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CHARACTERIZATION OF STRICT CONVEXITY FOR LOCALLY LIPSCHITZ FUNCTIONS

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ABSTRACT. The first goal of this paper is improvement of our previous result (Nonlinear Anal. TMA **57** (2004), 85–97), i.e., the characterization of convexity for regularly locally Lipschitz functions by means of the second-order upper (Dini) directional derivative.

Using the (Dini) type of generalized second-order directional derivative, we also provide the characterization of strict convexity for locally Lipschitz functions. As application of this characterization, we can obtain the second-order sufficient optimality condition introduced by Cominetti and Correa.

1. Introduction and preliminaries. Strict convexity plays a very important role in mathematics. For example, it is a well-known fact that it suffices to solve the inclusion $0 \in \partial f(x)$ to find a strict local minimum of the strict convex real function$f$ defined on an open subset of X, where $\partial f(x)$ denotes the convex subdifferential of f at x [**26**] and the symbol X is reserved for a real Banach space with the norm $\|\cdot\|$ in this paper.

We can give the following characterization of strict convexity by means of classical second-order directional derivatives. This result is proved, e.g., in [**7**] for $X = \mathbf{R}$, but we note that the dimension of X is not important in this case. By S_X, we will mean the set $\{h \in X : \|h\| = 1\}$. For $x \in X$, $(u, v) \in X^2$, we denote

$$f''(x; u, v) = \lim_{t \to 0} \frac{\langle \bigtriangledown f(x + tu) - \bigtriangledown f(x), v\rangle}{t},$$

where $\bigtriangledown f(x)$ is the symbol for the Gâteaux derivative of f at x.

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Theorem 1.1. *Let U be an open convex subset of X and $f : U \mapsto \mathbf{R}$ a twice differentiable function on U. Then f is strictly convex on U if and only if both of the following conditions are satisfied.*

(i) $f''(x; h, h) \geq 0$, *for all* $x \in U$, *for all* $h \in S_X$.

(ii) *the set* $\{z : f''(z; h, h) > 0\}$ *is dense in* $I_{x,h} = \{x+th : t \in \mathbf{R}\} \cap U$, *for every* $x \in U$ *and for every* $h \in S_X$.

The main aim of this paper is to generalize Theorem 1.1 via a certain type of generalized second-order derivative. The reasons why we consider just this type are given in Section 2. In Section 3, we give the characterization of convexity for regularly locally Lipschitz functions via the second-order upper (Dini) directional derivative. This characterization is, in a certain sense, more precise than it was stated before, cf. e.g., [**2, 12, 15, 16, 29, 31**], especially see comments and historical remarks in [**31**, page 610] and [**2**, page 96].

Throughout this paper, we use the symbol (a, b) for the open interval with endpoints $a, b \in X$, $a \neq b$. In the case $X = \mathbf{R}$, we assume that $a < b$.

If A, B are subsets of $\mathbf{R}$, then $A \leq B$ means that $a \leq b$ whenever $a \in A$ and $b \in B$. $A < B$, $A \leq c$, $A \geq c$, $A > c$, where $c \in \mathbf{R}$, have the analogous meaning. For $A, B \subset X$, we set $A - B := \{a - b : a \in A, b \in B\}$.

X^* denotes a topological dual space of a Banach space X, and $\langle x^*, x\rangle$ is a canonical pair, where $x^* \in X^*$, $x \in X$.

The effective domain of the set-valued mapping $F : X \hookrightarrow Y$ is denoted by $D(F)$, i.e., $D(F) = \{x \in X : F(x) \neq \varnothing\}$.

We recall several basic facts about set-valued mappings, above all about their monotonicity. For more information, see e.g., [**1, 26**].

Let A be a subset of X. A set-valued mapping $F : A \hookrightarrow X^*$ is said to be *monotone* provided that

$$\langle x^* - y^*, x - y\rangle \geq 0$$

whenever $x, y \in A$ and $x^* \in F(x), y^* \in F(y)$, and *strictly monotone* if this inequality is sharp whenever $x \neq y$.

A monotone set-valued mapping is a special case of cyclically monotone set-valued mappings.

Let n be a positive integer, $n \geq 2$, and A a subset of X. A set-valued mapping $F : A \hookrightarrow X^*$ is said to be *n-cyclically monotone* provided that

$$\sum_{1\leq k\leq n} \langle x_k^*, x_k - x_{k-1}\rangle \geq 0$$

whenever $x_0, x_1, \ldots, x_n \in X$, $x_n = x_0$, and $x_k^* \in F(x_k)$ for every $k = 1, 2, \ldots, n$.

We say that F is *cyclically monotone* if it is n-cyclically monotone for every $n \in \mathbf{N}$, $n \geq 2$.

Remark 1.1. Of course, in the case of one dimension, the notions monotone mapping and cyclically monotone mapping coincide.

A set-valued mapping $F : A \hookrightarrow X^*$, $A \subset X$, is said to be *maximal monotone* if it is monotone and the graph

$$\mathrm{Gr}\,(F) = \{(x^*, x) \in X^* \times X, x^* \in F(x)\}$$

is maximal in the family of the graphs of monotone set-valued mappings from A into subsets of X^*, ordered by inclusion.

Remark 1.2. Let X be an Hilbert space, and let U, V be subsets of X. Then $F : U \hookrightarrow V$ is a (maximal) monotone if and only if F^{-1} is a (maximal) monotone.

Remark 1.3. Let $(a, b) \subset \mathbf{R}$. If $f : (a, b) \mapsto \mathbf{R}$ is a nondecreasing and continuous function, then f is maximal monotone.

A set-valued mapping F from a topological space A into nonempty subsets of a Hausdorff locally convex space Y is called *cusco* if $F(t)$ is compact and convex for each $t \in A$ and F is upper-semicontinuous, i.e., $\{t \in A : F(t) \subset U\}$ is an open subset of X for each open subset U of Y). Further, F is said to be *minimal cusco on* A if its graph does not contain the graph of any other cusco on A. Maximal monotone set-valued mapping is the well-known example of minimal cusco mapping, see e.g., [**18, 26**].

Proposition 1.1 [**25**, Theorem 3.2]. *Let A be a nonempty open subset of X and F a set-valued mapping from A into nonempty subsets of X^* which is minimal weak* cusco. If F possesses a densely defined monotone selection, then F is monotone.*

Now, we recall some assertions about the Clarke generalized gradient which was introduced in [**10**] and about locally Lipschitz functions.

Let $f : X \mapsto \mathbf{R}$ be Lipschitz near x, and let $v \in X$. *The Clarke upper and lower generalized directional derivatives of f at x in the direction* v are defined, respectively, by

$$f^\circ(x;v) = \limsup_{\substack{y\to x\\ t\downarrow 0}} \frac{f(y+tv)-f(y)}{t},$$
$$f_\circ(x;v) = \liminf_{\substack{y\to x\\ t\downarrow 0}} \frac{f(y+tv)-f(y)}{t},$$

and the *Clarke generalized gradient of f at x* is defined by

$$\partial_c f(x) = \{x^* \in X^* : \langle x^*, v\rangle \le f^\circ(x,v), \text{ for all } v \in X\}.$$

Proposition 1.2 [**11**, Proposition 2.1.1]. *Let U be an open convex subset of X, and let $f : U \mapsto \mathbf{R}$ be Lipschitz. Then the set-valued mapping $F : U \hookrightarrow X^* : x \hookrightarrow \partial_c f(x)$ is cusco.*

A function $f : X \mapsto \mathbf{R}$ is *strictly differentiable* at a point $\widehat{x}$ if $f(x') = f(x) + \langle \bigtriangledown f(\widehat{x}), x'-x\rangle = o(\|x'-x\|)$, i.e.,

$$\frac{f(x') - f(x) - \langle \bigtriangledown f(\widehat{x}), x'-x\rangle}{\|x'-x\|} \to 0 \quad \text{as } x', x \to \widehat{x} \text{ with } x' \ne x.$$

Proposition 1.3 [**11**, Proposition 2.2.4]. *Let $f : X \mapsto \mathbf{R}$ be Lipschitz near $x \in X$. Then f is strictly differentiable at x if and only if $\partial_c f(x)$ is a singleton.*

Proposition 1.4 [**26**, Proposition 3.26]. *Let $U \subset X$ be an open convex set. If a set-valued mapping F from U into subsets of X^* is* *maximal cyclically monotone with $D(F)$ nonempty, then there exists a proper convex lower semi-continuous function $f : U \mapsto \mathbf{R}^* = \mathbf{R}\cup\{+\infty\}$ such that $F = \partial_c f (= \partial f)$.*

Proposition 1.5 [**11**, Proposition 2.2.9]. *Let $U \subset X$ be an open convex set and $f : U \mapsto \mathbf{R}$ a locally Lipschitz function. Then f is (strictly) convex if and only if the set-valued mapping $x \hookrightarrow \partial_c f(x)$ is (strictly) monotone.*

2. Choice of the type of derivative. First, we note that for a set-valued mapping $F : \mathbf{R} \hookrightarrow \mathbf{R}$, we define

$$\liminf_{t\downarrow 0} F(t) := \liminf_{t\downarrow 0} \inf\{a : a \in F(t)\},$$

etc.

In [**22**], the following generalized second-order directional derivatives were introduced for a function $f : X \mapsto \mathbf{R}$ which is Lipschitz near x:

$$f''(x; u, v) = \liminf_{t\to 0} \frac{\langle \partial_c f(x + tu) - \partial_c f(x), v\rangle}{t},$$
$$f''_+(x; u, v) = \liminf_{t\downarrow 0} \frac{\langle \partial_c f(x + tu) - \partial_c f(x), v\rangle}{t}.$$

Let us recall that a locally Lipschitz function f is said to be *regular* at x provided that for every $v \in X$ the directional derivative $f'(x; v) = \lim_{t\downarrow 0}(f(x + tv) - f(x))/t$ exists, and moreover $f'(x; v) = f^\circ(x; v)$ for every $v \in X$. Note that if $f : X \mapsto \mathbf{R}$ is a regularly locally Lipschitz function near x, then it holds [**2**]:

$$f''_+(x; h, h) = \liminf_{t\downarrow 0} \frac{f'(x + th; h) - f'(x; h)}{t}. \tag{1}$$

Further, because a continuously differentiable function at the point x is strictly differentiable at this point, see e.g., [**11**, Section 2.2], we have by Proposition 1.3 that if $f : X \mapsto \mathbf{R}$ is a $C^{1,1}$ function near x, i.e., f is differentiable with a locally Lipschitz derivative near x, then

$$f''_+(x; u, v) = \liminf_{t\downarrow 0} \frac{\langle \bigtriangledown f(x + tu) - \bigtriangledown f(x), v\rangle}{t}. \tag{2}$$

Theorem 2.1 [**22**, Theorem 1.3]. *Let $U \subset X$ be an open convex set and $f : U \mapsto \mathbf{R}$ a locally Lipschitz function. Then the following are equivalent*:

(i) *f is convex.*

(ii) *$f''^{l}(x; h, h) \geq 0$, for all $x \in U$, for all $h \in S_X$.*

(iii) *$f''^{l}_{+}(x; h, h) \geq 0$, for all $x \in U$, for all $h \in S_X$.*

Theorem 2.2 [**23**, Theorem 3]. *Let $f : \mathbf{R}^n \mapsto \mathbf{R}$ be a $C^{1,1}$ function near x. If $\bigtriangledown f(x) = 0$, and $f''^{l}_{+}(x; h, h) > 0$, for all $h \in S_{\mathbf{R}^n}$, then f attains a strict local minimum at x.*

Analogously, we can define for a function $f : X \mapsto \mathbf{R}$ which is Lipschitz near x the "upper" form of the previous notions, i.e.,

$$f'^{U}(x; u, v) := \limsup_{t \to 0} \frac{\langle \partial_c f(x + tu) - \partial_c f(x), v \rangle}{t},$$
$$f'^{U}_{+}(x; u, v) := \limsup_{t \downarrow 0} \frac{\langle \partial_c f(x + tu) - \partial_c f(x), v \rangle}{t}.$$

Motivated by (1), in [**2**] we introduced the following generalized second-order directional derivative for a function $f : X \to \mathbf{R}$ which is regularly locally Lipschitz near $x \in X$, $(u, v) \in X \times X$, by

$$f'^{u}_{+}(x; u, v) = \limsup_{t \downarrow 0} \frac{f'(x + tu; v) - f'(x; v)}{t}.$$

We notice that because of possible sharp inequality $f'^{u}_{+}(x; h, h) < f'^{U}_{+}(x; h, h)$ (for example, considering a function $f : \mathbf{R} \mapsto \mathbf{R} : f(x) = |x|$, then $f'^{u}_{+}(0; 1, 1) = 0 < +\infty = f'^{U}_{+}(0; 1, 1)$), it is necessary to distinguish the notions of $f'^{U}(x; h, h)$ and $f'^{u}_{+}(x; h, h)$.

Thanks to the same arguments as preceded formula (2), for a function $f : X \mapsto \mathbf{R}$ which is $C^{1,1}$ near x, we have

$$f'^{U}_{+}(x; u, v) = \limsup_{t \downarrow 0} \frac{\langle \bigtriangledown f(x + tu) - \bigtriangledown f(x), v \rangle}{t} = f'^{u}_{+}(x; u, v).$$

Since

$$(-f)'^{l}(x; u, v) = -f'^{u}(x; u, v), \quad \text{and} \quad (-f)'^{l}_{+}(x; u, v) = -f'^{u}_{+}(x; u, v),$$

it is easy to derive the characterization of the concavity for a locally Lipschitz function in terms of $f'^{U}(x;h,h)$ and $f'^{U}_{+}(x;h,h)$ (compare with Theorem 2.1), and a sufficient optimality condition for a strict local maximum for a $C^{1,1}$ function with $f'^{u}_{+}(x;h,h)$ (compare with Theorem 2.2).

The minimal cusco structure of the set-valued mapping $x \hookrightarrow \partial_c f(x)$ of regularly locally Lipschitz functions (for more details see [**6, 21**]) yields [**2**, Theorem 3.2] the following result.

Theorem 2.3. *Let $U \subset X$ be an open convex set and $f : U \mapsto \mathbf{R}$ a regularly locally Lipschitz function. Then f is convex if and only if*

$$f'^{u}_{+}(x;h,h) \geq 0, \quad \textit{for all} \quad x \in U, \quad \textit{for all} \quad h \in S_X.$$

Various other generalized second-order directional derivatives have been introduced and deeply studied during the last 25 years, see e.g., [**4, 8, 9, 12, 14–17, 20, 27, 29–31** and references therein]. In terms of many of them, characterizations of convexity and second-order optimality conditions (both sufficient and necessary) were stated.

Therefore, much attention was focused, in [**2, 3, 22, 24**], on relations between them and a (Dini) type of directional derivative which was mentioned in this section (in particular, for the discussion on Theorem 2.3, see [**2**, Section 5]). We hope that these considerations together with relatively comfortable calculus explain our choice of directional derivatives enough.

3. Convexity of regular functions. Using a certain condition for a real function of a real variable to be nondecreasing (Lemma 3.4) and some properties of regularly locally Lipschitz functions, we improve our previous (see Theorem 2.3 in this paper) characterization of convexity for regularly locally Lipschitz functions via $f'^{u}_{+}(x;h,h)$.

Lemma 3.1 [**2**, Lemma 2.1]. *Let $f : X \mapsto \mathbf{R}$ be a regularly locally Lipschitz function and $x, y \in X$, $x \neq y$. Then, the function $g : (0,1) \mapsto \mathbf{R}$ defined by $g(t) = f(x + t(y-x))$ is regularly locally Lipschitz, and*

$$\partial_c g(t) = \langle \partial_c f(x + t(y-x)), y-x \rangle, \quad \textit{for all } t \in (0,1).$$

Lemma 3.2 [**19**]. *Let $f : X \mapsto \mathbf{R}$ be a Lipschitz function on an open set containing the line segment $[x, y]$. Then there exists a point $u \in (x, y)$ such that*

$$f(y) - f(x) \in \langle \partial_c f(u), y - x \rangle.$$

Lemma 3.3. *Let $f : X \mapsto \mathbf{R}$ be a regularly locally Lipschitz function, and let $x \in X$, $h \in S_X$. Then*

$$\limsup_{t \uparrow 0} f'(x + th; h) \le f'(x; h) = \limsup_{t \downarrow 0} f'(x + th; h). \tag{3}$$

Proof. Because of the upper semi-continuity of the mapping $z \to f^\circ(z; h)$ and regularity of f, it holds that

$$\limsup_{t \uparrow 0} f'(x + th; h) \le \limsup_{y \to x} f^\circ(y; h) \le f^\circ(x; h) = f'(x; h),$$

and

$$\limsup_{t \downarrow 0} f'(x + th; h) \le f'(x; h).$$

Thus, for the proof of formula (3), it suffices to show that

$$f'(x; h) \le \limsup_{t \downarrow 0} f'(x + th; h). \tag{4}$$

Thanks to Lemma 3.2, for every $s > 0$ there exists a $t \in (0, s)$ satisfying

$$\frac{f(x + sh) - f(x)}{s} \in \langle \partial_c f(x + th), h \rangle \le f^\circ(x + th; h) = f'(x + th; h).$$

Limiting for $s \downarrow 0$, we obtain inequality (4). □

In the following lemma, we denote the Dini right upper derivative of function $f : \mathbf{R} \mapsto \mathbf{R}$ at $x \in \mathbf{R}$ by $\overline{f'}(x)$, i.e.,

$$\overline{f'}(x) = \limsup_{t \downarrow 0} \frac{f(x + t) - f(x)}{t}.$$

Lemma 3.4 [**28**, page 135]. *Suppose that the function* $f : \mathbf{R} \mapsto \mathbf{R}$ *satisfies the following conditions*:

(1) *at each point* x,

$$\limsup_{t\uparrow 0} f(x+t) \le f(x),$$

(2) $\overline{f'}(x) \ge 0$ *almost everywhere* (*in the Lebesgue sense*),

(3) $\overline{f'}(x) > -\infty$ *everywhere except possibly at the points* x *of a denumerable set, at each point of which the inequality*

$$f(x) \le \limsup_{t\downarrow 0} f(x+t)$$

holds.

Then f *is nondecreasing.*

Theorem 3.1. *Let* $U \subset X$ *be an open convex set and* $f : U \mapsto \mathbf{R}$ *a regularly locally Lipschitz function. Then* f *is convex if and only if for every* $x \in U$ *and* $h \in S_X$, *the set* $I_{x,h} = \{y \in X : y = x+th, t \in \mathbf{R}\} \cap U$ *satisfies the following conditions*:

(i) $f_+^{\prime u}(y; h, h) \ge 0$ *almost everywhere on* $I_{x,h}$,

(ii) $f_+^{\prime u}(y; h, h) > -\infty$ *everywhere except possibly at the points of a denumerable subset of* $I_{x,h}$.

Proof. First, if f is convex, then the set-valued mapping $x \hookrightarrow \partial_c f(x)$ is monotone on U. Since $f'(x+th; h) \in \langle \partial_c f(x+th), h\rangle$, for every $t \in \mathbf{R}$ satisfying $x + th \in U$, $f_+^{\prime u}(y; h, h) \ge 0$ for every $y \in I_{x,h}$ and therefore conditions (i) and (ii) are satisfied.

Conversely, it suffices to show that f is convex on $I_{x,h}$ for every $x \in U$ and $h \in S_X$.

We take arbitrary $x \in U$ and $h \in S_X$, and let us consider a function $g : (a, b) \mapsto \mathbf{R} : g(t) = f(x + th)$, where $(a, b) = \{t \in \mathbf{R} : x + th \in U\}$. Due to Lemma 3.1, g is regularly locally Lipschitz. Hence,

$$g^\circ(t; 1) = g'(t; 1) = f'(x + th; h), \quad \text{for all } t \in (a, b).$$

Then

$$(5)\qquad \begin{aligned} g_+'^u(t;1,1) &= \limsup_{\tau\downarrow 0} \frac{g'(t+\tau;1)-g'(t;1)}{\tau} \\ &= \limsup_{\tau\downarrow 0} \frac{f'(x+th+\tau h;h)-f'(x+th;h)}{\tau}, \\ &\qquad \text{for all } t\in(a,b). \end{aligned}$$

It is obvious that f is convex on the line $I_{x,h}$ if and only if g is convex. Following Lemma 3.3, formula (5) and conditions (i) and (ii), we obtain that a function $t \to g^\circ(t;1)$ is nondecreasing by Lemma 3.4. Because of $g^\circ(t;1) = \max \partial_c g(t)$, a function $t \to g^\circ(t,1)$ is a selection of $\partial_c g(t)$. Since the set-valued mapping $t \hookrightarrow \partial_c g(t)$ is minimal cusco, Proposition 1.1 yields that this mapping is monotone. Proposition 1.5 completes the proof. □

Corollary 3.1. *Let $U \subset X$ be an open convex set and $f : U \mapsto \mathbf{R}$ a regularly locally Lipschitz function. Then f is convex if and only if*

$$f_+'^u(x;h,h) \geq 0$$

holds for every $[x,h] \in U \times S_X$ except possibly at the points $[x,h] \in U \times S_X$ of a denumerable set.

4. Strict monotonicity. In order to obtain our main result (Theorem 4.2), we derive the characterization of strict monotonicity for set-valued mappings in terms of the following derivatives.

Definition 4.1. Let $F : X \hookrightarrow X^*$ be a set-valued mapping, $x,u,v \in X$. Then

$$(6)\qquad \begin{aligned} F^l(x;u,v) &:= \liminf_{t\to 0} \frac{\langle F(x+tu)-F(x),v\rangle}{t}, \\ F_+^l(x;u,v) &:= \liminf_{t\downarrow 0} \frac{\langle F(x+tu)-F(x),v\rangle}{t}, \\ F_-^l(x;u,v) &:= \liminf_{t\uparrow 0} \frac{\langle F(x+tu)-F(x),v\rangle}{t}, \\ F^u(x;u,v) &:= \limsup_{t\to 0} \frac{\langle F(x+tu)-F(x),v\rangle}{t}. \end{aligned}$$

For a single-valued version of Lemma 4.1, see e.g., [**28**, page 134]. The proof of Lemma 4.1 uses the ideas of the proof of Lemma 2.1 in [**24**].

Lemma 4.1. *Let* $(a,b) \subset \mathbf{R}$*, and let* $F : (a,b) \hookrightarrow \mathbf{R}$ *be a set-valued mapping. Then* F *is monotone if and only if*

$$F^l(x) = F^l(x;1,1) \geq 0, \quad \textit{for all } x \in (a,b). \tag{7}$$

For the proof of Lemma 4.1, we will use the following well-known fact:

Fact 4.1. *Let* $\beta \in \mathbf{R}$*, and let* $M \subset \mathbf{R}$*,* $M \leq \beta$ *be a set satisfying*

(i) $M \neq \varnothing$,

(ii) (*for all* $x \in M$) ($x < \beta \Rightarrow$ (*there exists* $y \in M$), $y > x$),

(iii) $\sup M \in M$.

Then $\beta \in M$.

Proof of Lemma 4.1. 1. We suppose that the set-valued map F is monotone. We fix $x \in (a,b)$ and consider an arbitrary $t \in \mathbf{R}$, $t \neq 0$, satisfying $x+t \in (a,b)$. Then, for any $q \in F(x+t)$ and for every $p \in F(x)$,

$$(q-p)t \geq 0.$$

Therefore,

$$\frac{q-p}{t} \geq 0.$$

Hence,

$$F^l(x) \geq 0.$$

2. We suppose that condition (6) is true. Consider arbitrary $\alpha, \beta \in (a,b)$, $\alpha < \beta$. Fix an arbitrary $\varepsilon > 0$, and put

$$M := \{x \in [\alpha,\beta] : F(x) - F(\alpha) > -\varepsilon(x-\alpha)\}. \tag{7}$$

Due to condition (6), for each $x \in (a,b)$, there exists a $\delta_x > 0$ such that, for every $0 < t < \delta_x$,

$$F(x+t) - F(x) > -\varepsilon t \tag{8}$$

and

$$F(x) - F(x-t) > -\varepsilon t. \tag{9}$$

Now we show that M satisfies the conditions of the previous fact. Setting $x = \alpha$ in (8), it is straightforward to verify condition (i). For condition (ii), we fix $x \in M$ with $x < \beta$. Due to inequalities in (7) and (8), we obtain, for $0 < t < \delta_x$,

$$\begin{aligned} F(x+t) - F(\alpha) \subset (F(x+t) - F(x)) + (F(x) - F(\alpha)) \\ > -\varepsilon t - \varepsilon(x-\alpha) = -\varepsilon(x+t-\alpha). \end{aligned}$$

Finally, put $x = \sup M$. By the property of the supremum, we can find $0 < t < \delta_x$ satisfying $x - t \in M$. Then by (9) and (7),

$$\begin{aligned} F(x) - F(\alpha) \subset (F(x) - F(x-t)) + (F(x-t) - F(\alpha)) \\ > -\varepsilon t - \varepsilon(x-t-\alpha) = -\varepsilon(x-\alpha). \end{aligned}$$

It means that $x \in M$ and also condition (iii) of Fact 4.1 is verified. Thus,

$$F(\beta) - F(\alpha) > -\varepsilon(\beta - \alpha).$$

Since ε was arbitrary, we have $F(\beta) \geq F(\alpha)$. □

In an analogous way, we can obtain the "upper" form of Lemma 4.1:

Lemma 4.2. *Let* $(a,b) \subset \mathbf{R}$ *and* $F : (a,b) \hookrightarrow \mathbf{R}$ *be a set-valued mapping. Then* $F(x) \geq F(y)$ *for every* $x, y \in (a,b)$ *satisfying* $x \leq y$ *if and only if*

$$F^u(x) = F^u(x; 1, 1) \leq 0, \quad \textit{for all } x \in (a,b).$$

The following lemma follows immediately from Lemmas 4.1 and 4.2:

Lemma 4.3. *Let* $(a,b) \subset \mathbf{R}$*, and let* $F : (a,b) \hookrightarrow \mathbf{R}$ *be a set-valued map satisfying*

$$F^l(x) \geq 0 \quad \textit{and} \quad F^u(x) \leq 0, \quad \textit{for every } x \in (a,b).$$

Then F is a single-valued constant function.

Now, let us give the characterization of strict monotonicity for set-valued mappings from $\mathbf{R}$ into subsets of $\mathbf{R}$ at first.

Proposition 4.1. *Let $(a,b) \subset \mathbf{R}$, and let $F : (a,b) \hookrightarrow \mathbf{R}$ be a set-valued mapping. Then the following conditions are equivalent*

(i) *F is strictly monotone.*

(ii) *$F^l(x) \geq 0$ for every $x \in (a,b)$, and $\{x : F^u(x) > 0\}$ is dense in (a,b).*

(iii) *$\alpha < \beta \Rightarrow \sup F(\alpha) < \inf F(\beta)$ for every $\alpha, \beta \in (a,b)$.*

Proof. The implication (iii) $\Rightarrow$ (i) is clear. It suffices to show (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii).

(i) $\Rightarrow$ (ii). If F is strictly monotone, then $F^l(x) \geq 0$ for every $x \in (a,b)$. Suppose now that there exists an open interval $(c,d) \subset (a,b)$ with the property $F^u(x) \leq 0$ whenever $x \in (c,d)$. Lemma 4.3 implies that F is a single-valued constant function on (c,d), but it is a contradiction. Hence, the set $\{x : F^u(x) > 0\}$ is dense in (a,b).

(ii) $\Rightarrow$ (iii). Since $F^l(x) \geq 0$ for every $x \in (a,b)$, Lemma 4.1 implies that F is monotone. Fix arbitrary $\alpha, \beta \in (a,b)$, $\alpha < \beta$. Using the density of $\{x : F^u(x) > 0\}$ in (α, β), we obtain $z_1, z_2 \in (\alpha, \beta)$, $z_1 < z_2$ and $u \in F(z_1)$, $v \in F(z_2)$ such that $u < v$. Subsequently,

$$F(\alpha) \leq u < v \leq F(\beta).$$

Thus, $\sup F(\alpha) < \inf F(\beta)$. □

In higher dimensions, the characterization of strict monotonicity for set-valued mappings can be expressed as follows.

Proposition 4.2. *Let $U \subset X$ be an open convex set, and let $F : U \hookrightarrow X^*$ be a set-valued mapping. Then F is strictly monotone if and only if both of the following conditions are satisfied.*

(i) $F^l(x;h,h) \geq 0$, *for all* $x \in U$, *for all* $h \in S_X$,

(ii) *the set* $\{z : F^u(z;h,h) > 0\}$ *is dense in* $I_{x,h} = \{x + th : t \in \mathbf{R}\} \cap U$, *for every* $x \in U$ *and for every* $h \in S_X$.

Proof. We fix arbitrary $x \in U$ and $h \in S_X$, and consider the corresponding set $I_{x,h}$. Let all $t \in \mathbf{R}$ which work in $I_{x,h}$ form an interval (a,b). Since $x \in U$ and $h \in S_X$ are arbitrary, it suffices to show that, for every $t, s \in \mathbf{R}$ satisfying $t \in (a,b)$ and $(t+s) \in (a,b)$, we have

$$s\langle F(x+(t+s)h) - F(x+th), h\rangle > 0 \tag{10}$$

if and only if

$$F^l(x+th,h,h) \geq 0, \quad \text{for all } t \in (a,b) \tag{11}$$

and the set

$$\{t \in (a,b) : F^u(x+th,h,h) > 0\} \text{ is dense in } (a,b). \tag{12}$$

Consider $G : (a,b) \hookrightarrow \mathbf{R}$ defined by $G(t) = \langle F(x+th), h\rangle$ for every $t \in (a,b)$. Due to Proposition 4.1, G is strictly monotone, i.e., inequality (10) holds, if and only if $G^l(t) \geq 0$ for every $t \in (a,b)$, i.e., condition (11) is satisfied, and the set $\{t \in (a,b) : G^u(t) > 0\}$ is dense in (a,b), i.e., condition (12) is satisfied. □

We notice that

$$F^l(x;h,h) = \min\{F^l_+(x;h,h), F^l_-(x;h,h)\},$$

and

$$F^l_+(x;-h,-h) = F^l_-(x;h,h), \quad \text{for all } x \in U, \text{ for all } h \in S_X.$$

Then, due to the symmetry of S_X, condition (i) in Proposition 4.2 is equivalent to the following condition:

$$F^l_+(x;h,h) > 0, \quad \text{for all } x \in U, \text{ for all } h \in S_X,$$

and we have

Theorem 4.1. *Let $U \subset X$ be an open convex set, and let $F : U \hookrightarrow X^*$ be a set-valued mapping. Then F is strictly monotone if and only if both of the following conditions are satisfied.*

(i) *$F^l_+(x;h,h) \geq 0$, for all $x \in U$, for all $h \in S_X$,*

(ii) *the set $\{z : F^u(z;h,h) > 0\}$ is dense in $I_{x,h} = \{x+th : t \in \mathbf{R}\} \cap U$, for every $x \in U$ and for every $h \in S_X$.*

Setting $F \equiv \partial_c f$ in Theorem 4.1 and following Proposition 1.5, we obtain the generalization of Theorem 1.1:

Theorem 4.2. *Let $U \subset X$ be an open convex set, and let $f : U \to \mathbf{R}$ be a locally Lipschitz function. Then f is strictly convex if and only if both of the following conditions are satisfied.*

(i) *$f'^l_+(x;h,h) \geq 0$, for all $x \in U$, for all $h \in S_X$,*

(ii) *the set $\{z : f'^U(z;h,h) > 0\}$ is dense in $I_{x,h} = \{x+th : t \in \mathbf{R}\} \cap U$, for every $x \in U$ and for every $h \in S_X$.*

As an application of Theorem 4.2, we can obtain the following second-order sufficient optimality condition which was introduced in [**12**, Proposition 5.2] (see also, e.g., [**8**, Proposition 6.2], [**30**, Theorem 5.1(ii)], [**31**, Theorem 4.2(ii)]).

Corollary 4.1. *Let $f : \mathbf{R}^n \mapsto \mathbf{R}$ be a $C^{1,1}$ function near x. If $\bigtriangledown f(x) = 0$, and*

$$f'^l_\infty(x;h,h) = \liminf_{y\to x, t\downarrow 0} \frac{\bigtriangledown f(y+th) - \bigtriangledown f(y)}{t} > 0, \quad \textit{for all } h \in S_{\mathbf{R}^n},$$

then f attains a strict local minimum at x.

Proof. The function $(y,h) \to f'^l_\infty(y;h,h)$ is lower semi-continuous on $W \times \mathbf{R}^n$ [**30**, Proposition 2.3], where W is an open neighborhood of x such that f satisfies the $C^{1,1}$ property there. Then, for every $h \in S_{\mathbf{R}^n}$, there exist an open neighborhood $U(h) \subset \mathbf{R}^n$ of x and an open neighborhood $V(h) \subset \mathbf{R}^n$ of h satisfying $f'^l_+(y;h',h') > 0$ for every $y \in U(h)$ and for every $h' \in V(h)$. The compactness of $S_{\mathbf{R}^n}$ yields that there exist points $h_1, h_2, \dots, h_k \in S_{\mathbf{R}^n}$ such that $S_{\mathbf{R}^n} \subset \cup_{i=1}^k V(h_i)$. Then, for every $y \in \cap_{i=1}^k U(h_i)$ and for every $h \in S_{\mathbf{R}^n}$, we have $f_+^{\prime l}(y; h, h) > 0$ and thus f is strictly convex due to Theorem 4.2. Since $0 \in \partial_c f(x)$, f attains a strict local minimum at x. $\square$

Remark 4.1. Of course, Corollary 4.1 follows immediately from Theorem 2.2, but the second-order condition in terms of $f_\infty^{\prime l}(x; h, h)$ in Corollary 4.1 ensures strict convexity of f near x in contrast to the second-order condition in terms of $f_+^{\prime l}(x; h, h)$ in Theorem 2.2.

5. Discussion of Theorem 4.2. In this section, we will discuss whether or under which next assumptions conditions (i) and (ii), respectively, in Theorem 4.2 could be weakened.

5.1. Condition (i). There arises the question of whether or not $f_+^{\prime l}(x; h, h)$ can be replaced by $f_+^{\prime U}(x; h, h)$ in condition (i) of Theorem 4.2.

Of course, the necessary condition of strict convexity is then also satisfied, but the following example shows that there is a nonconvex function satisfying the replaced condition (i) in terms of $f_+^{\prime U}(x; h, h)$ and condition (ii) of Theorem 4.2.

Example 5.1. Consider a function $f : (0, +\infty) \to \mathbf{R}$ such that for every $n \in \mathbf{N} \cup \{0\}$ we have $f(x) = (x - n)^2 + n$ whenever $x \in (n, n+1]$. Then $f_+^{\prime U}(x; 1, 1) = f_+^{\prime U}(x, -1, -1) = 2$ for every $x \in (0, +\infty)$, but f is not convex.

It remains an open question whether $f_+^{\prime l}(x; h, h)$ can be replaced by $f_+^{\prime U}(x; h, h)$ in condition (i) of Theorem 4.2 for regularly locally Lipschitz functions. Nevertheless, for this class of functions, we are able, thanks to Corollary 3.1, to state a tighter form of Theorem 4.2 (note that $f_+^{\prime l}(x; h, h) \le f_+^{\prime u}(x; h, h) \le f_+^{\prime U}(x; h, h)$):

Theorem 5.1. *Let $U \subset X$ be an open convex set, and let $f : U \to \mathbf{R}$ be a regularly locally Lipschitz function. Then f is strictly convex if and only if both the following conditions are satisfied.*

(i) $f'^u_+(x;h,h) \geq 0$, *for all* $[x,h] \in U \times S_X$ *except possibly at the points* $[x,h] \in U \times S_X$ *of a denumerable set,*

(ii) *the set* $\{z : f'^U(z;h,h) > 0\}$ *is dense in* $I_{x,h} = \{x+th : t \in \mathbf{R}\} \cap U$ *and for every* $h \in S_X$.

Proof. If f is strictly convex, then conditions (i) and (ii) are satisfied by means of Corollary 3.1 and Theorem 4.2, respectively.

If condition (i) holds, then f is convex by Corollary 3.1. So, it suffices to show that the convexity of a locally Lipschitz function f together with condition (ii) imply strict convexity of f. But the last follows immediately from Proposition 1.5. □

5.2. Condition (ii). There is also a question whether $f'^U(z;h,h)$ can be replaced by $f'^l_+(z;h,h)$ in condition (ii) of Theorem 4.2. Because of $f'^U(z;h,h) \geq f'^l_+(z;h,h)$, the sufficient condition of strict convexity is immediately satisfied.

On the other hand, we will prove that there exists a strictly convex function for which $f'^l_+(\cdot;1,1) \equiv 0$.

At first, we show that there is a strictly monotone set-valued mapping satisfying $F^l_+(\cdot;1,1) \equiv 0$.

Example 5.2. Let us consider a function $g : \mathbf{R} \to \mathbf{R}$,

$$g(x) = \begin{cases} c(x) & \text{if } x \in [0,1], \\ x & \text{otherwise,} \end{cases}$$

where c is the Cantor function, see for example [**13**, Chapter 8, Example 1]. We recall some properties of g.

1. g maps the Cantor set C on the segment $[0,1]$.

2. $x \in C$ if and only if we can express x in ternary expansion as follows:

$$x = 0, b_1 b_2 b_3 \cdots,$$

where b_i is either 0 or 2 for every $i \in \mathbf{N}$. Then

$$0, \frac{b_1}{2}\frac{b_2}{2}\frac{b_3}{2}\cdots$$

is the expression of $g(x)$ in binary expansion.

3. Since g is a nondecreasing and continuous function, g is a maximal monotone due to Remark 1.3.

Now we consider a set-valued mapping F defined by $\mathrm{Gr}\,(F) = (\mathrm{Gr}\,(g))^{-1}$. Property 1 of g implies that $D(F) = \mathbf{R}$.

Using Property 2 of g, we describe the set-valued map F more exactly on the open interval $(0,1)$. For this, we determine a maximal selection u of F.

For every $x \in (0,1)$, we can find a sequence $\{d_n(x)\}_{n=1}^{+\infty}$ with the properties

(a) $d_n(x) \in \{0,1\}$ for every $n \in \mathbf{N}$.

(b) $x = \sum_{n=1}^{+\infty}(d_n(x)/2^n)$.

(c) $M_0(x) = \{n \in \mathbf{N} : d_n(x) = 0\}$ is an infinite set.

Define

$$u(x) = 2\sum_{n=1}^{+\infty} \frac{d_n(x)}{3^n},$$

and set

$$M_1(x) = \mathbf{N} \setminus M_0(x) (= \{n \in \mathbf{N}, d_n(x) = 1\}).$$

According to the properties of $M_1(x)$, we have

(A) If $M_1(x)$ is an infinite set, then $F(x) = \{u(x)\}$.

(B) If $M_1(x)$ is a finite set, then $F(x) = [u(x) - (1/3^{n_1(x)}), u(x)]$,

where $n_1(x) = \max M_1(x)$.

Since F^{-1} is a function, one has that F is strictly monotone on $(0,1)$.

We show that $F_+^l(x) = F_+^l(x;1,1) = 0$ on $(0,1)$. Thanks to the monotonicity of F, it suffices, for every $x \in (0,1)$ and for every $0 < \varepsilon < 1-x$, to find $x_\varepsilon \in (x,1)$ such that $x_\varepsilon - x < \varepsilon$, and

$$\frac{u(x_\varepsilon) - u(x)}{x_\varepsilon - x} < \varepsilon.$$

Fix $x \in (0,1)$ and $0 < \varepsilon < 1-x$. Then there exists a $k \in M_0(x)$,

$$\frac{1}{2^k} < 2\left(\frac{2}{3}\right)^k < \varepsilon.$$

Define $x_\varepsilon = x + (1/2^k)$; then $u(x_\varepsilon) = u(x) + (2/3^k)$. Hence, $x_\varepsilon - x < \varepsilon$, and

$$\frac{u(x_\varepsilon) - u(x)}{x_\varepsilon - x} = \frac{2/3^k}{1/2^k} = 2\left(\frac{2}{3}\right)^k < \varepsilon,$$

which is what we wanted.

So, $F^u(z;h,h)$ cannot be replaced by $F^l_+(z;h,h)$ (and thus not even by $F^l(z;h,h)$) in condition (ii) of Theorem 4.1. Deeper insight into Example 5.2 demonstrates that the corresponding change, i.e., $f'^l_+(z;h,h)$ instead of $f'^U(z;h,h)$, is not possible in condition (ii) of Theorem 4.2 (recall that Theorem 4.2 is a special case of Theorem 4.1).

Example 5.3. We follow Example 5.2. Property 3 of g and Remark 1.2 imply that F is maximal monotone. By Remark 1.1, F is also maximal cyclically monotone. Applying Proposition 1.4, we obtain a proper convex lower semi-continuous function $f : (0,1) \mapsto \mathbf{R}^*$ such that $F = \partial f$. Since $D(\partial f) = (0,1)$, f is a real function in fact.

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REFERENCES

1. J.P. Aubin and A. Cellina, *Differential inclusions*, Springer-Verlag, Berlin, 1984.

2. D. Bednařík and K. Pastor, *On characterization of convexity for regularly locally Lipschitz functions*, Nonlinear Anal. TMA **57** (2004), 85–97.

3. ———, *Elimination of strict convergence in optimization*, SIAM J. Control Optim. **43** (2004), 1063–1077.

4. A. Ben-Tal and J. Zowe, *Directional derivatives in nonsmooth optimization*, J. Optim. Theory Appl. **47** (1985), 483–490.

5. J.M. Borwein, *Minimal cuscos and subgradients of Lipschitz functions*, in *Fixed point theory and its applications*, J.-B. Baillon and M. Thera, eds., Pitman Lecture Notes Math., Longman, Essex, 1991.

6. J.M. Borwien and W.B. Moors, *Essentially strictly differentiable Lipschitz functions*, J. Funct. Anal. **149** (1997), 305–351.

7. N. Bourbaki, *Éléments de mathématique*, Livre **IV**: Fonctions d'une variable réele, Hermann & C^{ie}, Paris, 1951.

8. W.L. Chan, L.R. Huang and K.F. Ng, *On generalized second-order derivatives and Taylor expansions in nonsmooth analysis*, SIAM J. Control Optim. **32** (1994), 591–611.

9. R.W. Chaney, *Second-order sufficiency conditions for nondifferentiable programming problems*, SIAM J. Control Optim. **20** (1982), 20–33.

10. F.H. Clarke, *Necessary conditions for nonsmooth problems in optimal control and the calculus of variations*, Ph.D. thesis, University of Washington, 1973.

11. ———, *Optimization and nonsmooth analysis*, John Wiley and Sons, New York, 1983.

12. R. Cominetti and R. Correa, *A generalized second-order derivative in nonsmooth optimization*, SIAM J. Control Optim. **28** (1990), 789–809.

13. B.R. Gelbaum and J.H. Olmsted, *Counterexamples in analysis*, Holden Day, Amsterodam, 1964.

14. J.B. Hiriart-Urruty, J.J. Strodiot and V.H. Nguyen, *Generalized Hessian matrix and second-order optimality conditions for problems with $C^{1,1}$ data*, Appl. Math. Optim. **11** (1984), 43–56.

15. L.R. Huang and K.F. Ng, *On lower bounds of the second-order directional derivatives of Ben-Tal-Zowe and Chaney*, Math. Oper. Res. **22** (1997), 747–753.

16. V. Jeyakumar and X.Q. Yang, *Approximate generalized Hessians and Taylor's expansions for continously Gâteaux differentiable functions*, Nonlinear Anal. TMA **36** (1999), 353–368.

17. B. Jiménez and V. Novo, *First and second-order sufficient conditions for strict minimality in nonsmooth vector optimization*, J. Math. Anal. Appl. **284** (2003), 496–510.

18. L. Jokl, *Minimal convex-valued weak* usco correspondences and the Radon-Nikodym property*, Comment. Math. Univ. Carolin. **28** (1987), 353–375.

19. G. Lebourg, *Generic differentiability of Lipschitz functions*, Trans. Amer. Math. Soc. **256** (1979), 125–144.

20. P. Michel and J.P. Penot, *Calcul sous-différential pour des fonctions Lipschitziennes et non-Lipschiziennes*, C.R. Acad. Sci. Paris **298** (1985), 269–272.

21. W.B. Moors, *A characterization of minimalsubdifferential mappings of locally Lipschitz functions*, Set-Valued Anal. **3** (1995), 129–141.

22. K. Pastor, *On relations among the generalized second-order directional derivatives*, Discuss. Math. Differ. Incl. Control Optim. **21** (2001), 235–247.

23. ———, *Generalized second-order sufficient optimality conditions*, J. Electr. Engin. **53** (2002), 76–78.

24. ———, *Convexity and generalized second-order derivatives for locally Lipschitz functions*, Nonlinear Anal. TMA **60** (2005), 547–555.

25. K. Pastor and D. Bednařík, *On monotone minimal cuscos*, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. **40** (2001), 185–194.

26. R.R. Phelps, *Convex functions, monotone operators and differentiability*, Springer Verlag, Berlin, 1993.

27. J.J. Rückmann and A. Shapiro, *Second-order optimality conditions in generalized semi-infinite programming*, Set-Valued Anal. **9** (2001), 169–186.

28. B.S. Thomson, *Real functions*, Lect. Notes Math. **1170**, Springer-Verlag, Berlin, 1985.

29. X.Q. Yang, *Generalized second-order characterizations of convex functions*, J. Optim. Theory Appl. **82** (1994), 173–180.

30. ———, *On second-order directional derivatives*, Nonlinear Anal. TMA **26** (1996), 55–66.

31. ———, *On relations and applications of generalized second-order directional derivatives*, Nonlinear Anal. TMA **36** (1999), 595–614.

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