpha} : \alpha \in \mathcal{A},\, n = 1, 2, \dots\}).$$

Proposition 4.2. *Let K be a formally real field. Suppose that H is a closed subset of $\mathcal{X}(K)$ such that $\lambda_K|_H$ is a bijection onto $M(K)$, and suppose that Y is a closed subset of $\mathcal{X}(K)$ such that $Y = \cap_{\alpha \in \mathcal{A}} H_K(\alpha)$, where $\mathcal{A} \subset \mathbf{E}(K)$, i.e., $\mathcal{A}$ is a subset of units of the real holomorphy ring of K. Let $Y_0 = H \cap Y$. Then there exists an extension L of K such that the map*

$$\rho^{-1}_{L/K}(Y_0) \xrightarrow{\lambda_L} M(L)$$

is a bijection.

Proof. Note that if $\alpha \in \mathbf{E}(K)$, P, $Q \in \mathcal{X}(K)$ and $\lambda_K(P) = \lambda_K(Q)$, then $\alpha \in P$ if and only if $\alpha \in Q$.

Since $\lambda_K|_H$ is a bijection, we have that for every $P \in Y$ there exists exactly one $Q \in H$ such that $\lambda_K(P) = \lambda_K(Q)$. But then $Q \in Y$ and therefore $Q \in Y_0$. Thus, the mapping $\pi : Y \to Y_0$, which assigns to every $P \in Y$ the unique $Q \in H$ such that $\lambda_K(P) = \lambda_K(Q)$, is well defined.

Let L be the field constructed in Proposition 4.1 for Y. The diagram

$$\begin{tikzcd}
\mathcal{X}(L) \arrow[rr, "\lambda_L"] \arrow[ddd, "\rho_{L/K}(bij.)"'] & & M(L) \arrow[ddd, "\omega_{L/K}"] \\
& \rho^{-1}_{L/K}(Y_0) \arrow[ul, "id"'] \arrow[ur, "\lambda_L"] \arrow[d, "\rho_{L/K}(bij.)"'] & \\
& Y_0 \arrow[dr, "\lambda_K(inj.)"] & \\
Y \arrow[ur, "\pi"] \arrow[rr, "\lambda_K"'] & & M(K)
\end{tikzcd}$$

commutes. Let $\xi \in M(L)$. Choose an order $P^L \in \mathcal{X}(L)$ with $\lambda_L(P) = \xi$, and denote $P_0^L := \rho^{-1}_{L/K} \circ \pi \circ \rho_{L/K}(P^L)$. Then $P_0^L \in \rho^{-1}_{L/K}(Y_0)$ and $\lambda_L(P_0^L) = \xi$; hence, $\lambda_L|_{\rho^{-1}_{L/K}(Y_0)}$ is a surjection.

Now suppose that P_0^L and Q_0^L are two different orders in $\rho^{-1}_{L/K}(Y_0)$. Then $\rho_{L/K}(P_0^L) \neq \rho_{L/K}(Q_0^L)$ in Y_0, and by the injectivity of $\lambda_K|_{Y_0}$ we have $\lambda_K(\rho_{L/K}(P_0^L)) \neq \lambda_K(\rho_{L/K}(Q_0^L))$. Thus, $\lambda_L(P_0^L) \neq \lambda_L(Q_0^L)$ which proves the injectivity of λ_L on $\rho^{-1}_{L/K}(Y_0)$. $\square$

Remark 4.3. Since Y_0 is a closed subspace of $\mathcal{X}(K)$ and $\rho_{L/K}$ is continuous, $\rho^{-1}_{L/K}(Y_0)$ is a compact space. Since $M(L)$ is Hausdorff and λ_L is continuous, $\lambda_L|_{\rho^{-1}_{L/K}(Y_0)}$ is a homeomorphism. Therefore, $\lambda_K \circ \rho^{-1}_{L/K}$ is a homeomorphism from Y_0 onto $M(L)$.

Now we are in the position to prove the main theorem.

Theorem 4.4. *Every Boolean space is realized as a space of* **R***-places of some formally real field* L.

Proof. Take Y_0 to be any Boolean space and view it as a closed subspace of the Cantor cube $D_{\mathfrak{m}}$.

Let K be the field constructed in Theorem 3.2, so that $M(K) \cong D_{\mathfrak{m}} \cong H_K(X)$ and the map $\lambda_K : H_K(X) \to M(K)$ is a bijection. Now we can consider Y_0 as a closed subset of $H_K(X)$. By [**4**], K is an SAP field, in particular the Harrison subbasis is a basis of $\mathcal{X}(K)$. The complement $[Y_0]^c_{H_K(X)}$ of Y_0 in $H_K(X)$ is an open set; thus, there exists a subset $\mathcal{B} \subset K$ such that

$$[Y_0]^c_{H_K(X)} = \bigcup_{\beta \in \mathcal{B}} H_K(-\beta).$$

Take $\beta \in \mathcal{B}$. Note that

$$\lambda_K(H_K(\beta) \cap H_K(X)) \cap \lambda_K(H_K(-\beta) \cap H_K(X)) = \varnothing.$$

By the Separation Criterion [**8**, Proposition 9.13], there exists an $\alpha \in K$ which is a unit in the ring $A(P)$ for every $P \in [H_K(\beta) \cap H_K(X)] \cup [H_K(-\beta) \cap H_K(X)] = H_K(X)$ and $H_K(\beta) \cap H_K(X) \subset H_K(\alpha)$ and $H_K(-\beta) \cap H_K(X) \subset H_K(-\alpha)$. Since $\lambda_K : H_K(X) \to M(K)$ is a bijection, we have that α is a unit in $A(P)$ for every order P of K, and therefore $\alpha \in \mathbf{E}(K)$.

We shall show that $H_K(\beta) \cap H_K(X) = H_K(\alpha) \cap H_K(X)$. Since $H_K(-\beta) \cap H_K(X) \subset H_K(-\alpha)$ and $H_K(-\beta) \subset H_K(X)$, we have $H_K(-\beta) \subset H_K(-\alpha)$. Therefore, $H_K(\alpha) \subset H_K(\beta)$ and thus $H_K(\alpha) \cap H_K(X) \subset H_K(\beta) \cap H_K(X)$. The converse inclusion is obvious. Repeating the argument for arbitrary $\beta \in \mathcal{B}$ we get a subset $\mathcal{A} \subset \mathbf{E}(K)$ such that

$$Y_0 = \bigcap_{\beta \in \mathcal{B}} H_K(\beta) \cap H_K(X) = \bigcap_{\alpha \in \mathcal{A}} H_K(\alpha) \cap H_K(X).$$

Now it suffices to use Proposition 4.2, taking the set $\cap_{\alpha \in \mathcal{A}} H_K(\alpha)$ as Y, and the set $H_K(X)$ as H. $\square$

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