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MULTIPLE DIRICHLET SERIES INTERPOLATING BELL NUMBERS AND STIRLING NUMBERS

HIROFUMI TSUMURA

ABSTRACT. The number of ways that a set of n elements can be partitioned into nonempty subsets is called the nth Bell number. An interpolation Dirichlet series of Bell numbers is well-known classically. In this paper, as its generalization, we construct a certain multiple Dirichlet series which interpolates Bell numbers. As another example, we construct a multiple Dirichlet series which interpolates Stirling numbers of the second kind.

1. Introduction. Let $\mathbf{N}$ be the set of natural numbers, $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, $\mathbf{Z}$ the ring of rational integers, $\mathbf{Z}^* = \mathbf{Z} \setminus \{0\}$, $\mathbf{Q}$ the field of rational numbers, $\mathbf{R}$ the field of real numbers, and $\mathbf{C}$ the field of complex numbers.

The nth Bell number β_n is defined as the number of ways that a set of n elements can be partitioned into nonempty subsets, see [**2**]. Formally we let $\beta_0 = 1$. Then $\{\beta_n \mid n \in \mathbf{N}_0\}$ satisfy

$$\beta_n = \sum_{j=0}^{n-1} \binom{n-1}{j} \beta_j, \quad n \in \mathbf{N}.$$

By this relation, we can calculate that $\beta_1 = 1$, $\beta_2 = 2$, $\beta_3 = 5$, $\beta_4 = 15$, $\beta_5 = 52$, etc. In the twentieth century, various properties of $\{\beta_n\}$ were studied, for example,

$$e^{e^x - 1} = \sum_{n=0}^{\infty} \beta_n \frac{x^n}{n!}; \tag{1}$$

$$\frac{1}{e} \sum_{k=1}^{\infty} \frac{k^n}{k!} = \beta_n, \quad n \in \mathbf{N}, \tag{2}$$

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see, for example, [**1, 4, 7**]. Indeed, we can easily see that (2) is derived from (1).

Now we define

$$\mathcal{Z}_1(s) := \frac{1}{e}\sum_{k=1}^{\infty}\frac{1}{k!\,k^s}, \quad s \in \mathbf{C}. \tag{3}$$

Then (2) means $\mathcal{Z}_1(-n) = \beta_n$ for $n \in \mathbf{N}$. This result looks like the well-known formula $\zeta(1-n) = -B_n/n$, $n \in \mathbf{N}$, concerning the Riemann zeta-function $\zeta(s)$ and the Bernoulli numbers $\{B_n\}$.

In this decade, as a multiple analogue of the Riemann zeta value, the multiple zeta value

$$\sum_{k_1,\dots,k_r \geq 1}\frac{1}{k_1^{p_1}(k_1+k_2)^{p_2}\cdots(k_1+\cdots+k_r)^{p_r}}$$

$(p_1, p_2, \dots, p_r \in \mathbf{N};\ p_r \geq 2)$ has been actively investigated in various branches of mathematics and physics. From these considerations, it is interesting to study multiple analogues of $\mathcal{Z}_1$. Hence, for $s \in \mathbf{C}$ with $\Re s > 1$, $r \in \mathbf{N}$ with $r \geq 2$, $q_1 \in \mathbf{C}$ and $p_2, \dots, p_{r-1} \in \mathbf{N}$, we define

$$\begin{aligned}\mathcal{Z}_r&(q_1, 2p_2, \dots, 2p_{r-1}, s)\\ &:= \frac{1}{e}\sum_{\mathbf{k}\in\Phi_r}\frac{\operatorname{sgn}\left(\sum_{j=1}^{r} k_j\right)}{k_1!k_1^{q_1}\prod_{d=2}^{r-1}\left(\sum_{j=1}^{d} k_j\right)^{2p_d}\left|\sum_{j=1}^{r} k_j\right|^{s}},\end{aligned} \tag{4}$$

where $\operatorname{sgn}(x) = 1$, respectively $= -1$, if $x > 0$, respectively $x < 0$, and

$$\Phi_r = \left\{\mathbf{k} = (k_j) \in \mathbf{N}\times(\mathbf{Z}^*)^{r-1}\ \middle|\ \sum_{j=1}^{d} k_j \neq 0,\ 2 \leq d \leq r\right\}. \tag{5}$$

We will prove the following theorem by using a method similar to that introduced in [**3**].

Theorem 1.1. *For $r \in \mathbf{N}$, with $r \geq 2$, $q_1 \in \mathbf{C}$ and $p_2, \dots, p_{r-1} \in \mathbf{N}$,*

$$\mathcal{Z}_r(q_1, 2p_2, \dots, 2p_{r-1}, s) = (-1)^{r+1}\mathcal{Z}_1\left(q_1 + 2\sum_{j=2}^{r-1} p_j + s\right) \tag{6}$$

for $s \in \mathbf{C}$ with $\Re s > 1$. Note that the lefthand side of (6) *can be analytically continued for all $s \in \mathbf{C}$ by this equation. In particular, for $n \in \mathbf{N} \cup \{0\}$, $q_1 \in \mathbf{Z}$ and $p_2, \dots, p_r \in \mathbf{N}$,*

$$\mathcal{Z}_r\left(q_1, 2p_2, \dots, 2p_{r-1}, -n - q_1 - 2\sum_{j=2}^{r-1} p_j\right) = (-1)^{r+1}\,\beta_n. \tag{7}$$

From Theorem 1.1, we have, for example,

$$\mathcal{Z}_2(2,-3) = -\beta_1 = -1;$$
$$\mathcal{Z}_3(-1,2,-4) = \beta_3 = 5.$$

As another example, we consider Stirling numbers of the second kind $\{S(n,m) \mid n, m \in \mathbf{N}_0\}$ defined by

$$\frac{(e^x-1)^m}{m!} = \sum_{n=0}^{m} S(n,m)\frac{x^n}{n!} \tag{8}$$

(see, for example, [**4**, Section 1]). It is known that

$$S(n,m) = \frac{(-1)^m}{m!}\sum_{k=0}^{m}(-1)^k\binom{m}{k}k^n, \quad n, m \in \mathbf{N}_0, \tag{9}$$

and

$$\beta_n = \sum_{j=1}^{n} S(n,j), \quad n \in \mathbf{N}.$$

As well as $\mathcal{Z}_r$, we consider a certain multiple Dirichlet series which interpolates $S(n,m)$ (see Theorem 3.1).

2. Main result and its proof. First we consider a more general situation as follows. Let

$$\mathfrak{P}_1(s) = \sum_{k=1}^{\infty}\frac{a_k}{k^s} \tag{10}$$

be a Dirichlet series which is absolutely convergent for all $s \in \mathbf{C}$. As a multiple analogue of (10), we define

$$(11)\quad \mathfrak{P}_r(q_1, 2p_2, \dots, 2p_{r-1}, s) := \sum_{\mathbf{k}\in\Phi_r} \frac{a_{k_1}\mathrm{sgn}\,(\sum_{j=1}^r k_j)}{k_1^{q_1}\prod_{d=2}^{r-1}\left(\sum_{j=1}^d k_j\right)^{2p_d}\left|\sum_{j=1}^r k_j\right|^s}$$

for $s \in \mathbf{C}$ with $\Re s > 1$, $r \in \mathbf{N}$ with $r \geq 2$, $q_1 \in \mathbf{C}$ and $p_2, \dots, p_{r-1} \in \mathbf{N}$. Then we aim to prove the following theorem.

Theorem 2.1. *For $r \in \mathbf{N}$ with $r \geq 2$, $q_1 \in \mathbf{C}$ and $p_2, \dots, p_r \in \mathbf{N}$,*

$$(12)\qquad \mathfrak{P}_r(q_1, 2p_2, \dots, 2p_{r-1}, s) = (-1)^{r+1}\mathfrak{P}_1\left(q_1 + 2\sum_{j=2}^{r-1} p_j + s\right).$$

Note that the lefthand side of (12) *can be analytically continued for all $s \in \mathbf{C}$ by this equation.*

We can see that Theorem 1.1 is a special case of Theorem 2.1. In the rest of this section, we will give a proof of this theorem.

We define

$$(13)\quad \mathcal{A}_r(\theta; q_1, 2p_2, \dots, 2p_r) := \sum_{\mathbf{k}\in\Phi_r} \frac{ak_1\mathrm{sgn}\,(\sum_{j=1}^r k_j)\sin(|\sum_{j=1}^r k_j|\theta)}{k_1^{q_1}\prod_{d=2}^{r}(\sum_{j=1}^d k_j)^{2p_d}}, \quad \theta \in \mathbf{R},$$

for $r \in \mathbf{N}$, $q_1 \in \mathbf{C}$ and $p_2, \dots, p_r \in \mathbf{N}$, where

$$\Phi_r = \left\{\mathbf{k} = (k_j) \in \mathbf{N} \times (\mathbf{Z}^*)^{r-1} \,\middle|\, \sum_{j=1}^d k_j \neq 0,\ 2 \leq d \leq r\right\}.$$

We have already proved the following.

Lemma 2.2 [**3**, Lemma 7.6]. *For $r \in \mathbf{N}$ and $p_1, \dots, p_r \in \mathbf{N}$,*

$$\sum_{\mathbf{k}\in\Psi_r} \frac{1}{\prod_{d=1}^r\left(\sum_{j=1}^d k_j\right)^{2p_d}} \leq \prod_{d=1}^r\{2\zeta(2p_d)\} < \infty,$$

where

$$\Psi_r := \left\{ \mathbf{k} = (k_j) \in (\mathbf{Z}^*)^r \,\middle|\, \sum_{j=1}^{d} k_j \neq 0,\ 1 \leq d \leq r \right\}.$$

From this lemma and the assumption on $\mathfrak{P}_1(s)$, we see that the righthand side of (13) is absolutely and uniformly convergent with respect to $\theta \in \mathbf{R}$. We will prove the following.

Lemma 2.3. *For $r \in \mathbf{N}$ with $r \geq 2$, $q_1 \in \mathbf{C}$ and $p_2, \ldots, p_r \in \mathbf{N}$,*

$$\mathcal{A}_r(\theta; q_1, 2p_2, \ldots, 2p_r) + (-1)^r \mathcal{A}_1\left(\theta; q_1 + 2\sum_{j=2}^{r} p_j\right) = 0, \quad \theta \in \mathbf{R}. \tag{14}$$

Proof. We prove the assertion by induction on $r \geq 2$. Let

$$\Omega_r = \left\{ \mathbf{k} = (k_j) \in \mathbf{N} \times (\mathbf{Z}^*)^{r-2} \times \mathbf{Z} \,\middle|\, \sum_{j=1}^{d} k_j \neq 0,\ 2 \leq d \leq r \right\}. \tag{15}$$

In the case $r = 2$, we see that

$$\begin{aligned}
&\sum_{(k_1,k_2)\in\Omega_2} \frac{a_{k_1} \operatorname{sgn}(k_1+k_2) \sin(|k_1+k_2|\theta)}{k_1^{q_1}(k_1+k_2)^{2p_2}} \\
&\qquad = \sum_{k_1\in\mathbf{N}} \sum_{\substack{k_2\in\mathbf{Z} \\ k_1+k_2\neq 0}} \frac{a_{k_1} \operatorname{sgn}(k_1+k_2) \sin(|k_1+k_2|\theta)}{k_1^{q_1}(k_1+k_2)^{2p_2}} \\
&\qquad = \sum_{k_1\in\mathbf{N}} \frac{a_{k_1}}{k_1^{q_1}} \sum_{N\in\mathbf{Z}^*} \frac{\operatorname{sgn}(N) \sin(|N|\theta)}{N^{2p_2}} = 0
\end{aligned}$$

by putting $N = k_1 + k_2$ and changing the index N into $-N$. The lefthand side is equal to

$$\sum_{k_1\in\mathbf{N}} \sum_{\substack{k_2\in\mathbf{Z}^* \\ k_1+k_2\neq 0}} \frac{a_{k_1} \operatorname{sgn}(k_1+k_2) \sin(|k_1+k_2|\theta)}{k_1^{q_1}(k_1+k_2)^{2p_2}} + \sum_{k_1\in\mathbf{N}} \frac{a_{k_1} \sin(k_1\theta)}{k_1^{q_1+2p_2}},$$

by dividing into the case with $k_2 \neq 0$ and $k_2 = 0$. Therefore, we see that the case $r = 2$ holds.

Next we assume that the case of $r-1$ holds (for $r > 2$) and consider the case of r.

$$\begin{aligned}
&\sum_{(k_j)\in\Omega_r} \frac{a_{k_1}\operatorname{sgn}\left(\sum_{j=1}^{r} k_j\right)\sin(|\sum_{j=1}^{r} k_j|\theta)}{k_1^{q_1}\prod_{d=2}^{r}(\sum_{j=1}^{d} k_j)^{2p_d}} \\
&\qquad = \sum_{\substack{(k_j)\in\Omega_{r-1}\\ k_{r-1}\neq 0}} \frac{a_{k_1}}{k_1^{q_1}\prod_{d=2}^{r-1}(\sum_{j=1}^{d} k_j)^{2p_d}} \sum_{N\neq 0} \frac{\operatorname{sgn}(N)\sin(|N|\theta)}{N^{2p_r}} \\
&\qquad = 0
\end{aligned}$$

by putting $N = k_1 + \cdots + k_r$ and changing the index N into $-N$. The lefthand side is equal to

$$\begin{aligned}
&\sum_{(k_j)\in\Phi_r} \frac{a_{k_1}\operatorname{sgn}\left(\sum_{j=1}^{r} k_j\right)\sin(|\sum_{j=1}^{r} k_j|\theta)}{k_1^{q_1}\prod_{d=2}^{r}(\sum_{j=1}^{d} k_j)^{2p_d}} \\
&\qquad + \sum_{(k_j)\in\Phi_{r-1}} \frac{a_{k_1}\operatorname{sgn}\left(\sum_{j=1}^{r-1} k_j\right)\sin(|\sum_{j=1}^{r-1} k_j|\theta)}{k_1^{q_1}\prod_{d=2}^{r-2}(\sum_{j=1}^{d} k_j)^{2p_d}\cdot(\sum_{j=1}^{r-1} k_j)^{2p_{r-1}+2p_r}}
\end{aligned}$$

by dividing into the case with $k_r \neq 0$ and $k_r = 0$. By the assumption of induction, the second term is equal to

$$-(-1)^{r-1}\sum_{k=1}^{\infty}\frac{a_k\sin(k\theta)}{k^{q_1+2\sum_{d=2}^{r}p_d}}.$$

Thus, we see that the case of r holds. By induction, we complete the proof of this lemma. $\square$

In order to complete the proof of Theorem 2.1, we quote the following fact which was given in our previous paper [**3**].

Let $\mathcal{D}(s)$ be the Dirichlet series defined by

$$\mathcal{D}(s) = \sum_{m=1}^{\infty}\frac{c_m}{m^s}, \tag{16}$$

where $\{c_n\} \subset \mathbf{C}$. Let $\Re s = \rho$, $\rho \in \mathbf{R}$, be the abscissa of convergence of $\mathcal{D}(s)$. We further assume that $\rho < 1$.

Theorem 2.4 [**3**, Theorem 3.1]. *Assume that*

$$\sum_{m=1}^{\infty} c_m \sin(mt) = 0 \tag{17}$$

is boundedly convergent for $t > 0$ *and that, for* $\rho < s < 1$,

$$\lim_{\lambda \to \infty} \sum_{m=1}^{\infty} c_m \int_{\lambda}^{\infty} t^{s-1} \sin(mt)\, dt = 0. \tag{18}$$

Then $\mathcal{D}(s)$ *can be continued meromorphically to* $\mathbf{C}$, *and actually* $\mathcal{D}(s) = 0$ *for all* $s \in \mathbf{C}$.

Using this result, we can give the proof of Theorem 2.1 as follows.

Proof of Theorem 2.1. By Lemma 2.3, we have

$$\mathcal{A}_r(\theta; q_1, 2p_2, \dots, 2p_r) + (-1)^r \mathcal{A}_1\left(\theta; q_1 + 2\sum_{j=2}^{r} p_j\right) = 0. \tag{19}$$

We can rewrite $\mathcal{A}_r(\theta; q_1, 2p_2, \dots, 2p_r)$ to

$$\begin{aligned} &\mathcal{A}_r(\theta; q_1, 2p_2, \dots, 2p_r) \\ &\quad = \sum_{k \in \Phi_r} \frac{a_{k_1} \operatorname{sgn}\left(\sum_{j=1}^{r} k_j\right) \sin\left(\left|\sum_{j=1}^{r} k_j\right| \theta\right)}{k_1^{q_1} \prod_{d=2}^{r} \left(\sum_{j=1}^{d} k_j\right)^{2p_d}} \\ &\quad = \sum_{m=1}^{\infty} \left\{ \sum_{\substack{k \in \Phi_r \\ |k_1 + \cdots + k_r| = m}} \frac{a_{k_1} \operatorname{sgn}\left(\sum_{j=1}^{r} k_j\right)}{k_1^{q_1} \prod_{d=2}^{r} \left(\sum_{j=1}^{d} k_j\right)^{2p_d}} \right\} \sin(m\theta), \end{aligned} \tag{20}$$

which is absolutely and uniformly convergent with respect to $\theta \in \mathbf{R}$. We put

$$\begin{aligned} \mathcal{D}(s) = &\sum_{k \in \Phi_r} \frac{a_{k_1} \operatorname{sgn}\left(\sum_{j=1}^{r} k_j\right)}{k_1^{q_1} \prod_{d=2}^{r-1} \left(\sum_{j=1}^{d} k_j\right)^{2p_d} \left|\sum_{j=1}^{r} k_j\right|^{2p_r + s}} \\ &+ \sum_{k=1}^{\infty} \frac{a_k}{k^{q_1 + 2(\sum_{j=2}^{r} p_j) + s}}. \end{aligned} \tag{21}$$

Then, by applying Theorem 2.4 with (19) and (20), we see that $\mathcal{D}(s)$ can be continued meromorphically to $\mathbf{C}$, and actually $\mathcal{D}(s) = 0$ for all $s \in \mathbf{C}$. Hence by (10), (11) and (21), we obtain

$$\mathfrak{P}_r(q_1, 2p_2, \ldots, 2p_{r-1}, 2p_r + s) = (-1)^{r+1}\mathfrak{P}_1\left(q_1 + 2\sum_{j=2}^{r} p_j + s\right)$$

for all $s \in \mathbf{C}$. Replacing $2p_r + s$ with s, we complete the proof of Theorem 2.1. $\square$

Note that if we let $\mathfrak{P}_r = \mathcal{Z}_r$ in Theorem 2.1, then we immediately obtain the proof of Theorem 1.1.

3. Multiple series interpolating Stirling numbers. As another example, we consider

$$\mathcal{W}_1(s; m) = \frac{(-1)^m}{m!}\sum_{k=1}^{m} \frac{(-1)^k \binom{m}{k}}{k^s}, \quad s \in \mathbf{C}. \tag{22}$$

Then (9) means that $\mathcal{W}_1(-n; m) = S(n, m)$ for $m, n \in \mathbf{N}$. Of course, $\mathcal{W}_1(s; m)$ is absolutely convergent for all $s \in \mathbf{C}$. Furthermore, we define

$$\begin{aligned} &\mathcal{W}_r(q_1, 2p_2, \ldots, 2p_{r-1}, s; m) \\ &\quad := \frac{(-1)^m}{m!}\sum_{\mathbf{k}\in\Phi_{r,m}} \frac{(-1)^{k_1}\binom{m}{k_1}\operatorname{sgn}\left(\sum_{j=1}^{d} k_j\right)}{k_1^{q_1}\prod_{d=2}^{r-1}\left(\sum_{j=1}^{d} k_j\right)^{2p_d}\left|\sum_{j=1}^{r} k_j\right|^{s}}, \end{aligned} \tag{23}$$

for $s \in \mathbf{C}$ with $\Re s > 1$, where $r \in \mathbf{N}$ with $r \geq 2$, $m \in \mathbf{N}$, $q_1 \in \mathbf{C}$ and $p_2, \ldots, p_{r-1} \in \mathbf{N}$, where

$$\Phi_{r,m} = \left\{\mathbf{k} = (k_j) \in \mathbf{N} \times (\mathbf{Z}^*)^{r-1} \;\middle|\; 1 \leq k_1 \leq m,\ \sum_{j=1}^{d} k_j \neq 0,\ 2 \leq d \leq r\right\}.$$

Then, from Theorem 2.1, we obtain the following.

Theorem 3.1. *For $r \in \mathbf{N}$ with $r \geq 2$, $m \in \mathbf{N}$ and $q_1 \in \mathbf{C}$, $p_2, \ldots, p_{r-1} \in \mathbf{N}$,*

$$\begin{aligned}(24)\quad & \mathcal{W}_r(q_1, 2p_2, \ldots, 2p_{r-1}, s; m) \\ & \qquad = (-1)^{r+1}\mathcal{W}_1\Bigg(q_1 + 2\sum_{j=2}^{r-1} p_j + s; m\Bigg).\end{aligned}$$

Note that the lefthand side of (24) *can be analytically continued for all $s \in \mathbf{C}$ by this equation. In particular, for $n \in \mathbf{N}$, $q_1 \in \mathbf{Z}$ and $p_2, \ldots, p_{r-1} \in \mathbf{N}$,*

$$(25)\quad \mathcal{W}_r\Bigg(q_1, 2p_2, \ldots, 2p_{r-1}, -n - 2\sum_{j=1}^{r-1} p_j; m\Bigg) = (-1)^{r+1}\, S(n,m).$$

Remark. We can also apply the above method to other Dirichlet series. Indeed, by considering certain Dirichlet series involving hyperbolic triangle functions, we can give some multiple analogues of Ramanujan's result ([**6**]).

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Department of Mathematics and Information Sciences, Tokyo Metropolitan University, Minami-Ohsawa, Hachioji, 192-0397 Tokyo, Japan
Email address: tsumura@tmu.ac.jp