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# COMPLETE CONTINUITY PROPERTIES FOR THE FREMLIN PROJECTIVE TENSOR PRODUCT OF ORLICZ SEQUENCE SPACES AND BANACH LATTICES

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ABSTRACT. In this paper, we show that if $\varphi$ is an Orlicz function then $\ell_\varphi \widehat{\otimes}_F X$, the Fremlin projective tensor product of an Orlicz sequence space $\ell_\varphi$ and a Banach lattice $X$, has the complete continuity property (respectively, the analytic complete continuity property) if and only if both $\ell_\varphi$ and $X$ have the same property.

**1. Introduction.** Randrianantoanina and Saab in [**14**] showed that, for a finite measure $\mu$, the Bochner function space $L_p(\mu, X)$, $1 < p < \infty$, has the complete continuity property whenever the Banach space $X$ does. Bu in [**1**] showed that, for a $\sigma$-finite measure $\mu$, the Bochner function space $L_p(\mu, X)$, $1 \le p < \infty$, has the analytic complete continuity property whenever the Banach space $X$ does. Moreover, Randrianantoanina in [**13**] obtained several results about the complete continuity property and the analytic complete continuity property in the Köthe-Bochner function space $E(X)$, where $E$ is a Köthe function space and $X$ is a Banach space. In 2004, Dowling in [**5**] showed that $L_p[0,1]\widehat{\otimes}_\pi X$, the projective tensor product of $L_p[0,1]$, $1 < p < \infty$, and a Banach space $X$, has the complete continuity property and the analytic complete continuity property whenever $X$ has the same property. In addition, he also showed in [**6**] that if $U$ is a Banach space with an unconditional basis and $X$ is a Banach space, then $U\widehat{\otimes}_\pi X$, the projective tensor product of $U$ and $X$, has the complete continuity property and the analytic complete continuity property if and only if both $U$ and $X$ have the same property.

Fremlin in [**7, 8**] investigated the positive projective tensor product $X\widehat{\otimes}_F Y$ of Banach lattices $X$ and $Y$, for convenience, called the Fremlin projective tensor product. Dowling in [**6**] showed that $\ell_p\widehat{\otimes}_F X$, $1 < p <$

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$\infty$, has the complete continuity property and the analytic complete continuity property whenever the Banach lattice $X$ has the same property. In this paper, by using the sequential representation of $\ell_\varphi \widehat{\otimes}_F X$ given in [**3**], we show that $\ell_\varphi \widehat{\otimes}_F X$, where $\ell_\varphi$ is an Orlicz sequence space and $X$ is a Banach lattice, has the complete continuity property and the analytic complete continuity property whenever both $\ell_\varphi$ and $X$ have the same property.

**Notations.** For a Banach space $X$, $X^*$ denotes its dual space, and $B_X$ denotes its closed unit ball. For a Banach lattice $X$, $X^+$ denotes its positive cone. For $\overline{x} = (x_i)_i \in X^{\mathbf{N}}$ and $n \in \mathbf{N}$, we write

$$\overline{x}(\geq n) = (0, \dots, 0, x_n, x_{n+1}, \dots).$$

**2. Concepts and definitions.** A bounded linear operator from a Banach space $X$ to a Banach space $Y$ is called *completely continuous* (or *Dunford-Pettis*) if it takes weakly null sequences in $X$ into norm null sequences in $Y$. A Banach space $X$ is said to have the *complete continuity property* (CCP in short) if, for any finite measure space $(\Omega, \Sigma, \nu)$, each bounded linear operator from $L_1(\nu)$ to $X$ is completely continuous, or equivalently, if for any finite measure space $(\Omega, \Sigma, \nu)$, each countably additive $\nu$-continuous $X$-valued measure $\mu$ of bounded variation has a relatively compact range, i.e., $\{\mu(A) : A \in \Sigma\}$ is a relatively compact subset of $X$ (see [**12**]).

Let $X$ be a complex Banach space, $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$ be the unit circle of $\mathbf{C}$, let $\mathcal{B}$ be the $\sigma$-algebra of Borel subsets of $\mathbf{T}$ and let $\lambda$ be the normalized Lebesgue measure on $\mathbf{T}$. A countably additive $X$-valued measure $\mu$ of bounded variation is called *analytic* if its Fourier coefficients $\widehat{\mu}(n) = \int_{\mathbf{T}} e^{-int}\, d\mu(t) = 0$ for all $n < 0$. A complex Banach space $X$ is said to have the *analytic complete continuity property* if each analytic $X$-valued measure has a relatively compact range (see [**15**]).

Next let us introduce the complete continuity properties associated with subsets of a discrete abelian group. Let $G$ be a compact metrizable abelian group, let $\mathcal{B}(G)$ be the $\sigma$-algebra of Borel subsets of $G$, let $\lambda$ be a normalized Haar measure on $G$, and let $\Gamma$ be the dual group of $G$. For a real or complex Banach space $X$, we denote by $L_1(G, X)$, respectively $L_\infty(G, X)$, the Banach space of (all equivalence classes of) $\lambda$-Bochner integrable functions on $G$ with values in $X$, respectively (all equivalence classes of) $\lambda$-measurable $X$-valued functions that are essentially bounded. If $\mu$ is a countably additive $X$-valued measure on $\mathcal{B}(G)$, we say that it is *of bounded variation* if $\sup \sum_{A\in\pi} \|\mu(A)\| < \infty$, where the supremum is taken over all finite measurable partitions of $G$. The measure $\mu$ is said to be *of bounded average range* if there is a positive constant $c$ so that $\|\mu(A)\| \le c\lambda(A)$ for every $A \in \mathcal{B}(G)$. We denote by $\mathcal{M}_1(G, X)$ the space of all $X$-valued measures on $\mathcal{B}(G)$ that are of bounded variation and by $\mathcal{M}_\infty(G, X)$ the space of all $X$-valued measures on $\mathcal{B}(G)$ that are of bounded average range.

For $\gamma \in \Gamma$ and $f \in L_1(G, X)$, we define the *Fourier coefficient* of $f$ at $\gamma$ by

$$\widehat{f}(\gamma) = \int_G f(t)\overline{\gamma}(t)\,d\lambda(t).$$

Similarly, for $\mu \in \mathcal{M}_1(G, X)$, we define the *Fourier coefficient* of $\mu$ at $\gamma$ by

$$\widehat{\mu}(\gamma) = \int_G \overline{\gamma}(t)\,d\mu(t).$$

Let $\Lambda$ be a subset of $\Gamma$. A measure $\mu$ in $\mathcal{M}_1(G, X)$ is called a $\Lambda$-*measure* if $\widehat{\mu}(\gamma) = 0$ for all $\gamma \notin \Lambda$. A Banach space $X$ is said to have *the type* I-$\Lambda$-*complete continuity property* (I-$\Lambda$-CCP in short) if every $\Lambda$-measure $\mu$ in $\mathcal{M}_\infty(G, X)$ has a relatively compact range, and to have *the type* II-$\Lambda$-*complete continuity property* (II-$\Lambda$-CCP in short) if every $\Lambda$-measure $\mu$ in $\mathcal{M}_1(G, X)$ has a relatively compact range (see [**16**]; also see [**4**]).

*Remark* 1. If $G = \{-1, 1\}^{\mathbf{N}}$, the Cantor group, then $\Gamma = \{-1, 1\}^{(\mathbf{N})}$ and Fourier coefficients of measures on $\mathcal{B}(G)$ with values in a real Banach space are well-defined. If $\Lambda = \Gamma$, then I-$\Lambda$-CCP and II-$\Lambda$-CCP are equivalent, and equivalent to the usual complete continuity property (see [**5, 15**]).

*Remark* 2. If $G = \mathbf{T}$, then $\Gamma = \mathbf{Z}$ and Fourier coefficients of measures on $\mathcal{B}(G)$ with values in a complex Banach space are well defined. If $\Lambda = \mathbf{Z}$, then I-$\Lambda$-CCP and II-$\Lambda$-CCP are equivalent, and equivalent to the usual complete continuity property. If $\Lambda = \mathbf{N} \cup \{0\}$, then I-$\Lambda$-

CCP and II-$\Lambda$-CCP are equivalent, and equivalent to the usual analytic complete continuity property (see [**5, 15**]).

**3. CCP for Orlicz sequence spaces.** An *Orlicz function* is a continuous nondecreasing and convex function $\varphi : [0, +\infty) \to [0, +\infty)$ such that $\varphi(t) = 0$ only at $t = 0$ and $\lim_{t\to+\infty} \varphi(t) = +\infty$. Every Orlicz function $\varphi$ has a right derivative $p$ and

$$\varphi(t) = \int_0^t p(u)\, du, \quad t \geq 0.$$

If $p$ satisfies $p(0) = 0$ and $\lim_{t\to+\infty} p(t) = +\infty$ (these restrictions exclude only the case that (when $\varphi(t)$ is equivalent to $t$) $\ell_\varphi = \ell_1$), then the right inverse $q$ of $p$,

$$q(s) = \sup\{t : p(t) \leq s\}, \quad s \geq 0,$$

is a right continuous nondecreasing function such that $q(0) = 0$ and $q(s) > 0$ whenever $s > 0$. Define

$$\varphi^*(s) = \int_0^s q(u)\, du, \quad s \geq 0.$$

Then $\varphi^*$ is also an Orlicz function and $q$ is its right derivative. $\varphi^*$ is called the *function complementary* to $\varphi$ (see [**9**, page 147]).

We say that an Orlicz function $\varphi$ *has its complementary function* if its right derivative $p$ satisfies $p(0) = 0$ and $\lim_{t\to+\infty} p(t) = +\infty$. An Orlicz function $\varphi$ is said to satisfy the $\Delta_2$-*condition* (*at zero*) if there exist $K > 0$ and $t_0 > 0$ such that $\varphi(2t) \leq K\varphi(t)$ for every $0 < t \leq t_0$.

The *Orlicz sequence space* $\ell_\varphi$ is defined by

$$\ell_\varphi = \Big\{ a = (a_i)_i \in \mathbf{R}^{\mathbf{N}} : \sum_{i=1}^{\infty} \varphi(|\lambda a_i|) < +\infty \text{ for some } \lambda > 0 \Big\}.$$

The space $\ell_\varphi$, equipped with the norm

$$\|a\|_{\ell_\varphi} = \inf \Big\{ \lambda > 0 : \sum_{i=1}^{\infty} \varphi\Big(\frac{|a_i|}{\lambda}\Big) \leq 1 \Big\}, \quad a = (a_i)_i \in \ell_\varphi,$$

is a Banach space (see [**9**, Chapter 4]). For a Banach space $X$, the $X$-valued Orlicz sequence space $\ell_\varphi(X)$ is defined by

$$\ell_\varphi(X) = \Big\{\overline{x} = (x_i)_i \in X^{\mathbf{N}} : \sum_{i=1}^{\infty} \varphi(\|\lambda x_i\|) < +\infty \text{ for some } \lambda > 0\Big\}.$$

Then $\ell_\varphi(X)$, equipped with the norm

$$\|\overline{x}\|_{\ell_\varphi(X)} = \inf\Big\{\lambda > 0 : \sum_{i=1}^{\infty} \varphi\Big(\frac{\|x_i\|}{\lambda}\Big) \le 1\Big\}, \quad \overline{x} = (x_i)_i \in \ell_\varphi(X),$$

is a Banach space. It follows from [**9**, page 138, Proposition 4.a.4] that an Orlicz function $\varphi$ satisfies the $\Delta_2$-condition if and only if $\lim_n \|\overline{x}(\ge n)\|_{\ell_\varphi(X)} = 0$ for each $\overline{x} = (x_i)_i \in \ell_\varphi(X)$.

First we characterize compact subsets of $\ell_\varphi(X)$ as follows.

**Proposition 1.** *Let $\varphi$ be an Orlicz function satisfying the $\Delta_2$-condition. Then a subset $B$ of $\ell_\varphi(X)$ is relatively compact if and only if for each $i \in \mathbf{N}$, the set $\{x_i : \overline{x} = (x_i)_i \in B\}$ is a relatively compact subset of $X$, and*

$$\lim_n \sup\big\{\|\overline{x}(\ge n)\|_{\ell_\varphi(X)} : \overline{x} \in B\big\} = 0. \tag{1}$$

*Proof.* Suppose that $B$ is a relatively compact subset of $\ell_\varphi(X)$. It is easy to see that $\{x_i : \overline{x} = (x_i)_i \in B\}$ is a relatively compact subset of $X$ for each $i \in \mathbf{N}$. Next assume that (1) does not hold. Note that $\lim_n \|\overline{x}(\ge n)\|_{\ell_\varphi(X)} = 0$ for each $\overline{x} \in \ell_\varphi(X)$ since $\varphi$ satisfies the $\Delta_2$-condition. There exist an $\varepsilon_0 > 0$, $\overline{x}^{(k)} = (x_i^{(k)})_i \in B$ for each $k \in \mathbf{N}$, and a subsequence $n_1 < m_1 < n_2 < m_2 < \cdots$ such that

$$\Big\|\overline{x}^{(k)}(\ge n_k)\Big\|_{\ell_\varphi(X)} \ge \varepsilon_0, \quad k = 1, 2, \dots,$$

and

$$\Big\|\overline{x}^{(k)}(\ge m)\Big\|_{\ell_\varphi(X)} \le \varepsilon_0/2, \quad m > m_k,\ k = 1, 2, \dots.$$

For each $k, j \in \mathbf{N}$ with $k > j$,

$$\begin{aligned}\left\|\overline{x}^{(k)} - \overline{x}^{(j)}\right\|_{\ell_\varphi(X)} &\geq \left\|\overline{x}^{(k)}(\geq n_k) - \overline{x}^{(j)}(\geq n_k)\right\|_{\ell_\varphi(X)} \\ &\geq \left\|\overline{x}^{(k)}(\geq n_k)\right\|_{\ell_\varphi(X)} - \left\|\overline{x}^{(j)}(\geq n_k)\right\|_{\ell_\varphi(X)} \\ &\geq \varepsilon_0 - \varepsilon_0/2 = \varepsilon_0/2.\end{aligned}$$

Thus the sequence $\{\overline{x}^{(k)}\}_1^\infty$ in $B$ cannot have any limit point in $\ell_\varphi(X)$, which shows that $B$ is not a relatively compact subset of $\ell_\varphi(X)$. This contradiction shows that (1) holds.

On the other hand, suppose that each $\{x_i : \overline{x} = (x_i)_i \in B\}$ is a relatively compact subset of $X$ and (1) holds. Take a sequence $\{\overline{x}^{(m)}\}_1^\infty$ in $B$. By the diagonal method, there exists a subsequence $\{\overline{x}^{(m_k)}\}_1^\infty$ of $\{\overline{x}^{(m)}\}_1^\infty$ such that

$$\lim_k x_i^{(m_k)} \text{ exists in } X \text{ for each } i \in N. \tag{2}$$

For each $\varepsilon > 0$, there exists, by (1), an $n_0 \in \mathbf{N}$ such that

$$\|\overline{x}(\geq n_0 + 1)\|_{\ell_\varphi(X)} < \varepsilon/4, \quad \overline{x} \in B.$$

By (2), there exists a $k_0 \in \mathbf{N}$ such that for each $k, j \in \mathbf{N}$ with $k, j > k_0$,

$$\left\|x_i^{(m_k)} - x_i^{(m_j)}\right\|_X < \varepsilon/2n_0, \quad i = 1, 2, \dots, n_0.$$

Thus for each $k, j \in \mathbf{N}$ with $k, j > k_0$,

$$\begin{aligned}\left\|\overline{x}^{(m_k)} - \overline{x}^{(m_j)}\right\|_{\ell_\varphi(X)} &\leq \sum_{i=1}^{n_0} \left\|x_i^{(m_k)} - x_i^{(m_j)}\right\|_X \\ &\quad + \left\|\overline{x}^{(m_k)}(\geq n_0 + 1)\right\|_{\ell_\varphi(X)} \\ &\quad + \left\|\overline{x}^{(m_j)}(\geq n_0 + 1)\right\|_{\ell_\varphi(X)} \\ &< \varepsilon/2 + \varepsilon/4 + \varepsilon/4 = \varepsilon.\end{aligned}$$

Therefore $\{\overline{x}^{(m_k)}\}_1^\infty$ is a Cauchy sequence in $\ell_\varphi(X)$ and hence a convergent sequence in $\ell_\varphi(X)$. This shows that $B$ is a relatively compact subset of $\ell_\varphi(X)$. □

**Corollary 2.** *Let $\varphi$ be an Orlicz function satisfying the $\Delta_2$-condition. Then a subset $B$ of $\ell_\varphi$ is relatively compact if and only if for each $i \in \mathbf{N}$, the set $\{a_i : a = (a_i)_i \in B\}$ is a bounded subset of $\mathbf{R}$, and $\lim_n \sup\{\|a(\geq n)\|_{\ell_\varphi} : a = (a_i)_i \in B\} = 0$.*

Next we present a characterization of $\ell_\varphi(X)$ having types of CCP.

**Theorem 3.** *Let $\varphi$ be an Orlicz function satisfying the $\Delta_2$-condition. Let $G$ be a compact metrizable abelian group, $\Gamma$ the dual group of $G$ and $\Lambda$ a subset of $\Gamma$. Then $\ell_\varphi(X)$ has I-$\Lambda$-CCP, respectively II-$\Lambda$-CCP, if and only if both $\ell_\varphi$ and $X$ have I-$\Lambda$-CCP, respectively II-$\Lambda$-CCP.*

*Proof.* We will give the proof for II-$\Lambda$-CCP. The similar proof for I-$\Lambda$-CCP will be omitted. We need only to show that if $\ell_\varphi$ and $X$ have II-$\Lambda$-CCP then $\ell_\varphi(X)$ has II-$\Lambda$-CCP.

Let $\mu : \mathcal{B}(G) \to \ell_\varphi(X)$ be a $\lambda$-continuous, $\Lambda$-measure of bounded variation. We want to show that $\{\mu(E) : E \in \mathcal{B}(G)\}$, the range of $\mu$, is a relatively compact subset of $\ell_\varphi(X)$. For each $i \in \mathbf{N}$, define

$$\mu_i : \mathcal{B}(G) \longrightarrow X, \qquad E \longmapsto \mu(E)_i,$$

where $\mu(E)_i$ is the $i$th coordinator of $\mu(E)$. Note that for each $E \in \mathcal{B}(G)$, $\|\mu_i(E)\|_X \leq \|\mu(E)\|_{\ell_\varphi(X)}$. Each $\mu_i$ is also a $\lambda$-continuous, $\Lambda$-measure of bounded variation. Since $X$ has II-$\Lambda$-CCP,

$$\{\mu(E)_i = \mu_i(E) : E \in \mathcal{B}(G)\} \tag{3}$$

is a relatively compact subset of $X$ for each $i \in \mathbf{N}$. Define

$$\widetilde{\mu} : \mathcal{B}(G) \longrightarrow \ell_\varphi, \qquad E \longmapsto (\|\mu(E)_i\|_X)_i.$$

Note that, for each $E \in \mathcal{B}(G)$, $\|\widetilde{\mu}(E)\|_{\ell_\varphi} = \|\mu(E)\|_{\ell_\varphi(X)}$. Thus $\widetilde{\mu}$ is a $\lambda$-continuous, $\Lambda$-measure of bounded variation. Since $\ell_\varphi$ has II-$\Lambda$-CCP, $\{\widetilde{\mu}(E) : E \in \mathcal{B}(G)\}$ is a relatively compact subset of $\ell_\varphi$. By Corollary 2,

$$\lim_n \sup\left\{\left\|(0, \ldots, 0, \|\mu(E)_n\|_X, \|\mu(E)_{n+1}\|_X, \ldots)\right\|_{\ell_\varphi} : E \in \mathcal{B}(G)\right\} = 0.$$

That is,
(4)
$$\lim_n \sup\left\{\left\|\left(0,\dots,0,\mu(E)_n,\mu(E)_{n+1},\dots\right)\right\|_{\ell_\varphi(X)} : E\in\mathcal{B}(G)\right\}=0.$$

It follows from (3), (4) and Proposition 1 that the set $\{\mu(E) : E \in \mathcal{B}(G)\}$ is a relatively compact subset of $\ell_\varphi(X)$. □

**4. CCP for the Fremlin projective tensor product.** Throughout this section, $X$ will be a Banach lattice. For an Orlicz function $\varphi$, define

$$\ell_\varphi^\varepsilon(X)=\left\{\overline{x}=(x_i)_i\in X^{\mathbf{N}} : \left(x^*(|x_i|)\right)_i\in\ell_\varphi,\ \text{for all } x^*\in X^{*+}\right\}$$

and

$$\|\overline{x}\|_{\ell_\varphi^\varepsilon(X)}=\sup\left\{\left\|\left(x^*(|x_i|)\right)_i\right\|_{\ell_\varphi} : x^*\in B_{X^{*+}}\right\},\quad \text{for all } \overline{x}=(x_i)_i\in\ell_\varphi^\varepsilon(X).$$

Then $\ell_\varphi^\varepsilon(X)$ is a Banach lattice (see [**3**]). For an Orlicz function $\varphi$ that has its complementary function $\varphi^*$, define

$$\ell_\varphi^\pi(X)=\left\{\overline{x}=(x_i)_i\in X^{\mathbf{N}} : \sum_{i=1}^\infty x_i^*(|x_i|)<+\infty,\quad \text{for all } (x_i^*)_i\in\ell_{\varphi^*}^\varepsilon(X^*)^+\right\}$$

and

$$\|\overline{x}\|_{\ell_\varphi^\pi(X)}=\sup\left\{\sum_{i=1}^\infty x_i^*(|x_i|) : (x_i^*)_i\in B_{\ell_{\varphi^*}^\varepsilon(X^*)^+}\right\},\ \text{for all } \overline{x}\in\ell_\varphi^\pi(X).$$

Then $\ell_\varphi^\pi(X)$ is a Banach lattice (see [**3**]).

For Banach lattices $X$ and $Y$, the *projective cone* on the algebraic tensor product $X\otimes Y$ is defined to be (see [**11**, page 229])

$$C_p=\left\{\sum_{k=1}^n x_k\otimes y_k : n\in\mathbf{N},\ x_k\in X^+,\ y_k\in Y^+\right\}.$$

Fremlin [**7, 8**] introduced the *positive projective tensor norm* on $X \otimes Y$ as follows:

$$\|u\|_{|\pi|} = \sup\left\{|\sum_{k=1}^{n}\psi(x_k, y_k)| : u = \sum_{k=1}^{n} x_k \otimes y_k \in X \otimes Y,\ \psi \in M\right\},$$

where $M$ is the set of all positive bilinear functionals $\psi$ on $X \times Y$ with $\|\psi\| \leq 1$. Let $X\widehat{\otimes}_F Y$ denote the completion of $X \otimes Y$ with respect to the norm $\|\cdot\|_{|\pi|}$. Then $X\widehat{\otimes}_F Y$ with $C_p$ as its positive cone is a Banach lattice (see [**7, 8**]), called the *Fremlin projective tensor product* of $X$ and $Y$. Bu, Buskes and Lai in [**3**] gave a sequential representation of $\ell_\varphi\widehat{\otimes}_F X$ as follows.

**Proposition 4.** *Let $X$ be a Banach lattice and $\varphi$ an Orlicz function that has its complementary function. If $\varphi$ satisfies the $\Delta_2$-condition, then $\ell_\varphi\widehat{\otimes}_F X$ is isometrically lattice isomorphic to $\ell_\varphi^\pi(X)$.*

Recall that a Banach space $X$ is said to *semi-embed into* a Banach space $Y$ if there is a one-to-one continuous linear operator from $X$ to $Y$ such that the image of the closed unit ball of $X$ is closed in $Y$ (see [**10**]). A Banach space property $\mathcal{P}$ is said (i) to be *separably determined* if a Banach space $X$ has $\mathcal{P}$ whenever every separable closed subspace of $X$ has $\mathcal{P}$, and (ii) to be *separably semi-embeddably stable* if a separable Banach space $X$ has $\mathcal{P}$ whenever $X$ semi-embeds into a Banach space $Y$ with $\mathcal{P}$. Conditions (i) and (ii) imply that if a Banach space $X$ has $\mathcal{P}$, then so does every closed subspace of $X$. Conditions (i) and (ii) also imply that if $X$ and $Y$ are isomorphic Banach spaces and $Y$ has $\mathcal{P}$, then $X$ has $\mathcal{P}$, see [**6**].

**Theorem 5.** *Let $X$ be a Banach lattice, and let $\varphi$ be an Orlicz function that has its complementary function and satisfies the $\Delta_2$-condition. Let $\mathcal{P}$ be a Banach space property such that $\mathcal{P}$ is separably determined and separably semi-embeddably stable. If $\ell_\varphi(X)$ has $\mathcal{P}$, then $\ell_\varphi\widehat{\otimes}_F X$ also has $\mathcal{P}$.*

*Proof.* To show $\ell_\varphi\widehat{\otimes}_F X$ has $\mathcal{P}$, it suffices to show that every separable closed sublattice $S$ of $\ell_\varphi\widehat{\otimes}_F X$ has $\mathcal{P}$. By [**3**, Proposition 6.3], there exists a separable closed sublattice $Z$ of $X$ such that $S$ is a sublattice of $\ell_\varphi \widehat{\otimes}_F Z$. By Proposition 4, to show $S$ has $\mathcal{P}$, it suffices to show $\ell_\varphi^\pi(Z)$ has $\mathcal{P}$. By [**3**, Proposition 3.6], $\ell_\varphi^\pi(Z)$ semi-embeds into $\ell_\varphi(Z)$. Note that, for each $\overline{x} = (x_i)_i \in \ell_\varphi^\pi(Z)$, $\lim_n \|\overline{x}(\geq n)\|_{\ell_\varphi^\pi(Z)} = 0$ (see [**3**, Proposition 3.5]). Thus $\ell_\varphi^\pi(Z)$ is separable and hence, to show $\ell_\varphi^\pi(Z)$ has $\mathcal{P}$, it suffices to show $\ell_\varphi(Z)$ has $\mathcal{P}$. Note that $\ell_\varphi(Z)$ is a separable closed subspace of $\ell_\varphi(X)$. $\ell_\varphi(Z)$ has $\mathcal{P}$ and the proof is complete. □

Note that I-$\Lambda$-CCP and II-$\Lambda$-CCP are separably determined and separably semi-embeddably stable (see [**16**]). The following result is a consequence of Theorem 3 and Theorem 5.

**Corollary 6.** *Let $X$ be a Banach lattice, and let $\varphi$ be an Orlicz function that has its complementary function and satisfies the $\Delta_2$-condition. Let $G$ be a compact metrizable abelian group, $\Gamma$ the dual group of $G$ and $\Lambda$ a subset of $\Gamma$. Then $\ell_\varphi \widehat{\otimes}_F X$ has* I-$\Lambda$-CCP, *respectively* II-$\Lambda$-CCP, *if and only if both $\ell_\varphi$ and $X$ have* I-$\Lambda$-CCP, *respectively* II-$\Lambda$-CCP.

It is known that $c_0$ does not have the (analytic) complete continuity property. Thus, if $\ell_\varphi$ has the (analytic) complete continuity property, then $\ell_\varphi$ contains no copy of $c_0$ and hence, no copy of $\ell_\infty$. By [**9**, page 138, Proposition 4.a.4], $\varphi$ satisfies the $\Delta_2$-condition. Thus, we have our main results as follows.

**Corollary 7.** *Let $X$ be a Banach lattice, and let $\varphi$ be an Orlicz function that has its complementary function. Then $\ell_\varphi \widehat{\otimes}_F X$ has the complete continuity property, respectively the analytic complete continuity property, if and only if both $\ell_\varphi$ and $X$ have the same property.*

Note that if an Orlicz function $\varphi$ does not have its complementary function then $\varphi(t)$ is equivalent to $t$ and hence $\ell_\varphi = \ell_1$. In this case, $\ell_1 \widehat{\otimes}_F X = \ell_1 \widehat{\otimes}_\pi X$. Thus $\ell_1 \widehat{\otimes}_F X$ has the complete continuity property, respectively the analytic complete continuity property, whenever $X$ has the same property (see [**6**]). Now we can formulate Corollary 7 as follows.

**Corollary 8.** *Let $\varphi$ be an Orlicz function. Then $\ell_\varphi \widehat{\otimes}_F X$, the Fremlin projective tensor product of an Orlicz sequence space $\ell_\varphi$ and a Banach* *lattice $X$, has the complete continuity property, respectively the analytic complete continuity property, if and only if both $\ell_\varphi$ and $X$ have the same property.*

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