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SIMPLICITY OF THE PROJECTIVE UNITARY GROUP OF THE MULTIPLIER ALGEBRA OF A SIMPLE STABLE NUCLEAR C^*-ALGEBRA

P.W. NG AND EFREN RUIZ

ABSTRACT. If $\mathcal{A}$ is a unital simple separable nuclear C^*-algebra with real rank zero, then $U(\mathcal{M}(\mathcal{A}\otimes\mathcal{K}))/\mathbf{T}$, given the quotient topology induced by the strict topology on $U(\mathcal{M}(\mathcal{A}\otimes\mathcal{K}))$, is a simple topological group. (Here, $U(\mathcal{M}(\mathcal{A}\otimes\mathcal{K}))$ is the unitary group of the multiplier algebra of the stabilization of $\mathcal{A}$. $\mathbf{T}$ is the subgroup of scalar unitaries.)

If $\mathcal{A}$ is a unital simple separable C^*-algebra, then $U(\mathcal{A})/\mathbf{T}$, given the quotient topology induced by the relative weak topology on $U(\mathcal{A})$, is a simple topological group.

1. Introduction. A topological group G is *simple* if it has no proper nontrivial closed normal subgroups. Simple topological groups play a fundamental role in many places. (Some examples are the connected simple Lie groups with trivial center, for which there is a complete classification. See, for example, [**5**].) In this paper, we study the simplicity of certain topological groups associated with simple C^*-algebras. Consider the case of the full matrix algebras $\mathbf{M}_n(\mathbf{C})$. In this case, the unitary group $U(\mathbf{M}_n(\mathbf{C}))$, given the norm topology, is *not* simple. However, when we take the quotient by the scalar unitaries, i.e., the center, we get the projective unitary group $U(\mathbf{M}_n(\mathbf{C}))/\mathbf{T}$ which if given the quotient topology induced by the norm topology of $U(\mathbf{M}_n(\mathbf{C}))$ is a simple topological group. We are interested in infinite dimensional generalizations of this result, which will necessarily involve interesting nonlocally compact topological groups.

The first infinite dimensional generalizations were due to Kadison who studied the case of von Neumann factors. In [**6**], Kadison showed that if $\mathcal{M}$ is a type II_1 factor or a type III factor then $U(\mathcal{M})/\mathbf{T}$, given the quotient topology induced by the norm topology on $U(\mathcal{M})$, is a simple topological group. However, if $\mathcal{M}$ is a type I factor or a

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type II_∞ factor then $U(\mathcal{M})/\mathbf{T}$, given the same topology as before, has a unique proper nontrivial closed normal subgroup.

We note that the above results all involve the (quotient topology induced by the) *norm topology* on a von Neumann algebra. If we were to use one of the weaker von Neumann algebra topologies, then the picture becomes simpler. (We note that for a von Neumann algebra, the weak, weak*, strong, σ-strong, strong*, and σ-strong* topologies all coincide on the unitary group. See [**9**].) In particular, the following theorem follows immediately from the work of Kadison in [**6**]:

Theorem 1.1. *If $\mathcal{M}$ is a von Neumann factor, then $U(\mathcal{M})/\mathbf{T}$, given the quotient topology induced by the* weak**-topology on $U(\mathcal{M})$, is a simple topological group.*

Our first goal is to study multiplier algebra versions of the above result. The multiplier algebra $\mathcal{M}(\mathcal{B})$ of a C^*-algebra $\mathcal{B}$ is the largest unital C^*-algebra containing $\mathcal{B}$ as an essential ideal. $\mathcal{M}(\mathcal{B})$ encodes the extension theory of $\mathcal{B}$ and is an important object in K-theory as well as classification theory [**1, 8**].

$\mathcal{M}(\mathcal{B})$ sits in between $\mathcal{B}$ and its second dual von Neumann algebra $\mathcal{B}^{**}$, and (like a von Neumann algebra), $\mathcal{M}(\mathcal{B})$ has more than one interesting natural topology. In particular, $\mathcal{M}(\mathcal{B})$ has another natural topology (other than the norm topology) called the "strict topology." The strict topology on $\mathcal{M}(\mathcal{B})$ is the topology on $\mathcal{M}(\mathcal{B})$ induced by the family of semi-norms $\{\|.\|_b\}_{b\in\mathcal{B}}$, where for all $b \in \mathcal{B}$ and $m \in \mathcal{M}(\mathcal{B})$, $\|m\|_b =_{df} \|mb\| + \|bm\|$. The strict topology on $\mathcal{M}(\mathcal{B})$ plays a role similar to the weak* topology on the von Neumann algebra $\mathcal{B}^{**}$. For example, just as $\mathcal{B}^{**}$ is the weak* topology closure of $\mathcal{B}$, $\mathcal{M}(\mathcal{B})$ is the strict topology closure of $\mathcal{B}$. (See [**1, 8, 11**].)

Our first result is the following:

Theorem 1.2. *Let $\mathcal{A}$ be a unital separable simple nuclear C^*-algebra with real rank zero. Let $U(\mathcal{M}(\mathcal{A}\otimes\mathcal{K}))$ be the unitary group of the multiplier algebra of the stabilization of $\mathcal{A}$, given the strict topology. Then $U(\mathcal{M}(\mathcal{A}\otimes\mathcal{K}))/\mathbf{T}$, given the quotient topology induced by the strict topology on $U(\mathcal{M}(\mathcal{A}\otimes\mathcal{K}))$, is a simple topological group.*

We note that the above theorem would not be true if we had used the norm topology instead of the strict topology. For example, there exists a unital simple AF-algebra $\mathcal{A}$ such that $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))/\mathbf{T}$, with the quotient topology induced by the *norm* topology on $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))$, has (uncountably) infinitely many distinct proper closed normal subgroups.

In another direction, de la Harpe and Skandalis, and Elliott and Rordam, proved the following [**2, 4**]:

Theorem 1.3. *Let $\mathcal{A}$ be a simple unital separable C^*-algebra. Suppose that either*

i) *$\mathcal{A}$ has real rank zero, stable rank one and weak unperforation, or*

ii) *$\mathcal{A}$ is purely infinite.*

Then $U(\mathcal{A})_0/\mathbf{T}$, given the quotient topology induced by the norm topology on $U(\mathcal{A})_0$, is a simple topological group. (*Here, $U(\mathcal{A})_0$ is the group of unitaries in $\mathcal{A}$ that are in the connected component of the identity.*)

A natural question is whether or not the above theorem is true for arbitrary simple unital separable C^*-algebra $\mathcal{A}$. Thomsen showed that this is not the case. Specifically, Thomsen gave an example of a unital simple AI-algebra $\mathcal{A}$ with real rank one such that $U(\mathcal{A})_0/\mathbf{T}$, with the quotient topology induced by the norm topology on $U(\mathcal{A})_0$, is not a simple topological group [**4, 10**].

We show, however, that by modifying the topology, $U(\mathcal{A})/\mathbf{T}$ becomes a simple topological group. In particular, we will replace the norm topology by the *relative weak topology* on $U(\mathcal{A})$. For a unital C^*-algebra $\mathcal{A}$, the *relative weak topology* on $\mathcal{A}$ is the weak topology given by all the linear functionals in $\mathcal{A}^*$. Our second result is the following:

Theorem 1.4. *Let $\mathcal{A}$ be a unital separable simple C^*-algebra. Then $U(\mathcal{A})/\mathbf{T}$, given the quotient topology induced by the relative weak topology on $U(\mathcal{A})$, is a simple topological group.*

We end this introduction with a question:

Question 1.5. *Let $\mathcal{A}$ be a unital separable simple nuclear C^*-algebra.*

Then is it the case that $U(\mathcal{M}(\mathcal{A}\otimes\mathcal{K}))/\mathbf{T}$, given the quotient topology induced by the strict topology on $U(\mathcal{M}(\mathcal{A}\otimes\mathcal{K}))$, is a simple topological group?

2. The projective unitary group of the multiplier algebra of a nuclear C^*-algebra.

Towards our goal, we first need some definitions and results from the theory of absorbing extensions.

Definition 2.1. Let $\mathcal{C}$ be a unital separable C^*-algebra and let $\mathcal{A}$ be a unital simple separable C^*-algebra. Let $\pi : \mathcal{C} \to \mathbf{B}(\mathbf{H})$ be a unital essential $*$-representation. Then a *Kasparov extension* of $\mathcal{A}\otimes\mathcal{K}$ by $\mathcal{C}$ is a unital $*$-homomorphism $\phi : \mathcal{C} \to \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ given by $\phi : c \mapsto 1_{\mathcal{A}}\otimes\pi(c)$ for every $c \in \mathcal{C}$.

Definition 2.2. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and $\phi, \psi : \mathcal{A} \to \mathcal{B}$ be $*$-homomorphisms. ϕ and ψ are said to be *approximately unitarily equivalent* if there exists a sequence of unitaries $\{u_n\}_{n=1}^{\infty}$ in $\mathcal{M}(\mathcal{B})$ such that

$$\lim_{n\to\infty} \|u_n\phi(a)u_n^* - \psi(a)\| = 0$$

for all $a \in \mathcal{A}$.

For our purposes, we need only the following result. See [**3**] for a proof of the result.

Theorem 2.1. *Let $\mathcal{C}$ be a unital separable C^*-algebra, and let $\mathcal{A}$ be a unital simple separable nuclear C^*-algebra. Let $\phi : \mathcal{C} \to \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ be a Kasparov extension of $\mathcal{A}\otimes\mathcal{K}$ by $\mathcal{C}$, and let $\psi : \mathcal{C} \to \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ be any unital $*$-homomorphism. Let $S_1, S_2 \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ be isometries such that $S_1(S_1)^* + S_2(S_2)^* = 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$. (In other words, S_1, S_2 generate a unital copy of the Cuntz algebra O_2 in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$.) Consider the unital $*$-homomorphism $\Psi : \mathcal{C} \to \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ that is given by*

$$\Phi(c) =_{df} S_1\phi(c)(S_1)^* + S_2\psi(c)(S_2)^*$$

for all $c \in \mathcal{C}$. (In the terminology of extension theory, Φ is the BDF-sum of ϕ and ψ.) Then Φ and ϕ are approximately unitarily equivalent.

Lemma 2.2. *Let $\mathcal{A}$ be a unital separable simple nuclear C^*-algebra. Then the set of all unitaries, with finite spectrum and with nonzero spectral projections all Murray-von Neumann equivalent to $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$, is strictly dense in $U(\mathcal{M}(\mathcal{A}\otimes\mathcal{K}))$.*

Proof. Let $\mathbf{T}$ be the unit circle, i.e., $\mathbf{T} = S^1$, and let $\pi : C(\mathbf{T}) \to \mathbf{B}(\mathcal{H})$ be a unital essential $*$-homomorphism. Let $\phi : C(\mathbf{T}) \to \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ be the unital $*$-homomorphism that is given by $\phi = 1 \otimes \pi$. In other words, ϕ is a Kasparov extension of $\mathcal{A} \otimes \mathcal{K}$ by $C(\mathbf{T})$.

Let U be an element of $U(\mathcal{M}(\mathcal{A}\otimes\mathcal{K}))$. It suffices to show that U can be approximated in the strict topology by unitaries (in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$) with finite spectrum and with nonzero spectral projections all Murray–von Neumann equivalent to $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$. So let $\varepsilon > 0$ be given, and let $\mathcal{F}$ be a finite subset of $\mathcal{A} \otimes \mathcal{K}$. Contracting $\varepsilon > 0$ if necessary, we may assume that the elements of $\mathcal{F}$ all have norm less than or equal to one. Let $\{e_{i,j}\}_{1\le i,j<\infty}$ be a system of matrix units for $\mathcal{K}$. Therefore, $\{\sum_{i=1}^{n}(1_{\mathcal{A}} \otimes e_{i,i})\}_{n=1}^{\infty}$ is an approximate unit for $\mathcal{A}\otimes\mathcal{K}$, consisting of projections. Hence, there exists an $N \ge 1$ such that

i) For all $b \in \mathcal{F}$ and for all $n \ge N$, $(\sum_{i=1}^{n} 1_{\mathcal{A}} \otimes e_{i,i}) b (\sum_{i=1}^{n} 1_{\mathcal{A}} \otimes e_{i,i})$, $(\sum_{i=1}^{n} 1_{\mathcal{A}} \otimes e_{i,i}) b$, $b(\sum_{i=1}^{n} 1_{\mathcal{A}} \otimes e_{i,i})$ and b are all within $\varepsilon/100$ of each other.

Choose $M \ge N$ such that

ii) for all $m \ge M$ and for all $n \le N$, $(\sum_{i=1}^{m} 1_{\mathcal{A}} \otimes e_{i,i}) U (\sum_{i=1}^{n} 1_{\mathcal{A}} \otimes e_{i,i})$ is within $\varepsilon/100$ of $U(\sum_{i=1}^{n} 1_{\mathcal{A}} \otimes e_{i,i})$; and

iii) for all $m \ge M$ and for all $n \le N$, $(\sum_{i=1}^{n} 1_{\mathcal{A}} \otimes e_{i,i}) U (\sum_{i=1}^{m} 1_{\mathcal{A}} \otimes e_{i,i})$ is within $\varepsilon/100$ of $(\sum_{i=1}^{n} 1_{\mathcal{A}} \otimes e_{i,i})U$.

We collectively denote the above statements by "$(*)$."

Let $S \in 1_{\mathcal{A}} \otimes \mathbf{B}(\mathcal{H})$ be an isometry with range projection $\sum_{i=1}^{2M} 1_{\mathcal{A}} \otimes e_{i,i} + \sum_{i=M+1}^{\infty} 1_{\mathcal{A}} \otimes e_{2i,2i}$ such that $S(\sum_{i=1}^{2M} 1_{\mathcal{A}} \otimes e_{i,i}) = (\sum_{i=1}^{2M} 1_{\mathcal{A}} \otimes e_{i,i})S = \sum_{i=1}^{2M} 1_{\mathcal{A}} \otimes e_{i,i}$. Let $T \in 1_{\mathcal{A}} \otimes \mathbf{B}(\mathcal{H})$ be an isometry with range projection $\sum_{i=M}^{\infty} 1_{\mathcal{A}} \otimes e_{2i+1,2i+1}$. Then $SS^* + TT^* = 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$. (In other words, S, T generate a unital copy of O_2 in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$.)

Let $\pi : C(\mathbf{T}) \to \mathbf{B}(\mathcal{H})$ be a unital essential $*$-representation. Let $\phi : C(\mathbf{T}) \to \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ be the unital injective $*$-homomorphism given by $\phi = 1_{\mathcal{A}} \otimes \pi$. Then ϕ is a Kasparov extension. Let id : $\mathbf{T} \to \mathbf{T}$

be the identity map $\mathrm{id}(\alpha) = \alpha$. (So id is the generator of $C(\mathbf{T})$.) Let $\psi : C(\mathbf{T}) \to \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ be the unique unital $*$-homomorphism given by $\psi(\mathrm{id}) = U$. Let $\Phi : C(\mathbf{T}) \to \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ be the unital injective $*$-homomorphism given by $\Phi(f) =_{df} S\psi(f)S^* + T\phi(f)T^*$ for all $f \in C(\mathbf{T})$. Then by Theorem 2.1, there exists a sequence of unitaries $\{U_n\}_{n=1}^{\infty}$ in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ such that

$$U_n\phi(f)(U_n)^* \longrightarrow \Phi(f)$$

in norm for all $f \in C(\mathbf{T})$.

Choose $L \geq 1$ such that

$$\|(U_L)^*\phi(\mathrm{id})U_L - \Phi(id)\| < \varepsilon/2.$$

Hence,

$$\|(U_L)^*\phi(\mathrm{id})U_L - (SUS^* + T\phi(\mathrm{id})T^*)\| < \varepsilon/20.$$

By the definition of T, we have that

$$T^* \sum_{i=1}^{N} 1_{\mathcal{A}} \otimes e_{i,i} = \bigg(\sum_{i=1}^{N} 1_{\mathcal{A}} \otimes e_{i,i}\bigg) T = 0.$$

Hence, we must have that

$$\|((U_L)^*\phi(\mathrm{id})U_L - SUS^*)\sum_{i=1}^{N} 1_{\mathcal{A}} \otimes e_{i,i}\| < \varepsilon/20.$$

We denote the above inequality by "$(**)$."

From the definition of S and from $(*)$, we see that

$$\begin{aligned}
&\Big\|(SUS^* - U)\sum_{i=1}^{N} 1_{\mathcal{A}} \otimes e_{i,i}\Big\| \\
&\quad = \Big\|SU\Big(\sum_{i=1}^{N} 1_{\mathcal{A}} \otimes e_{i,i}\Big) - U\Big(\sum_{i=1}^{N} 1_{\mathcal{A}} \otimes e_{i,i}\Big)\Big\| \\
&\quad \leq \Big\|SU\Big(\sum_{i=1}^{N} 1_{\mathcal{A}} \otimes e_{i,i}\Big) - \Big(\sum_{i=1}^{M} 1_{\mathcal{A}} \otimes e_{i,i}\Big)U(\sum_{i=1}^{N} 1_{\mathcal{A}} \otimes e_{i,i})\Big\| \\
&\qquad + \Big\|\Big(\sum_{i=1}^{M} 1_{\mathcal{A}} \otimes e_{i,i}\Big)U\Big(\sum_{i=1}^{N} 1_{\mathcal{A}} \otimes e_{i,i}\Big) - U\Big(\sum_{i=1}^{N} 1_{\mathcal{A}} \otimes e_{i,i}\Big)\Big\| \\
&\quad < \Big\|\Big(S - \Big(\sum_{i=1}^{M} 1_{\mathcal{A}} \otimes e_{i,i}\Big)\Big)U\Big(\sum_{i=1}^{N} 1_{\mathcal{A}} \otimes e_{i,i}\Big)\Big\| + \varepsilon/20 \\
&\quad = \Big\|\Big(S^*S - S^*\Big(\sum_{i=1}^{M} 1_{\mathcal{A}} \otimes e_{i,i}\Big)\Big)U\Big(\sum_{i=1}^{N} 1_{\mathcal{A}} \otimes e_{i,i}\Big)\Big\| + \varepsilon/20 \\
&\quad = \Big\|\Big(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - \Big(\sum_{i=1}^{M} 1_{\mathcal{A}} \otimes e_{i,i}\Big)\Big)U\Big(\sum_{i=1}^{N} 1_{\mathcal{A}} \otimes e_{i,i}\Big)\| + \varepsilon/20 \\
&\quad < \varepsilon/20 + \varepsilon/20 = \varepsilon/10.
\end{aligned}$$

From this and $(**)$, we have that

$$\|((U_L)^*\phi(id)U_L - U)\sum_{i=1}^{N} 1_{\mathcal{A}} \otimes e_{i,i}\| < \varepsilon/20 + \varepsilon/10 = 3\varepsilon/20.$$

We denote the above statement by "$(***)$."

By $(*)$, we have that, for all $b \in \mathcal{F}$,

$$\Big\|\Big(\sum_{i=1}^{N} 1_{\mathcal{A}} \otimes e_{i,i}\Big)b - b\Big\| < \varepsilon/20.$$

From this and $(***)$ (and since the elements of $\mathcal{F}$ all have norm less than or equal to one), we get the following for all $b \in \mathcal{F}$:

$$\begin{aligned}
&\|((U_L)^*\phi(id)U_L - U)b\| \\
&\qquad \leq \left\|((U_L)^*\phi(\mathrm{id})U_L - U)\left(b - \left(\sum_{i=1}^{N} 1_{\mathcal{A}} \otimes e_{i,i}\right)b\right)\right\| \\
&\qquad\quad + \left\|((U_L)^*\phi(\mathrm{id})U_L - U)\left(\left(\sum_{i=1}^{N} 1_{\mathcal{A}} \otimes e_{i,i}\right)b\right)\right\| \\
&\qquad < \varepsilon/10 + 3\varepsilon/20 = 5\varepsilon/20 = \varepsilon/4.
\end{aligned}$$

By a similar argument,

$$\|b((U_L)^*\phi(\mathrm{id})U_L - U)\| < \varepsilon/4$$

for all $b \in \mathcal{F}$. Hence,

$$\|(U_L)^*\phi(\mathrm{id})U_L - U\|_b < \varepsilon/2$$

for all $b \in \mathcal{F}$. We denote this inequality by "$(* * **)$."

Now any unitary in $1_{\mathcal{A}} \otimes \mathbf{B}(\mathcal{H})$ can be approximated "arbitrarily close" in the strict topology by unitaries in $1_{\mathcal{A}} \otimes \mathbf{B}(\mathcal{H})$ with finite spectrum and with nonzero spectral projections all Murray–von Neumann equivalent to $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$. Hence, let $\{V_n\}_{n=1}^{\infty}$ be a sequence of finite spectrum unitaries in $1_{\mathcal{A}} \otimes \mathbf{B}(\mathcal{H})$, with nonzero spectral projections all Murray–von Neumann equivalent to $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$, such that $V_n \to \phi(\mathrm{id})$ in the strict topology on $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$. (Note that $\mathcal{A}$ is assumed to be separable.) Hence, $(U_L)^*V_nU_L \to (U_L)^*\phi(\mathrm{id})U_L$ in the strict topology on $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$. Choose $N' \geq 1$ such that for all $n \geq N'$, for all $b \in \mathcal{F}$,

$$\|(U_L)^*V_nU_L - (U_L)^*\phi(\mathrm{id})U_L\|_b < \varepsilon/2.$$

From this and $(* * **)$, we have that for all $b \in \mathcal{F}$,

$$\|(U_L)^*V_{N'}U_L - U\|_b < \varepsilon.$$

Since ε, $\mathcal{F}$ were arbitrary, U can be approximated "arbitrarily close" in the strict topology by unitaries in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ with finite spectrum and nonzero spectral projections all Murray–von Neumann equivalent to $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$. □

Lemma 2.3. *For every $\varepsilon > 0$, there exists a $\delta > 0$ such that, for any unital C^*-algebra $\mathcal{A}$, if*

i) $p_1, p_2, \dots, p_n$ *are pairwise orthogonal projections in* $\mathcal{A}$,

ii) $q_1, q_2, \dots, q_n$ *are pairwise orthogonal projections in* $\mathcal{A}$,

iii) $\alpha_1, \alpha_2, \dots, \alpha_n$ *are scalars* (*complex numbers*) *with norm one,*

iv) $|\alpha_i - \alpha_j| \geq \varepsilon$ *for* $i \neq j$, *and*

v) $\|(\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n) - (\alpha_1 q_1 + \alpha_2 q_2 + \cdots + \alpha_n q_n)\| < \delta$,

then p_i is Murray–von Neumann equivalent to q_i in $\mathcal{A}$ for $1 \leq i \leq n$; moreover,

$$\|p_i - q_i\| < \varepsilon$$

for all i.

Theorem 2.4. *Let $\mathcal{A}$ be a unital simple separable nuclear C^*-algebra with real rank zero. Let $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))$ be the unitary group of the multiplier algebra of the stabilization of $\mathcal{A}$, given the strict topology. Then $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))/\mathbf{T}$ is a simple topological group.*

Proof. Let $G \subseteq U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))$ be a strictly closed normal subgroup such that G properly contains all the scalar unitaries (i.e., G contains all scalar unitaries, and G also contains a nonscalar unitary.) By Lemma 2.2, it suffices to show that G contains all unitaries (in $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))$) with finite spectrum and nonzero spectral projections all Murray–von Neumann equivalent to $1_{\mathcal{M}(\mathcal{A} \otimes \mathcal{K})}$.

Let u be a unitary in G such that u is not a scalar unitary. Since G contains the scalar unitaries, we may assume that 1 is a point in the spectrum of u.

Case 1. Suppose that the spectrum of u contains a point $\alpha \neq -1, 1, i, -i$.

Claim 1. *For every $\varepsilon > 0$, there exist pairwise orthogonal projections $p, q \in \mathcal{A} - \{0\}$, which are Murray–von Neumann equivalent in $\mathcal{A}$ such that the following hold*: *There is a unitary $w'' \in G$ with*

(i) $w''(1 - (p + q)) = (1 - (p + q))w'' = 1 - (p + q)$.

(ii) $w''p = pw'' = pw''p$ *is within* ε *of* $\alpha^2 p$.

(iii) $w''q = qw'' = qw''q$ *is within* ε *of* $\overline{\alpha}^2 q$.

Towards proving Claim 1, let $\varepsilon > 0$ be given. For simplicity, we may assume that $\varepsilon < 1/2$. Plug $\min\{|1-\alpha|/2, |1-\overline{\alpha}|/2, |\alpha - \overline{\alpha}|/2, \varepsilon/100\}$ into Lemma 2.3 to get a positive real number $\delta' > 0$. (δ' is the "δ" in Lemma 2.3.) Let $\delta =_{df} \min\{\varepsilon/1000, \delta'/1000, |1-\alpha|/1000, |1-\overline{\alpha}|/1000, |\alpha - \overline{\alpha}|/1000\}$. Contracting $\delta > 0$ if necessary, we may assume that for all $\gamma_1, \gamma_2 \in \mathbf{T}$, if $|\gamma_1 - \gamma_2| < 100\delta$ then $|(\gamma_1)^2 - (\gamma_2)^2| < \varepsilon/100$.

By [**12** Theorem 1.1], $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ has the (SP) property (i.e., every nonzero hereditary subalgebra of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ contains a nonzero projection). Hence, let O, O' be nonempty open neighborhoods (in the complex numbers) of 1, α respectively, let f, g be nonnegative continuous functions from $\mathbf{T}$ to $[0,1]$, and let P, Q be nonzero projections in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ such that

(a) $\overline{O} \cap \overline{O'} = \varnothing$,

(b) O and O' both have diameter strictly less than $\delta/2$.

(c) the support of f is contained in O.

(d) the support of g is contained in O'.

(e) P is contained in $\mathrm{Her}\,(f(u))$ (the hereditary subalgebra of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ generated by $f(u)$), and

(f) Q is contained in $\mathrm{Her}\,(g(u))$.

We collectively denote the above statements by "$(*)$."

Since $\mathcal{A} \otimes \mathcal{K}$ has real rank zero, the hereditary subalgebra $P(\mathcal{A} \otimes \mathcal{K})P$ (of $\mathcal{A} \otimes \mathcal{K}$) has an approximate unit consisting of projections. Similar for $Q(\mathcal{A} \otimes \mathcal{K})Q$. From this and [**12**, Lemma 1.2], there exist nonzero projections $p, q \in \mathcal{A} \otimes \mathcal{K}$ such that $p \leq P$, $q \leq Q$, and p and q are Murray–von Neumann equivalent in $\mathcal{A} \otimes \mathcal{K}$. From this and $(*)$, we must have that pu, up, pup and p are all within $\delta/2$ of each other. Similarly, qu, uq, quq and αq are all within $\delta/2$ of each other. Moreover, p and q must be orthogonal. We denote the above statements by "$(**)$."

Since $p, q \in \mathcal{A} \otimes \mathcal{K}$, $1_{\mathcal{M}(\mathcal{A} \otimes \mathcal{K})} - (p+q)$ is Murray–von Neumann equivalent (in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$) to $1_{\mathcal{M}(\mathcal{A} \otimes \mathcal{K})}$. Hence, there exists a unitary v in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ such that $vpv^* = v^*pv = q$ and $v(1_{\mathcal{M}(\mathcal{A} \otimes \mathcal{K})} - (p+q)) =$

$(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - (p+q))v = (1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - (p+q))$. Let $w =_{df} u^*v^*uv$. Since $u \in G$ and since G is a normal subgroup of $U(\mathcal{M}(\mathcal{A}\otimes\mathcal{K}))$, we must have that $w \in G$. Using the relations in $(*)$ and $(**)$, one can check that the following inequalities hold:

(i) $\|wp - \alpha p\| < 2\delta$,

(ii) $\|wq - \overline{\alpha} q\| < 2\delta$,

(iii) $\|w(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - (p+q)) - (1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - (p+q))\| < 4\delta$.

We collectively denote the above inequalities by "$(***)$."

From $(***)$, we get that

$$\|w - (\alpha p + \overline{\alpha} q + (1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - (p+q))\| < 8\delta.$$

We denote the above inequality by "$(****)$."

Recall that if $a, b \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ such that a is invertible in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ and $\|a - b\| < 1/\|a^{-1}\|$, then b is also invertible in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$. From this, $(****)$ and the definition of δ, it follows that the spectrum of w is contained in three pairwise disjoint open balls with centers 1, α and $\overline{\alpha}$. Since the spectrum is a compact set, we may assume that the closures of the three open balls are also pairwise disjoint. In particular, we can take the open balls to all have radius 8δ. Hence, there exist pairwise disjoint self-adjoint partial isometries $x, y, z \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$, and there exist pairwise disjoint projections $d, e, f \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ such that the following hold:

(i) $w = x + y + z$,

(ii) x, y, z "live in" open balls about $\alpha, \overline{\alpha}, 1$ (respectively), all with radius 8δ. (Of course, we are really applying the continuous functional calculus to w.)

(iii) $x^*x = xx^* = d$,

(iv) $y^*y = yy^* = e$,

(v) $z^*z = zz^* = f$,

(vi) $d + e + f = 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$.

We collectively denote the above statements by "$(+)$."

From $(+)$, we have that

$$\|w - (\alpha d + \overline{\alpha} e + f)\| < 8\delta.$$

From the above equation, $(***\,*)$, the definition of δ and Lemma 2.3, we have that d, e, f are Murray–von Neumann equivalent (in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$) to $p, q, 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - (p+q)$ respectively. Hence, there exists a unitary w' in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ such that $w'd(w')^* = p$, $w'e(w')^* = q$ and $w'f(w')^* = 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - (p+q)$. Hence, $w'x(w')^*$, $w'y(w')^*$, $w'z(w')^*$ are within 8δ of $\alpha p, \overline{\alpha}q, 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - (p+q)$, respectively. Therefore, $w'w(w')^*$ is an element of G such that

$$\|w'w(w')^* - (\alpha p + \overline{\alpha} q + (1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K}} - (p+q))\| < 8\delta.$$

Let us denote the statements in this paragraph by "(++)." Now, define a unitary w'' in G by

$$w'' =_{df} (w'w(w')^*)v(w'w(w')^*)^*v^*.$$

Recall that $vpv^* = v^*pv = q$ and $v(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - (p+q)) = (1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - (p+q))v = (1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - (p+q))$ (i.e., v flips p and q). From this, (+) and (++), we must have that w'' satisfies the statement of Claim 1. We have thus completed the proof of Claim 1.

Claim 2. *Let R be a projection in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ such that both R and $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - R$ are Murray–von Neumann equivalent (in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$) to $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$. Then $\alpha^2 R + \overline{\alpha}^2(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - R) \in G$.*

We will show that $\alpha^2 R + \overline{\alpha}^2(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - R)$ can be approximated "arbitrarily close" in the strict topology by unitaries in G.

Let $\varepsilon > 0$ be given, and let $S \subseteq \mathcal{A}\otimes\mathcal{K}$ be a finite subset. We will be approximating with respect to the strict topology semi-norms coming from elements of S (i.e., we will be approximating with respect to the semi-norms $\|m\|_b = \|mb\| + \|bm\|$ for $b \in S$). Contracting $\varepsilon > 0$ if necessary, we may assume that the elements of S all have norm less than or equal to one. Plug $\varepsilon/100$ into Claim 1. ($\varepsilon/100$ will now replace the ε in Claim 1.) Let $p, q \in \mathcal{A}\otimes\mathcal{K} - \{0\}$ be orthogonal projections such that p and q are Murray–von Neumann equivalent in $\mathcal{A}\otimes\mathcal{K}$, and such that there exists a unitary $w'' \in G$ which satisfies the statement of Claim 1 (with $\varepsilon/100$ replacing ε). Since R and $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - R$ are both Murray–von Neumann equivalent to the unit of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$, let $\{r_k\}_{k=1}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ be sequences of pairwise orthogonal projections in $\mathcal{A}\otimes\mathcal{K}$

such that, for all k, r_k and s_k are Murray–von Neumann equivalent (in $\mathcal{A} \otimes \mathcal{K}$) to p (and hence q), and such that

$$R = \sum_{k=1}^{\infty} r_k,$$

and

$$1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - R = \sum_{k=1}^{\infty} s_k,$$

where the sums converge in the strict topology on $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$.

Now choose $K \geq 1$ such that for all $k' \geq K$ and for all $b \in S$, $(\sum_{k=1}^{k'} r_k)b$ is within $\varepsilon/100$ of Rb; $b\sum_{k=1}^{k'} r_k$ is within $\varepsilon/100$ of bR; $(\sum_{k=1}^{k'} s_k)b$ is within $\varepsilon/100$ of $(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - R)b$; and $b(\sum_{k=1}^{k'} s_k)$ is within $\varepsilon/100$ of $b(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - R)$. Hence, we must have that, for every element c of norm less than or equal to one in $(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - \sum_{k=1}^{k'}(r_k + s_k))\mathcal{M}(\mathcal{A}\otimes \mathcal{K})(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - \sum_{k=1}^{k'}(r_k + s_k))$, and for every $b \in S$,

$$\left\| \sum_{k=1}^{k'} (\alpha^2 r_k + \overline{\alpha}^2 s_k) + c - (\alpha^2 R + \overline{\alpha}^2(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - R)) \right\|_b < 6\varepsilon/100.$$

We denote the above inequality by "$(+++)$."

Note that $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - (r_k + s_k)$ is Murray–von Neumann equivalent (in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$) to $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$, for all k. Hence, for every k, there exists a unitary x'_k in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ such that $x'_k p (x'_k)^* = r_k$, $x'_k q (x'_k)^* = s_k$ and $x'_k(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - (p+q))(x'_k)^* = 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - (r_k + s_k)$. Define a unitary w''' in G by

$$w''' =_{df} \prod_{k=1}^{K} x'_k w'' (x'_k)^*.$$

From $(+++)$, the definition of w'' and the definition of the x'_ks, we have that for all $b \in S$,

$$\|w''' - (\alpha^2 R + \overline{\alpha}^2(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - R))\|_b < \varepsilon.$$

Since $\varepsilon > 0$, S are arbitrary, and since G is strictly closed in $U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))$, we have that $\alpha^2 R + \overline{\alpha}^2(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} - R) \in G$. This completes the proof of Claim 2.

Claim 3. *Suppose that R is a projection in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ such that both R and $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}-R$ are Murray–von Neumann equivalent to $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$ (in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$). Then for every $\beta,\gamma\in\mathbf{T}$, $\beta R+\gamma(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}-R)\in G$.*

Let α be as before. Then by Claim 2, $\alpha^2 R+\overline{\alpha}^2(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}-R)\in G$. Let $\widehat{v}$ be a partial isometry in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ such that the initial projection of $\widehat{v}$ is R and the range projection is $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}-R$. Consider the unital $*$-subalgebra $\mathcal{C}\subseteq\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ that is generated by $R, 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}-R, \widehat{v}$. Then $\mathcal{C}\cong\mathbf{M}_2(\mathbf{C})$. Let $SU(\mathcal{C})\cong SU(\mathbf{M}_2(\mathbf{C}))$ be the unitaries with determinant one (i.e., it is a copy of $SU(2)$). Note that $\alpha^2R+\overline{\alpha}^2(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}-R)\in SU(\mathcal{C})$. Since the only proper normal subgroups of $SU(\mathcal{C})$ are contained in $\mathbf{T}$, and $\alpha^2R+\overline{\alpha}^2(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}-R)$ is not a scalar, the closed normal subgroup of $SU(\mathcal{C})$ generated by $\alpha^2R+\overline{\alpha}^2(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}-R)$ must be all of $SU(\mathcal{C})$. Hence, $SU(\mathcal{C})$ is contained in G. Since G contains all scalar unitaries, the unitary group of $\mathcal{C}$ ($U(\mathcal{C})$) is contained in G. Hence, for all $\beta,\gamma\in\mathbf{T}$, $\beta R+\gamma(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}-R)\in G$. This completes the proof of Claim 3.

We now finish the proof of the theorem for Case 1. Suppose that $P_1, P_2, \ldots, P_n$ are pairwise orthogonal projections in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ such that P_i is Murray–von Neumann equivalent to $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$ for all i. Suppose that $\alpha_1,\alpha_2,\ldots,\alpha_n\in\mathbf{T}$. By Claim 2, $\alpha_iP_i+(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}-P_i)\in G$ for all i. Multiplying all these unitaries together, we get that $\alpha_1P_1+\alpha_2P_2+\cdots+\alpha_nP_n\in G$. Since this unitary was arbitrary, we have shown that every unitary with finite spectrum, and with all nonzero spectral projections Murray–von Neumann equivalent to $1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$, is in G. From this and Lemma 2.2, we have that $G=U(\mathcal{M}(\mathcal{A}\otimes\mathcal{K}))$.

Case 2. Suppose that the spectrum of u is completely contained in $\{1,-1,-i,i\}$ and either $-i$ or i is in the spectrum of u. Suppose, for simplicity, that i is in the spectrum of u. By our hypothesis at the beginning (before Case 1), 1 is also in the spectrum of u. Using the argument of Claim 1 (specifically, the construction of w), we can find nonzero orthogonal projections $p,q\in\mathcal{A}\otimes\mathcal{K}$ such that p is Murray–von Neumann equivalent to q in $\mathcal{A}\otimes\mathcal{K}$ and such that $ip+-iq+(1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}-(p+q))\in G$. Then, following the arguments of Claims 2 and 3 and the rest of the proof of Case 1, we have that $G=U(\mathcal{M}(\mathcal{A}\otimes\mathcal{K}))$.

Case 3. Suppose that the spectrum of u is completely contained in $\{1, -1\}$. Since u is nonscalar, both -1 and 1 are in the spectrum of u. Therefore, since $\mathcal{A} \otimes \mathcal{K}$ is simple and has real rank zero, we can modify the arguments of Case 1, Claim 1, to find nonzero orthogonal projections p, q in $\mathcal{A} \otimes \mathcal{K}$ such that p, q are Murray–von Neumann equivalent in $\mathcal{A} \otimes \mathcal{K}$, and such that $p - q + (1 - (p + q)) \in G$. Then, by modifying the arguments of Case 1, Claims 2 and 3, we get that $G = U(\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))$. □

3. The projective unitary group of a unital simple separable C^*-algebra.

Theorem 3.1. *Let $\mathcal{A}$ be a unital separable simple C^*-algebra. Let $U(\mathcal{A})$ be the unitary group of $\mathcal{A}$, given the relative weak topology (i.e., the weak topology induced by the linear functionals in $\mathcal{A}^*$). Then $U(\mathcal{A})/\mathbf{T}$ is a simple topological group.*

Proof. If $\mathcal{A}$ is finite dimensional, then $\mathcal{A}$ would be a full matrix algebra, and the result would be immediate. Hence, we may assume that $\mathcal{A}$ is infinite dimensional.

Suppose that G is a closed normal subgroup of $U(\mathcal{A})$ (given relative weak topology) such that G properly contains all the scalar unitaries.

Suppose, to the contrary, that $G \neq U(\mathcal{A})$. Let $\mathcal{A}^{**}$ the be second dual von Neumann algebra, i.e., enveloping von Neumann algebra, of $\mathcal{A}$. Let G' be the closure of G in $U(\mathcal{A}^{**})$ (with weak*-topology). Note that the relative weak topology on $U(\mathcal{A})$ is the restriction of the weak*-topology from $U(\mathcal{A}^{**})$. Therefore, the closure of G in $U(\mathcal{A})$ with the relative weak topology is equal to the intersection of G' and $U(\mathcal{A})$. Hence, $G' \neq U(\mathcal{A}^{**})$ since $G \neq U(\mathcal{A})$.

Since G properly contains the scalar unitaries, there exists a unitary u in G which is not a scalar. For simplicity, let us assume that the spectrum of u contains $\{1, \alpha\}$, where $\alpha \neq -1, 1, -i, i$. (This corresponds to Case 1 in the proof of Theorem 2.4.) Now, since $\mathcal{A}$ is a simple infinite dimensional C^*-algebra which is weak*-dense in $\mathcal{A}^{**}$, we can modify the proof of Theorem 2.4, Claim 1 to get the following: For every $\varepsilon > 0$, there exist nonzero orthogonal projections $p, q \in \mathcal{A}^{**}$ which are Murray–von Neumann equivalent in $\mathcal{A}^{**}$ such that the following hold:

i) Both p and q are weak*-full in $\mathcal{A}^{**}$. (An element $a \in \mathcal{A}^{**}$ is weak*-full if the smallest weak*-closed C^*-algebra ideal generated by a is all of $\mathcal{A}^{**}$.)

For every pair of projections $r, s \in \mathcal{A}^{**}$ such that $r \leq p$ and $s \leq q$ and r, s Murray–von Neumann equivalent in $\mathcal{A}^{**}$, there exists a unitary $w \in G'$ such that the following are true:

(a) $w(1-(r+s)) = (1-(r+s))w = 1-(r+s)$,

(b) $wr = rw = rwr$ is within ε of $\alpha^2 r$,

(c) $ws = sw = sws$ is within ε of $\overline{\alpha}^2 s$.

We collectively denote the above statements by "(V)."

Now, since p is weak*-full in $\mathcal{A}^{**}$, we have that, for every projection $t \in \mathcal{A}^{**}$, there is a net $\{p_\lambda\}$ of pairwise orthogonal projections in $\mathcal{A}^{**}$ such that

i) for all λ, p_λ is Murray–von Neumann equivalent to a subprojection of p, and

ii) $t = \sum_\lambda p_\lambda$ where the sum converges in the weak*-topology of $\mathcal{A}^{**}$. The same holds for q. We collectively denote the above statements by "(VV)."

Using statements (V) and (VV), we can modify the arguments of Theorem 2.4, Claims 2 and 3, to show that if p', q' are orthogonal projections in $\mathcal{A}^{**}$ such that p' is Murray–von Neumann equivalent to q' in $\mathcal{A}^{**}$, and if $\beta \in \mathbf{T}$, then

$$\beta p' + \overline{\beta} q' + (1-(p'+q')) \in G'.$$

It follows that, if $p_1', p_2', \dots, p_n'$ are pairwise orthogonal, pairwise Murray–von Neumann equivalent projections in $\mathcal{A}^{**}$ and if $\beta_1, \beta_2, \dots, \beta_n \in \mathbf{T}$ with $\beta_1\beta_2\dots\beta_n = 1$, then

$$\beta_1 p_1' + \beta_2 p_2' + \cdots + \beta_n p_n' + \Big(1 - \sum_{j=1}^{n} p_j'\Big) \in G'.$$

But, since $\mathcal{A}$ is unital simple separable and infinite dimensional, the set of all such unitaries generate $U(\mathcal{A}^{**})$ as a topological group (see [**7**, Lemma 6.3.3 and Lemma 6.5.6]). Hence, $G' = U(\mathcal{A}^{**})$, which gives a contradiction. □

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Department of Mathematics, University of Louisiana, 217 Maxim D. Doucet Hall, P.O. Box 41010, Lafayette, Louisiana
Email address: png@louisiana.edu

Mathematics Department, University of Hawaii at Hilo, 200 W. Kawili St., Hilo, HI, 96720
Email address: ruize@hawaii.edu