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PERTURBED DISCRETE STURM-LIOUVILLE PROBLEMS AND ASSOCIATED SAMPLING THEOREMS

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ABSTRACT. We derive sampling expansions for discrete transforms whose kernels arise from perturbed discrete Sturm-Liouville problems with rank one perturbations. The kernels may be either solutions or Green's function of the problem. Due to perturbation, the multiplicities of the eigenvalues will be different from the classical case. We study the spectral analysis of the perturbed problem and derive sampling theorems. We follow the techniques established by Catchpole [**8**] and Stakgold [**16**]. The results are exhibited via illustrative examples.

1. Introduction. The connection between difference operators and sampling theory of signal processing has been established in [**2, 3, 5, 10, 11, 12**]. In sampling theory, analog signals are transformed into digital ones via interpolation formulae, cf., e.g., [**1, 15**]. This leads to several applications in communication theory, especially in the transmission of information. The sampling theorem of discrete transforms whose kernels arise from second order difference operators gives a generalized sampling principle. This principle also has been applied to derive general representations of some mathematical forms as those of [**7**], which have been generalized in [**4**]. Now let us mention some of the results of [**3**]. Let N be a fixed positive integer, and consider the eigenvalue problem

$$r^{-1}(n)\left\{\nabla\left[p(n)\Delta y(n)\right]+q(n)y(n)\right\}=\lambda y(n),\quad n=1,\dots,N, \tag{1.1}$$

$$U_1(y)=y(0)+a\,y(1)=0, \tag{1.2}$$

$$U_2(y)=y(N+1)+b\,y(N)=0, \tag{1.3}$$

where Δ is the forward difference operator, $\Delta y(n):=y(n+1)-y(n)$, and ∇ is the backward one, $\nabla y(n):=y(n)-y(n-1)$, a, b are real

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numbers and $\lambda \in \mathbf{C}$ is the eigenvalue parameter. The coefficients of (1.1) are assumed to be finite real-valued functions, $p(n) > 0$ for $n \geq 0$ and $r(n) > 0$ for $n > 0$. Let $\mathbf{I} := \{1, 2, \dots, N\}$, $\ell^2(\mathbf{Z}_N, r) := \ell^2(\mathbf{I}, r)$ be the space of all complex-valued functions $y = y(n) = (y(1), \dots, y(N))$ with the inner product $\langle y, z\rangle := \sum_{n=1}^{N} r(n)y(n)\overline{z}(n)$, and $\ell^2(\mathbf{Z}_N) := \ell^2(\mathbf{I}, 1)$. This eigenvalue problem defines a self-adjoint operator in $\ell^2(\mathbf{Z}_N, r)$, and it has been extensively studied, see e.g., [**6, 14**]. Let $\chi_1(\cdot, \lambda)$, $\chi_2(\cdot, \lambda)$ be nontrivial solutions of equation (1.1) such that $U_1(\chi_1(\cdot, \lambda)) = 0$, $U_2(\chi_2(\cdot, \lambda)) = 0$. The eigenvalues of the problem are N distinct real numbers which will be denoted by $\{\mu_k\}_{k=1}^{N}$. All eigenvalues are simple from algebraic and geometric points of view. The corresponding sequence of eigenfunctions is either $\{\chi_1(\cdot, \mu_k)\}_{k=1}^{N}$ or $\{\chi_2(\cdot, \mu_k)\}_{k=1}^{N}$. These two sequences are sets of real-valued functions, and there are nonzero real constants ν_k such that $\chi_2(\cdot, \mu_k) = \nu_k \chi_1(\cdot, \mu_k)$, $k = 1, \dots, N$. The set of eigenfunctions $\{\chi_1(\cdot, \mu_k)\}_{k=1}^{N}$ is an orthogonal basis of $\ell^2(\mathbf{Z}_N, r)$, cf. [**6, 14**]. Green's function of the problem (1.1)–(1.3) which is similar to that constructed in the case of differential operators, see e.g., [**14**, page 23], takes the form

$$g(n, m, \lambda) = \frac{1}{W(\lambda)} \begin{cases} \chi_1(n, \lambda)\chi_2(m, \lambda) & 1 \leq n \leq m \leq N, \\ \chi_2(n, \lambda)\chi_1(m, \lambda) & 1 \leq m \leq n \leq N, \end{cases} \tag{1.4}$$

where

$$W(\lambda) := p(N)[\chi_1(N, \lambda)\, \chi_2(N+1, \lambda) - \chi_1(N+1, \lambda)\, \chi_2(N, \lambda)].$$

Let $G(n, \lambda)$ be the function

$$G(n, \lambda) := \Pi(\lambda) G(n, m_0, \lambda),$$

where $m_0 \in \mathbf{I}$ is fixed and

$$\Pi(\lambda) = \begin{cases} \prod_{k=1}^{N} (1 - (\lambda/\mu_k)) & \text{if zero is not an eigenvalue,} \\ \lambda \prod_{k=2}^{N} (1 - (\lambda/\mu_k)) & \mu_1 = 0 \text{ is an eigenvalue.} \end{cases} \tag{1.5}$$

The main results of [**3**] may be summarized as follows.

Theorem 1.1. *If* $f(n) \in \ell^2(\mathbf{Z}_N, r)$ *and*

$$\begin{pmatrix} F(\lambda) \\ G(\lambda) \end{pmatrix} = \sum_{n=1}^{N} f(n) \begin{pmatrix} \chi_1(n, \lambda) \\ G(n, \lambda) \end{pmatrix} r(n), \quad \lambda \in \mathbf{C}, \tag{1.6}$$

then

$$\begin{pmatrix} F(\lambda) \\ G(\lambda) \end{pmatrix} = \sum_{k=1}^{N} \begin{pmatrix} F(\mu_k) \\ G(\mu_k) \end{pmatrix} \frac{\Pi(\lambda)}{(\lambda - \mu_k)\Pi'(\mu_k)}. \tag{1.7}$$

The aim of this paper is to investigate the sampling theory associated with perturbed second order difference operators. In the following section we define the second order perturbed discrete eigenvalue problem. In this problem, the difference equation (1.1) will have a rank one perturbation, see (2.1) below. As we will see, the properties of the perturbed problem will differ from those of the classical one, (1.1)–(1.3). For example, the eigenvalues are not necessarily simple, and the same initial conditions lead to an uncountable number of solutions. These changes will affect the sampling results since the derivation of (1.7) is based on the properties of (1.1)–(1.3). We will briefly study the spectral analysis of the perturbed problem in the next two sections. Section 4 contains the sampling analysis associated with the problem introduced in Section 2. The last section contains some illustrative examples.

2. Fundamental solutions. Consider the boundary-value problem which consists of the perturbed difference equation

$$\ell(y) = \nabla[p(n)\Delta y(n)] + q(n)y(n) + \sum_{i=1}^{N} r(n)r(i)y(i) = \lambda y(n), \quad n = 1, \dots, N, \tag{2.1}$$

together with conditions (1.2)–(1.3). The coefficients of the difference expression ℓ are assumed to be as in the previous section. Let D_L be the subspace of $\ell^2(\mathbf{Z}_N)$ defined by

$$D_L = \{y \in \ell^2(\mathbf{Z}_N) : \ell(y) \in \ell^2(\mathbf{Z}_N), \qquad U_1(y) = U_2(y) = 0\}. \tag{2.2}$$

We define the operator $L : D_L \to \ell^2(\mathbf{Z}_N)$ to be $Ly = \ell(y)$, $y \in D_L$. To prove that this operator is self-adjoint, it is sufficient to show that $\langle Ly, z\rangle = \langle y, Lz\rangle$ for y, $z \in D_L$. In the following lemma we will use summation by parts which may have the forms, cf. e.g., [**9**],

$$\sum_{k=a}^{b}(\Delta y_k)z_k = [y_{k+1}z_k]_{a-1}^{b} - \sum_{k=a}^{b} y_k \nabla z_k, \tag{2.3}$$

$$\sum_{k=a}^{b}(\nabla y_k)z_k = [y_k z_{k+1}]_{a-1}^{b} - \sum_{k=a}^{b} y_k \Delta z_k. \tag{2.4}$$

Lemma 2.1. *Problem* (2.1), (1.2)–(1.3) *is self-adjoint.*

Proof. Let y and z be in D_L. We show that $\langle \ell y, z\rangle = \langle y, \ell z\rangle$. Indeed,

$$\begin{aligned}\langle Ly, z\rangle = \langle \ell y, z\rangle &= \sum_{n=1}^{N} \nabla[p(n)\Delta y(n)]\overline{z}(n) + \sum_{n=1}^{N} q(n)y(n)\overline{z}(n)\\ &\quad + \sum_{n=1}^{N}\sum_{i=1}^{N} r(n)r(i)y(i)\overline{z}(n).\end{aligned} \tag{2.5}$$

Using summation by parts, we obtain

$$\begin{aligned}\sum_{n=1}^{N} \nabla\big[p(n)\Delta y(n)\big]\overline{z}(n) &= \Big[p(n)\big[y(n+1)\overline{z}(n) - y(n)\overline{z}(n+1)\big]\Big]_0^N\\ &\quad + \sum_{n=1}^{N} y(n)\nabla[p(n)\Delta\overline{z}(n)].\end{aligned} \tag{2.6}$$

Since y and z satisfy (1.2)–(1.3), then we deduce that

$$\Big[p(n)[y(n+1)\overline{z}(n) - y(n)\overline{z}(n+1)]\Big]_0^N = 0.$$

Substituting in (2.6), we obtain

$$\sum_{n=1}^{N} \nabla\big[p(n)\Delta y(n)\big]\overline{z}(n) = \sum_{n=1}^{N} y(n)\nabla[p(n)\Delta\overline{z}(n)]. \tag{2.7}$$

Combining (2.7) and (2.5), the lemma is proved. □

From the proof of the previous lemma, it is concluded that for any y and z in $\ell^2(\mathbf{Z}_N)$, we have the following Green's formula,

$$\langle \ell y, z\rangle - \langle y, \ell z\rangle = \Big[p(n)[y(n+1)\overline{z}(n) - y(n)\overline{z}(n+1)]\Big]_0^N. \tag{2.8}$$

Now let us find the general solution of the difference equation (2.1). Here we use the method of [**8**] established for integro-differential equations. Let $y(\cdot,\lambda)$ and $z(\cdot,\lambda)$ be any two solutions of

$$\nabla[p(n)\Delta y(n)] + q(n)y(n) = \lambda y(n), \quad \lambda \in \mathbf{C}. \tag{2.9}$$

Then, cf. [**14**],

$$W[y,z](n) = p(n-1)\begin{vmatrix} y(n-1,\lambda) & z(n-1,\lambda) \\ y(n,\lambda) & z(n,\lambda)\end{vmatrix}, \tag{2.10}$$

is independent of n. The Wronskian (Casoratian) $W[y,z](n)$ does not vanish if y, z are linearly independent; otherwise, it equals zero for all n. Let $P(\cdot,\lambda)$ be the unique solution of the inhomogeneous problem

$$\nabla[p(n)\Delta y(n)] + [q(n)-\lambda]y(n) = r(n), \quad y(1) = y(0) = 0. \tag{2.11}$$

Using the method of variation of parameters, cf. e.g., [**9**], we can show that if $y(\cdot,\lambda)$ and $z(\cdot,\lambda)$ are any two linearly independent solutions of (2.9), then

$$P(n,\lambda) = \sum_{j=1}^{n} \frac{r(j)}{W[y,z](1)}\Big(y(j,\lambda)z(n,\lambda) - z(j,\lambda)y(n,\lambda)\Big). \tag{2.12}$$

For convenience, we will set $\sum_{j=n}^{m} A_j = 0$ whenever $n > m$. Let $\varphi_1(n,\lambda)$, $\varphi_2(n,\lambda)$ denote the solutions of (2.9) determined by

$$\varphi_1(0,\lambda) = 1, \qquad \varphi_1(1,\lambda) = 0, \tag{2.13}$$

$$\varphi_2(0,\lambda) = 0, \qquad \varphi_2(1,\lambda) = 1. \tag{2.14}$$

Lemma 2.2. *A function $\varphi(\cdot,\lambda)$ is a solution of* (2.1) *determined by the initial conditions $\varphi(0,\lambda) = c_1$, $\varphi(1,\lambda) = c_2$ if and only if $\varphi(\cdot,\lambda)$ is a solution of the following equation*

$$\varphi = c_1\varphi_1 + c_2\varphi_2 - \langle \varphi, r\rangle P, \tag{2.15}$$

for arbitrary constants c_1 *and* c_2*.*

Proof. Let $\varphi(\cdot,\lambda)$ be a solution of (2.15). Since φ_1, φ_2 and P are solutions of (2.9) and (2.11), respectively. Then

$$\begin{aligned}\nabla[p(n)\Delta\varphi(n)] + q(n)\varphi(n) + r\langle\varphi, r\rangle \\ &= \lambda\, c_1\varphi_1 + \lambda\, c_2\varphi_2 - \langle\varphi, r\rangle\big(\lambda\, P + r\big) + r\langle\varphi, r\rangle \\ &= \lambda\left(c_1\varphi_1 + c_2\varphi_2 - \langle\varphi, r\rangle P\right) \\ &= \lambda\varphi.\end{aligned}$$

It is clear that $\varphi(0,\lambda) = c_1$ and $\varphi(1,\lambda) = c_2$. Conversely, any solution of (2.1) has the form $\varphi = c_1\varphi_1 + c_2\varphi_2 + u$, where u is a particular solution. Since φ satisfies (2.1), then

$$\nabla[p(n)\Delta\varphi(n)] + [q(n) - \lambda]\varphi(n) = -r(n)\,\sigma, \quad \sigma = \langle\varphi, r\rangle.$$

Putting $-r(n)\,\sigma$ instead of $r(n)$ in (2.11) and using (2.12), we get

$$\begin{aligned}u = -\sigma\sum_{1}^{n} &\frac{r(j)}{W[\varphi_1, \varphi_2](1)} \\ &\times \Big(\varphi_1(j,\lambda)\varphi_2(n,\lambda) - \varphi_2(j,\lambda)\varphi_1(n,\lambda)\Big) \\ = -\langle\varphi, r\rangle\, &P,\end{aligned}$$

which completes the proof. $\square$

Lemma 2.3. *If* $C(\lambda) = 1 + \langle P, r\rangle \neq 0$, *for some* λ, *then any solution of* (2.1), *according to that* λ, *has the form*

$$\varphi = c_1\varphi_1 + c_2\varphi_2 - \frac{c_1}{C(\lambda)}\langle\varphi_1, r\rangle P - \frac{c_2}{C(\lambda)}\langle\varphi_2, r\rangle P, \tag{2.16}$$

where c_1 *and* c_2 *are arbitrary constants.*

Proof. If φ is a solution of (2.15), then

$$\begin{aligned}\langle\varphi, r\rangle &= c_1\langle\varphi_1, r\rangle + c_2\langle\varphi_2, r\rangle - \langle\varphi, r\rangle\langle P, r\rangle \\ &= c_1\langle\varphi_1, r\rangle + c_2\langle\varphi_2, r\rangle - \langle\varphi, r\rangle(C(\lambda) - 1).\end{aligned}$$

Therefore,

$$C(\lambda)\langle\varphi, r\rangle = c_1\langle\varphi_1, r\rangle + c_2\langle\varphi_2, r\rangle.$$

If $C(\lambda) \neq 0$, then

$$\langle\varphi, r\rangle = \frac{c_1}{C(\lambda)}\langle\varphi_1, r\rangle + \frac{c_2}{C(\lambda)}\langle\varphi_2, r\rangle.$$

Substituting in (2.15), the required is obtained. □

Lemma 2.4. *Assume that* $C(\lambda) = 0$ *for some* λ. *Then*:

1. P *is a solution of* (2.1).
2. *If*

$$\langle\varphi_1, r\rangle = \langle\varphi_2, r\rangle = 0, \tag{2.17}$$

then any solution of (2.1) *has the form*

$$\varphi = c_1\varphi + c_2\varphi + \gamma\, P, \qquad c_1, c_2, \gamma \in \mathbf{C}, \tag{2.18}$$

with the initial conditions $\varphi(0) = c_1$, $\varphi(1) = c_2$, *and* $\gamma \in \mathbf{C}$ *is arbitrary.*

3. *If* (2.17) *does not hold, then* (2.1) *has the solution*

$$\varphi = \alpha\, v + \gamma\, P, \qquad v := \langle\varphi_2, r\rangle\varphi_1 - \langle\varphi_1, r\rangle\varphi_2, \quad \alpha, \gamma \in \mathbf{C}, \tag{2.19}$$

which satisfies the initial conditions $\varphi(0) = \alpha\langle\varphi_2, r\rangle$, $\varphi(1) = -\alpha\langle\varphi_1, r\rangle$.

Proof. 1. Since $C(\lambda) = 1 + \langle P, r\rangle = 0$, then P satisfies (2.1).

2. Because of (2.17), φ_1, φ_2 satisfy (2.1). Thus, (2.1) has three linearly independent solutions; namely, φ_1, φ_2, P, i.e., φ has the form (2.18).

3. Clearly every solution u of (2.9) is also a solution of (2.1) if and only if $\langle u, r\rangle = 0$. If (2.17) is not true, then the only solution, up to a

multiplicative constant, of both (2.9) and (2.1) is v. Thus, any solution of (2.1) has the form (2.19) with the indicated initial conditions. $\square$

Remark 2.5. As we see in the previous lemmas, if $C(\lambda) \neq 0$, equation (2.1) has two linearly independent solutions. If $C(\lambda) = 0$, then for the same initial conditions there are uncountably many solutions of (2.1) since α and γ of (2.18) and (2.19) can be chosen arbitrarily.

3. Green's function and an expansion theorem. In this section we investigate the eigenvalues of problem (2.1), (1.2)–(1.3) and their multiplicities. First we have the following result which is easy to prove.

Lemma 3.1. *The eigenvalues of problem* (2.1), (1.2)–(1.3) *are real, and eigenfunctions corresponding to different eigenvalues are orthogonal.*

Since equation (2.1) has, in some cases, three linearly independent solutions, the multiplicity of every eigenvalue is expected not to exceed three. We will see that, in the case under consideration, separate type boundary conditions, the multiplicity is at most two. We distinguish between two cases.

Case I. When $C(\lambda) \neq 0$. In this case we define the following fundamental set of solutions of (2.1) by

$$\phi_1 = \varphi_1 - \frac{\langle \varphi_1, r\rangle}{C(\lambda)} P, \qquad \phi_2 = \varphi_2 - \frac{\langle \varphi_2, r\rangle}{C(\lambda)} P. \tag{3.1}$$

This fundamental set satisfies the same initial conditions that $\{\varphi_1, \varphi_2\}$ satisfies. Using a technique similar to that of [**14**], we can show also that $W[\phi_1, \phi_2](n)$ is independent of n, and any two solutions y and z of (2.1) are linearly independent if and only if $W[y, z](n)$ does not vanish.

Theorem 3.2. *The real number λ is an eigenvalue of problem* (2.1), (1.2)–(1.3) *when $C(\lambda) \neq 0$, if and only if*

$$\Delta(\lambda) := \begin{vmatrix} U_1(\phi_1) & U_1(\phi_2) \\ U_2(\phi_1) & U_2(\phi_2) \end{vmatrix} = 0. \tag{3.2}$$

Moreover, if such a λ is an eigenvalue, then it is simple.

Proof. Let $C(\lambda) \neq 0$. Then any solution of (2.1) corresponding to λ has the form $\phi = c_1\phi_1 + c_2\phi_2$, where c_1 and c_2 are constants. This solution will be an eigenfunction if it satisfies the boundary conditions (1.2) and (1.3). Thus, λ is an eigenvalue if the following linear system of the unknowns c_1 and c_2 has a nontrivial solution.

$$c_1 U_i(\phi_1) + c_2 U_i(\phi_2) = 0, \quad i = 1, 2. \tag{3.3}$$

This will happen when and only when equation (3.2) is fulfilled. Now let λ be an eigenvalue of problem (2.1), (1.2)–(1.3), where $C(\lambda) \neq 0$. We prove that λ cannot have more than one linearly independent eigenfunction. Indeed, let ϕ_0^1 and ϕ_0^2 be two eigenfunctions corresponding to λ. Then there are constants $c_{11}, c_{12}, c_{21}, c_{22}$ such that

$$\phi_0^i = c_{i1}\varphi_1 + c_{i2}\varphi_2 - \frac{c_{i1}\langle\varphi_1, r\rangle}{C(\lambda)}P - \frac{c_{i2}\langle\varphi_2, r\rangle}{C(\lambda)}P, \quad i = 1, 2. \tag{3.4}$$

Let $\psi_i := c_{i1}\varphi_1 + c_{i2}\varphi_2$, $i = 1, 2$; then the functions ψ_1 and ψ_2 are solutions of the classical Sturm-Liouville problem (1.1)–(1.3) and

$$\psi_i = \phi_0^i + \frac{c_{i1}\langle\varphi_1, r\rangle}{C(\lambda)}P + \frac{c_{i2}\langle\varphi_2, r\rangle}{C(\lambda)}P, \quad i = 1, 2.$$

Since problem (1.1)–(1.3) has only simple eigenvalues and $(\psi_i(0), \psi_i(1)) = (c_{i1}, c_{i2})$, then the vectors (c_{11}, c_{12}), (c_{21}, c_{22}) are linearly dependent. Consequently, ϕ_0^1 and ϕ_0^2 are linearly dependent, proving the simplicity of λ. □

Now when $C(\lambda) \neq 0$, $\lambda \in \mathbf{C}$, we seek a solution which generates all eigenfunctions. Let $\Theta_1(\cdot, \lambda)$ and $\Theta_2(\cdot, \lambda)$ be the solutions of (2.1) determined by the initial conditions

$$\Theta_1(0, \lambda) = a, \quad \Theta_1(1, \lambda) = -1, \tag{3.5}$$

$$\Theta_2(0, \lambda) = b, \quad \Theta_2(1, \lambda) = -1. \tag{3.6}$$

Then

$$\Theta_1(n,\lambda) = -\begin{vmatrix} U_1(\phi_1) & U_1(\phi_2) \\ \phi_1(n,\lambda) & \phi_2(n,\lambda) \end{vmatrix} = a_2\,\phi_1(n,\lambda) - a_1\,\phi_2(n,\lambda), \tag{3.7}$$

$$\Theta_2(n,\lambda) = \begin{vmatrix} \phi_1(n,\lambda) & \phi_2(n,\lambda) \\ U_2(\phi_1) & U_2(\phi_2) \end{vmatrix} = U_2(\phi_2)\,\phi_1(n,\lambda) - U_2(\phi_1)\,\phi_2(n,\lambda). \tag{3.8}$$

Hence, from (3.2), the eigenvalues of problem (2.1), (1.2)–(1.3) are the zeros of either

$$\omega_1(\lambda) := U_2(\Theta_1) = \Theta_1(N+1,\lambda) + b\,\Theta_1(N,\lambda) = -\Delta(\lambda) = 0, \tag{3.9}$$

or

$$\omega_2(\lambda) := U_1(\Theta_2) = \Theta_2(0,\lambda) + a\,\Theta_2(1,\lambda) = \Delta(\lambda) = 0. \tag{3.10}$$

Since any solution of (2.1) has the form $c_1\,\Theta_1(\cdot,\lambda) + c_2\Theta_2(\cdot,\lambda)$, we get the following lemma.

Lemma 3.3. *Either* $\Theta_1(\cdot,\lambda)$ *or* $\Theta_2(\cdot,\lambda)$ *generates all eigenfunctions of the problem* (2.1), (1.2)–(1.3) *when* $C(\lambda) \neq 0$.

Proof. Using (3.7) and (3.8), it is not hard to see that

$$W[\Theta_1,\Theta_2](n) = \Delta(\lambda)W[\phi_1,\phi_2](n). \tag{3.11}$$

Since $W[\phi_1,\phi_2](n) = W[\phi_1,\phi_2](1) = p(0)$, we have

$$W[\Theta_1,\Theta_2](n) = \Delta(\lambda)\,p(0). \tag{3.12}$$

Thus, if $\lambda = \lambda_k$ is an eigenvalue, then $W[\Theta_1,\Theta_2](n) = 0$, and therefore

$$\Theta_1(n,\lambda_k) = c_k\,\Theta_2(n,\lambda_k), \quad 0 \neq c_k \in \mathbf{C}. \tag{3.13}$$

This means that any eigenfunction of (2.1), (1.2)–(1.3), can be generated by only one of $\Theta_1(\cdot,\lambda)$ and $\Theta_2(\cdot,\lambda)$. $\square$

Case II. When $C(\lambda) = 0$. In this case, according to the next lemma, we see that any real zero of $C(\lambda)$ is either not an eigenvalue or an eigenvalue with multiplicity less than or equals two.

Lemma 3.4. *Let $C(\lambda) = 0$ for some real λ. Then we have the following cases*:

1. *if* (2.17) *holds, then λ is a simple eigenvalue of* (2.1), (1.2)–(1.3) *if it is not an eigenvalue of the classical Sturm-Liouville problem* (1.1)–(1.3). *Otherwise it is a double eigenvalue.*

2. *If* (2.17) *is not true, then λ will be*

(a) *a double eigenvalue of* (2.1), (1.2)–(1.3) *if v satisfies the boundary conditions, i.e., $U_1(v) = U_2(v) = 0$,*

(b) *a simple eigenvalue if v satisfies one and only one of the boundary conditions,*

(c) *not an eigenvalue if v does not satisfy any of the boundary conditions.*

Proof. 1. Let $C(\lambda) = 0$ for some λ and (2.17) hold. We will see that P is an eigenfunction according to this λ. Since $P(0,\lambda) = P(1,\lambda) = 0$, then $U_1(P) = 0$. Moreover, $\langle \varphi_1, r\rangle = \langle \varphi_2, r\rangle = 0$, $W[\varphi_1,\varphi_2](1) = 1$, which, using (2.12), imply

$$U_2(P) = U_2(\varphi_1)\langle \varphi_2, r\rangle - U_1(\varphi_2)\langle \varphi_1, r\rangle = 0.$$

Thus, P satisfies the boundary conditions (1.2) and (1.3) in addition to equation (2.1). Hence, P is an eigenfunction of problem (2.1), (1.2)–(1.3). The eigenvalue λ will have another linearly independent eigenfunction φ if there are nonzero constants c_1 and c_2 such that $\varphi = c_1\varphi_1 + c_2\varphi_2 + \gamma P$ and the following equations are satisfied.

$$\begin{aligned} 0 = U_i(\varphi) &= c_1 U_i(\varphi_1) + c_2 U_i(\varphi_2) + \gamma U_i(P) \\ &= c_1 U_i(\varphi_1) + c_2 U_i(\varphi_2), \quad i = 1, 2. \end{aligned} \tag{3.14}$$

The above system will have a nontrivial solution if and only if λ is an eigenvalue of the classical Sturm-Liouville problem and it cannot have more than one linearly independent solution, since the classical Sturm-Liouville problem has only simple eigenvalues.

2. In this case any solution of (2.1) has the form $\varphi = \alpha\, v + \gamma\, P$. Thus, φ is an eigenfunction of (2.1), (1.2)–(1.3) if and only if

$$\Delta_c(\lambda) := \begin{vmatrix} U_1(v) & U_1(P) \\ U_2(v) & U_2(P) \end{vmatrix} = \begin{vmatrix} U_1(v) & 0 \\ U_2(v) & U_2(v) \end{vmatrix} = 0, \tag{3.15}$$

because $U_1(P) = 0$, $U_2(P) = U_2(v)$.

(a) Hence if v satisfies $U_1(v) = U_2(v) = 0$, the rank of the matrix corresponding to Δ_c is zero. Thus, there are two eigenfunctions which are v, P.

(b) If $v(n,\lambda)$ satisfies only one of the boundary conditions, then the rank will be one, i.e., there is only one linearly independent eigenfunction. If $U_1(v) = 0$, φ is an eigenfunction only if $\alpha = -\gamma$, i.e., the eigenfunction is $\alpha(v - P)$. If $U_2(v) = 0$, the corresponding eigenfunction is P.

(c) If $U_1(v) \neq 0 \neq U_2(v)$, from (3.15), λ is not an eigenvalue of (2.1), (1.2)–(1.3). $\square$

In the following we derive Green's function of the problem (2.1), (1.2)–(1.3). Thus, the equivalence between problem (2.1), (1.2)–(1.3) and a Fredholm-type difference operator with a symmetric kernel will be proved. This leads to the fact that the eigenfunctions of problem (2.1), (1.2)–(1.3) is a complete orthogonal set of $\ell^2(\mathbf{Z}_N)$. Here we use the technique of Stakgold [**16**]. First we want to find a solution of $(L - \lambda I)y(n) = f(n)$ when λ is not an eigenvalue of (2.1), (1.2)–(1.3), where I is the identity operator and $f(n) \in \ell^2(\mathbf{Z}_N)$ is given. Equivalently, we seek the solution of the problem

$$\begin{gathered} \nabla[p(n)\Delta y(n)] + (q(n) - \lambda)y(n) + \sum_{i=1}^{N} r(n)r(i)y(i) = f(n), \\ U_1(y) = U_2(y) = 0. \end{gathered} \tag{3.16}$$

Let $g(n,m,\lambda)$ be Green's function corresponding to $(L - \lambda I)y(n) = f(n)$ in the unperturbed case, i.e., any solution of

$$\begin{gathered} \nabla[p(n)\Delta y(n)] + (q(n) - \lambda)y(n) = f(n), \\ U_1(y) = U_2(y) = 0; \end{gathered} \tag{3.17}$$

λ is not an eigenvalue of (1.1)–(1.3) and has the form

$$y(n)=\sum_{m=1}^{N} g(n,m,\lambda)f(m).$$

From (1.4), we can get

(3.18)
$$g(n,m,\lambda)=\frac{1}{\Delta_0(\lambda)\,p(0)}\begin{cases}\varphi_1(m,\lambda)\varphi_2(n,\lambda) & 1\le m\le n\le N,\\ \varphi_1(n,\lambda)\varphi_2(m,\lambda) & 1\le n\le m\le N,\end{cases}$$

where

$$\Delta_0(\lambda):=\begin{vmatrix} U_1(\varphi_1) & U_1(\varphi_2)\\ U_2(\varphi_1) & U_2(\varphi_2)\end{vmatrix}.$$

In [**3**], it is shown that

$$g(n,m,\lambda)=\sum_{k=1}^{N}\frac{\varphi_k(n)\varphi_k(m)}{\mu_k-\lambda},\quad \lambda\neq\mu_k, \tag{3.19}$$

where $\{\varphi_k(n)\}_{k=1}^{N}$ is an orthonormal basis of eigenfunctions of (1.1)–(1.3).

Lemma 3.5. *If* λ *is not an eigenvalue of* (2.1), (1.2)–(1.3), *then*

$$1+\langle (A_\lambda r),r\rangle\neq 0,\qquad (A_\lambda r)(n):=\sum_{m=1}^{N} g(n,m,\lambda)r(m). \tag{3.20}$$

Proof. Assume that (3.20) does not hold. Since

$$\chi(n,\lambda)=\sum_{m=1}^{N} g(n,m,\lambda)r(m)$$

uniquely solves the problem

$$\nabla[p(n)\Delta y(n)]+(q(n)-\lambda)y(n)=r(n),\quad U_1(y)=U_2(y)=0,$$

and $\langle\chi,r\rangle=-1$. Then, $\chi(\cdot,\lambda)$ is an eigenfunction of (2.1), (1.2)–(1.3) corresponding to the eigenvalue λ, contradicting the assumption. $\square$

Theorem 3.6. *Assume that λ is not an eigenvalue of L. Problem* (3.16) *has a unique solution $y(n)$ which is given by*

$$y(n) = \sum_{m=1}^{N} G(n,m,\lambda)\, f(m), \tag{3.21}$$

where

$$G(n,m,\lambda) := g(n,m,\lambda) - \frac{(A_\lambda r)(n)(A_\lambda r)(m)}{1+\langle (A_\lambda r), r\rangle}, \quad \lambda \neq \lambda_k.$$

Proof. We first write (3.16) as

$$\nabla[p(n)\Delta y(n)] + q(n)y(n) = f(n) - \sigma r(n), \quad \sigma = \sum_{i=1}^{N} r(i)y(i). \tag{3.22}$$

Since $g(n,m,\lambda)$ is the Green's function of (3.17), then the solution of (3.22) is given by

$$\begin{aligned} y(n) &= \sum_{m=1}^{N} g(n,m,\lambda)f(m) - \sigma \sum_{m=1}^{N} g(n,m,\lambda)r(m) \\ &= (A_\lambda f)(n) - \sigma (A_\lambda r)(n). \end{aligned} \tag{3.23}$$

From (3.23), we have

$$\begin{aligned} \sum_{n=1}^{N} r(n)y(n) &= \sum_{n=1}^{N}(A_\lambda f)(n)r(n) - \sigma \sum_{n=1}^{N}(A_\lambda r)(n)r(n) \\ &= \langle (A_\lambda f), r\rangle - \sigma \left\langle ((A_\lambda r), r\right\rangle. \end{aligned} \tag{3.24}$$

Hence,

$$\sigma = \frac{\langle (A_\lambda f), r\rangle}{1+\langle (A_\lambda r), r\rangle}. \tag{3.25}$$

Substituting from (3.25) in (3.23), we obtain

$$y(n) = (A_\lambda f)(n) - \frac{(A_\lambda r)(n)\,\langle (A_\lambda f), r\rangle}{1+\langle (A_\lambda r), r\rangle}. \tag{3.26}$$

But

$$\langle (A_\lambda f), r\rangle = \sum_{n=1}^{N}(A_\lambda f)(n)r(n) = \sum_{n=1}^{N}\sum_{m=1}^{N} g(n,m,\lambda)f(m)r(n). \tag{3.27}$$

Then equation (3.26) can be rewritten as

$$\begin{aligned} y(n) &= \sum_{m=1}^{N} g(n,m,\lambda)f(m) \\ &\quad - \frac{\left(\sum_{m=1}^{N} g(n,m,\lambda)r(m)\right)\left(\sum_{n=1}^{N}\sum_{m=1}^{N} g(n,m,\lambda)f(m)r(n)\right)}{1+\langle (A_\lambda r), r\rangle} \\ &= \sum_{m=1}^{N}\left[g(n,m,\lambda) - \frac{(A_\lambda r)(n)(A_\lambda r)(m)}{1+\langle (A_\lambda r), r\rangle}\right] f(m), \end{aligned} \tag{3.28}$$

and the function $G(n,m,\lambda)$ is unique by construction. Since $g(n,m,\lambda)$ has simple poles at $\{\mu_k\}_{k=1}^{N}$, it remains to show that $G(n,m,\lambda)$ is defined, as a limit, at the eigenvalues of (1.1)–(1.3) which are not eigenvalues of (2.1), (1.2)–(1.3). Assume that μ_0 is an eigenvalue of the problem (1.1)–(1.3) and μ_0 is not an eigenvalue of (2.1), (1.2)–(1.3). Then from (3.19) we can find a neighborhood of μ_0, D_0 say, such that

$$g(n,m,\lambda) = -\frac{y_0(n)y_0(m)}{\lambda-\mu_0} + g_1(n,m,\lambda), \quad \lambda \in D_0^* = D_0 - \{\mu_0\}, \tag{3.29}$$

where $g_1(n,m,\lambda)$ is regular in D_0 and $y_0(n)$ is a normalized eigenfunction corresponding to μ_0. Substituting in $G(n,m,\lambda)$, we get for $\lambda \in D_0^*$,

$$\begin{aligned} G(n,m,\lambda) &= g_1 - \frac{y_0(n)y_0(m)}{\lambda-\mu_0} \\ &\quad - \frac{g_{21} - (\Omega/\lambda-\mu_0)\, g_{22} + (\Omega^2/(\lambda-\mu_0)^2)\, y_0(n)y_0(m)}{1+g_{11}-(\Omega^2/\lambda-\mu_0)}, \end{aligned} \tag{3.30}$$

where

$$g_1 = g_1(n,m,\lambda),\ g_{11} = \sum_{n=1}^{N}\sum_{m=1}^{N} g_1 r(n)r(m),\ g_{21} = \left(\sum_{m=1}^{N} g_1 r(m)\right)^2,$$

$$g_{22} = \left[y_0(n)+y_0(m)\right]\sum_{m=1}^{N} g_1 r(m), \qquad \Omega = \sum_{m=1}^{N} y_0(m)r(m).$$

Hence, we have for $\lambda \in D_0^*$,

$$G(n,m,\lambda) = g_1 - \frac{(\lambda-\mu_0)g_{21} - \Omega\, g_{22} - y_0(n)y_0(m)[1+g_{11}]}{(\lambda-\mu_0)[1+g_{11}] - \Omega^2}, \tag{3.31}$$

which could be defined at μ_0 if $\Omega \neq 0$. If

$$\Omega = \sum_{m=1}^{N} y_0(m)r(m) = \langle y_0, r\rangle = 0,$$

then $y_0(n)$ is an eigenfunction of the problem (2.1), (1.2)–(1.3) and μ_0 is a simple pole of $G(n,m,\lambda)$. □

The function $G(n,m,\lambda)$ is called a Green's function of the operator $L - \lambda I$.

Theorem 3.7. *The problem* (2.1), (1.2)–(1.3) *has exactly N eigenvalues, and the set of eigenfunctions is a complete orthogonal set in* $\ell^2(\mathbf{Z}_N)$.

Proof. Assume first that $\lambda = 0$ is not an eigenvalue of L. Let $G(n,m) \equiv G(n,m,0)$. The problem $L\,y(n) = f(n)$ has the solution

$$y(n) = \sum_{m=1}^{N} G(n,m)\,f(m), \quad f \in \ell^2(\mathbf{Z}_N). \tag{3.32}$$

If we let $f(m) = \lambda\,y(m)$, the eigenvalue problem $(L-\lambda I)y(n) = 0$ is equivalent to the system

$$y(n) = \lambda \sum_{m=1}^{N} G(n,m)\,y(m). \tag{3.33}$$

The geometric multiplicity of an eigenvalue cannot be higher than two. Equation (3.33) can be written as

$$\mathbf{G}\,y^{\top} = \mu\,y^{\top}, \qquad \mu = \frac{1}{\lambda}, \tag{3.34}$$

where $\mathbf{G} = (G(n,m))_{1\le n,m\le N}$, $y \in \ell^2(\mathbf{Z}_N)$ and $y^\top$ is the transpose of y. Hence, the two problems $(L-\lambda I)y_n = 0$, and (3.34) are equivalent. Since $G(n,m) = \overline{G}(m,n)$, then the transformation $\mathbf{G}$ is Hermitian, see [**13**, page 135]. From [**13**, pages 154–155], the algebraic multiplicity of each eigenvalue equals its geometric multiplicity; hence, problem (2.1), (1.2)–(1.3) has exactly N eigenvalues [**13**, page 105]. Therefore, we have a complete set of eigenfunctions. For the case $\lambda = 0$ is an eigenvalue, we replace the eigenvalue parameter λ by $\lambda - c$, where c is a constant different from all eigenvalues of L. The eigenvalue problem $(L-(\lambda-c)I)y = 0$, has the same eigenfunctions of L but zero is not an eigenvalue of it. □

The function $G(n,m)$ is called a Green's function of the operator L.

Lemma 3.8. *$G(n,m,\lambda)$ has the eigenfunction expansion*

$$G(n,m,\lambda) = \sum_{k=1}^{N} \frac{\phi_k(n)\phi_k(m)}{\lambda_k - \lambda}, \quad \lambda \ne \lambda_k, \tag{3.35}$$

where $\{\phi_k(\cdot)\}_{k=1}^N$ is a complete orthonormal set of eigenfunctions of (2.1), (1.2)–(1.3).

Proof. Since $\phi_k(\cdot)$ is an eigenfunction, for $\lambda \ne \lambda_k$, the equation

$$(L-\lambda I)\phi_k(n) = (\lambda_k - \lambda)\phi_k(n),$$

has the solution

$$\phi_k(n) = \sum_{m=1}^{N} G(n,m,\lambda)\,(\lambda_k-\lambda)\phi_k(m).$$

Thus,

$$\langle G(n,m,\lambda), \phi_k(m)\rangle = -\frac{\phi_k(n)}{\lambda - \lambda_k}, \quad \lambda \ne \lambda_k. \tag{3.36}$$

Since $G(n,m,\lambda) \in \ell^2(\mathbf{Z}_N)$ for all λ, then $G(n,m,\lambda)$ has the eigenfunction expansion (3.35). □

Since an eigenvalue λ_k may have more than one linearly independent eigenfunction, then expansion (3.35) may have the form

$$G(n,m,\lambda)=\sum_{k=1}^{s}\sum_{\nu=1}^{\nu_k}\frac{\phi_{k,\nu}(n)\phi_{k,\nu}(m)}{\lambda_k-\lambda},\quad \lambda\neq\lambda_k, \tag{3.37}$$

where ν_k is the multiplicity of λ_k, and $\{\lambda_k\}_{k=1}^{s}$ is the set of all different eigenvalues of (2.1), (1.2)–(1.3).

4. Sampling theorems. In the following we derive the sampling theorems of this paper. A discrete transform whose kernel is either $\Theta_1(\cdot,\lambda)$ or $\Theta_2(\cdot,\lambda)$, $\lambda\in\mathbf{C}$, will be sampled at the eigenvalues of (2.1), (1.2)–(1.3) via Lagrange-type interpolation expansion provided that $C(\lambda)\neq 0$. Let $C(\lambda)\neq 0$, and assume that the set of eigenvalues is denoted by $\{\lambda_k\}_{k=1}^{N}$.

Theorem 4.1. *Let $g(\cdot)\in\ell^2(\mathbf{Z}_N)$. Set*

$$\begin{aligned} F(\lambda)&=\sum_{n=1}^{N}\overline{g}(n)\Theta_1(n,\lambda),\\ F^*(\lambda)&=\sum_{n=1}^{N}\overline{g}(n)\Theta_2(n,\lambda),\quad \lambda\in\mathbf{C}. \end{aligned} \tag{4.1}$$

Hence, $F(\lambda),F^*(\lambda)$ can be reconstructed from their values at $\{\lambda_k\}_{k=1}^{N}$ via the interpolation expansions

$$\begin{aligned} F(\lambda)&=\sum_{k=1}^{N}F(\lambda_k)\frac{\omega_1(\lambda)}{(\lambda-\lambda_k)\omega_1'(\lambda_k)},\\ F^*(\lambda)&=\sum_{k=1}^{N}F^*(\lambda_k)\frac{\omega_2(\lambda)}{(\lambda-\lambda_k)\omega_2'(\lambda_k)}. \end{aligned} \tag{4.2}$$

Proof. We prove the theorem for the first transform in (4.1); the other is similar. Since $\{\Theta_1(\cdot,\lambda_k)\}_{k=1}^{N}$ forms a complete orthogonal set on $\ell^2(\mathbf{Z}_N)$, then applying Parseval's equality to the first transform in (4.1), we obtain

$$F(\lambda)=\sum_{k=1}^{N}\widehat{g}(k)\frac{\widehat{\Theta}_1(k,\lambda)}{\|\Theta_1(\cdot,\lambda_k)\|^2},\quad \lambda\in\mathbf{C}, \tag{4.3}$$

where

$$\widehat{g}(k)=\sum_{n=1}^{N}\overline{g}(n)\Theta_1(n,\lambda_k),\qquad \widehat{\Theta}_1(k,\lambda)=\sum_{n=1}^{N}\Theta_1(n,\lambda)\overline{\Theta}_1(n,\lambda_k). \tag{4.4}$$

From the definition of $F(\lambda)$, $\widehat{g}(k)=F(\lambda_k)$. Thus,

$$F(\lambda)=\sum_{k=1}^{N}F(\lambda_k)\frac{\widehat{\Theta}_1(k,\lambda)}{\|\Theta_1(\cdot,\lambda_k)\|^2}. \tag{4.5}$$

In Green's formula (2.8), if we let $y=\Theta_1(n,\lambda)$ and $z=\Theta_1(n,\lambda_k)$, then

$$\begin{aligned}(\lambda-\lambda_k)&\sum_{n=1}^{N}\Theta_1(n,\lambda)\overline{\Theta}_1(n,\lambda_k)\\&=\Big[p(n)[\Theta_1(n+1,\lambda)\overline{\Theta}_1(n,\lambda_k)-\Theta_1(n,\lambda)\overline{\Theta}_1(n+1,\lambda_k)]\Big]_0^N.\end{aligned}$$

Since both $\Theta_1(\cdot,\lambda)$ and $\Theta_1(\cdot,\lambda_k)$ satisfy (1.2) and $\Theta_1(\cdot,\lambda_k)$ satisfies (1.3), then

$$\begin{aligned}\sum_{n=1}^{N}\Theta_1(n,\lambda)\overline{\Theta}_1(n,\lambda_k)&=p(N)\overline{\Theta}_1(N,\lambda_k)\frac{[\Theta_1(N+1,\lambda)+b\,\Theta_1(N,\lambda)]}{(\lambda-\lambda_k)}\\&=p(N)\overline{\Theta}_1(N,\lambda_k)\frac{\omega_1(\lambda)}{(\lambda-\lambda_k)}.\end{aligned} \tag{4.6}$$

Letting $\lambda\to\lambda_k$ in (4.6), we obtain

$$\|\Theta_1(\cdot,\lambda_k)\|^2=\sum_{n=1}^{N}|\Theta_1(n,\lambda_k)|^2=p(N)\overline{\Theta}_1(N,\lambda_k)\,\omega_1'(\lambda_k). \tag{4.7}$$

Equation (4.7) also proves the simplicity of the zeros of $\omega_1(\lambda)$ since otherwise $\|\Theta_1(\cdot,\lambda_k)\|^2 = 0$, which implies $\Theta_1(\cdot,\lambda_k) \equiv 0$, contradicting the fact that $\Theta_1(\cdot,\lambda_k)$ is an eigenfunction. Combining (4.6), (4.7) and (4.5), one obtains the first expansion in (4.2). $\square$

Since there is no one single function that generates all eigenfunctions when $C(\lambda) = 0$, we will derive another theorem for transforms whose kernels are expressed in terms of Green's function. Let $m_0 \in [1, N]$ be such that $\phi_k(m_0) \neq 0$ for all k. Define the function $G_0(n,\lambda) \in \ell^2(\mathbf{Z}_N)$ to be

$$G_0(n,\lambda) := G(n, m_0, \lambda). \tag{4.8}$$

From Lemma 3.8 above, since $\{\phi_k(n)\}_{k=1}^{N}$ is a complete orthonormal set of $\ell^2(\mathbf{Z}_N)$, (3.37) can be viewed as the Fourier expansion of $G_0(n,\lambda)$ with the Fourier coefficients $\phi_k(m_0)/(\lambda_k - \lambda)$, $\lambda \neq \lambda_k$. The function $G_0(n,\lambda)$ is a meromorphic function with simple poles at the eigenvalues. The residue at each pole λ_k is

$$r_k = \sum_{\nu=1}^{\nu_k} \phi_{k,\nu}(n)\phi_{k,\nu}(m_0). \tag{4.9}$$

Define the entire function $\omega(\lambda)$ to be

$$\omega(\lambda) = \prod_{k=1}^{s} (\lambda - \lambda_k)\,. \tag{4.10}$$

The function

$$\Phi(n,\lambda) := \omega(\lambda)\, G_0(n,\lambda), \tag{4.11}$$

is an entire function of λ for each fixed n.

Theorem 4.2. *Let $g \in \ell^2(\mathbf{Z}_N)$ and*

$$F(\lambda) = \sum_{n=1}^{N} \overline{g}(n)\, \Phi(n,\lambda), \quad \lambda \in \mathbf{C}. \tag{4.12}$$

Then $F(\lambda)$ admits the sampling representation

$$F(\lambda)=\sum_{k=1}^{s}F(\lambda_k)\frac{\omega(\lambda)}{(\lambda-\lambda_k)\,\omega'(\lambda_k)},\quad \lambda\in\mathbf{C}. \tag{4.13}$$

Proof. Since both g and Φ are $\ell^2(\mathbf{Z}_N)$-functions, then

$$g(n)=\sum_{k=1}^{N}\langle g,\phi_k\rangle\,\phi_k(n),\qquad \Phi(n,\lambda)=\sum_{k=1}^{N}\langle \Phi,\phi_k\rangle\,\phi_k(n). \tag{4.14}$$

Using Parseval's identity, we get

$$F(\lambda)=\sum_{k=1}^{N}\overline{\langle g,\phi_k\rangle}\,\langle \Phi,\phi_k\rangle=\sum_{k=1}^{s}\sum_{\nu=1}^{\nu_k}\overline{\langle g,\phi_{k,\nu}\rangle}\,\langle \Phi,\phi_{k,\nu}\rangle\,. \tag{4.15}$$

From (3.37) and (4.11), we have

$$\langle \Phi,\phi_{k,\nu}\rangle=\frac{\omega(\lambda)}{\lambda_k-\lambda}\,\phi_{k,\nu}(m_0). \tag{4.16}$$

From (4.12) and (4.11), we obtain

$$F(\lambda)=\omega(\lambda)\sum_{n=1}^{N}\overline{g}(n)\,G_0(n,\lambda).$$

Therefore, using (4.9), we get

$$\begin{aligned} F(\lambda_k)&=\lim_{\lambda\to\lambda_k}\frac{\omega(\lambda)}{\lambda-\lambda_k}\sum_{n=1}^{N}(\lambda-\lambda_k)\overline{g}(n)G_0(n,\lambda)\\ &=-\omega'(\lambda_k)\sum_{k=1}^{\nu_k}\phi_{k,\nu}(m_0)\sum_{n=1}^{N}\overline{g}(n)\phi_{k,\nu}(n)\\ &=-\omega'(\lambda_k)\sum_{k=1}^{\nu_k}\phi_{k,\nu}(m_0)\overline{\langle g,\phi_{k,\nu}\rangle}. \end{aligned} \tag{4.17}$$

Substituting (4.16), (4.17) in (4.15), we arrive at (4.13). $\square$

5. Examples. In this section we give three worked examples exhibiting the results of the previous section. The cases $C(\lambda) \neq 0$ and $C(\lambda) = 0$ are both considered.

Example 5.1. Consider the eigenvalue problem

$$\nabla[p(n)\Delta y(n)] + 2y(n) + r(n)\sum_{i=1}^{N} r(i)y(i) = \lambda y(n), \tag{5.1}$$

$$U_1(y) = y(0) = 0, \qquad U_2(y) = y(N+1) = 0, \tag{5.2}$$

where $n = 1, \dots, N$, $(r(n)) = (1, 0, \dots, 0)$. After some computations we can see that

$$\begin{aligned} \varphi_1(n,\lambda) &= -\frac{\sin(n-1)\theta}{\sin\theta}, \\ \varphi_2(n,\lambda) &= \frac{\sin n\theta}{\sin\theta}, \\ W(\varphi_1,\varphi_2)(n) &= -1, \end{aligned} \tag{5.3}$$

$$P(n,\lambda) = \begin{cases} (\sin(n-1)\theta)/\sin\theta & n = 1, \dots, N, \\ 0 & n = 0. \end{cases} \tag{5.4}$$

Hence,

$$\langle P, r\rangle = 0, \qquad \langle \varphi_1, r\rangle = 0, \qquad \langle \varphi_2, r\rangle = 1, \qquad C(\lambda) = 1, \tag{5.5}$$

(5.6)
$$\begin{aligned} \phi_1(n,\lambda) &= -\frac{\sin(n-1)\theta}{\sin\theta}, \\ \phi_2(n,\lambda) &= \frac{\sin n\theta}{\sin\theta} - \frac{\sin(n-1)\theta}{\sin\theta} = \frac{\cos(n-1/2)\theta}{\cos(\theta/2)}, \end{aligned}$$

(5.7)
$$\Theta_1(n,\lambda) = -\frac{\sin n\theta}{\sin\theta}, \qquad \Theta_2(n,\lambda) = \frac{\sin(N-n+1)\theta}{\sin\theta},$$

where $\cos\theta = \lambda/2$. Thus,

$$\omega_1(\lambda) = -\frac{\sin(N+1)\theta}{\sin\theta} = -\omega_2(\lambda), \tag{5.8}$$

which gives

(5.9)
$$\lambda_k = 2\cos\left(\frac{k\pi}{N+1}\right), \quad \omega_1'(\lambda_k) = -\frac{(-1)^k(N+1)}{2\sin\left(k\pi/N+1\right)}, \quad k = 1, \dots, N.$$

Then the transform

$$F(\lambda) = \sum_{n=1}^{N} \overline{g}(n)\Theta_1(n,\lambda), \quad \lambda \in \mathbf{C}, \quad g \in \ell^2(\mathbf{Z}_N), \tag{5.10}$$

has the representation

$$F(\lambda) = \sum_{n=1}^{N} F(\lambda_k) \frac{\sin^2\left(k\pi/N+1\right)\,\sin\big((N+1)\theta - k\pi\big)}{(N+1)\left((\lambda/2) - \cos\left(k\pi/N+1\right)\right)\sin\theta}. \tag{5.11}$$

Example 5.2. Consider the problem (5.1)–(5.2) with $(r(n)) = (1, -1, \dots, 0)$. We have φ_1, φ_2 as in the previous example and

$$\begin{aligned} P(n,\lambda) &= \begin{cases} (\sin(n-1)\theta - \sin(n-2)\theta)/\sin\theta & n = 2, \dots, N, \\ 0 & n = 0, 1, \end{cases} \\ &= \begin{cases} (\cos(n-3/2)\theta)/\cos(\theta/2) & n = 2, \dots, N, \\ 0 & n = 0, 1, \end{cases} \end{aligned} \tag{5.12}$$

$$\langle P, r\rangle = -1, \quad \langle \varphi_1, r\rangle = 1, \quad \langle \varphi_2, r\rangle = 1 - 2\cos\theta, \quad C(\lambda) \equiv 0, \tag{5.13}$$

$$v(n,\lambda) = \frac{\sin n\theta}{\sin\theta} + (1 - 2\cos\theta)\,\frac{\sin(n-1)\theta}{\sin\theta} = \frac{\cos(n-3/2)\theta}{\cos(\theta/2)}. \tag{5.14}$$

Any solution of (5.1) has the form $\varphi = \alpha\, v + \gamma\, P$, and the eigenvalues are the zeros of

(5.15)
$$\begin{aligned} \Delta_c(\lambda) &= \begin{vmatrix} -\big(1 - 2\cos\theta\big) & 0 \\ \cos(n-1/2)\theta/(\cos(\theta/2)) & \cos(n-1/2)\theta/(\cos(\theta/2)) \end{vmatrix} \\ &= -\left(1 - 2\cos\theta\right)\frac{\cos(n-1/2)\theta}{\cos(\theta/2)}. \end{aligned}$$

Hence, the eigenvalues are

$$\begin{aligned} \lambda_k &= 2\cos\left(\frac{(2k+1)\pi}{2N-1}\right), \quad k = 0, \dots, N-2, \\ \lambda_{N-1} &= 2\cos\left(\frac{\pi}{3}\right) = 1. \end{aligned} \tag{5.16}$$

Green's function of the unperturbed problem is

$$g(n,m,\lambda) = \frac{1}{\sin\theta\sin(N+1)\theta}\begin{cases} \sin(m-1)\theta\sin n\theta & 1 \le m \le n \le N, \\ \sin(n-1)\theta\sin m\theta & 1 \le n \le m \le N. \end{cases} \tag{5.17}$$

and

$$\begin{aligned} (A_\lambda r)(n) &= \begin{cases} (-\sin\theta\sin n\theta)/(\sin\theta\sin(N+1)\theta) \\ \qquad\qquad 2 \le n \le N, \\ (-\sin(n-1)\theta\sin 2\theta)/(\sin\theta\sin(N+1)\theta) \\ \qquad\qquad 1 \le n \le 2 \le N, \end{cases} \\ &= -g(n,2,\lambda). \end{aligned} \tag{5.18}$$

Taking $m_0 = 2$, we get

$$G_0(n,\lambda) = G(n,2,\lambda) = \begin{cases} \sin\theta\ \sin n\theta/(\sin\theta\big(\sin(N+1)\theta + \sin 2\theta\big)) & 2 \le n \le N, \\ \sin(n-1)\theta\sin 2\theta/(\sin\theta\big(\sin(N+1)\theta + \sin 2\theta\big)) & 1 \le n \le 2 \le N. \end{cases} \tag{5.19}$$

We distinguish between two cases.

Case 1. If $N-2$, is not divisible by 3, i.e., $\lambda_{N-1} \neq \lambda_k$, $k = 0, 1, \dots, N-2$, then all eigenvalues are simple. Thus, the zeros of $\Delta_c(\lambda)$ and $\omega(\lambda)$ are the same and have the same multiplicity. So $\omega(\lambda) = c_0\,\Delta_c(\lambda)$, c_0 is a nonzero constant. Hence, we can replace $\omega(\lambda)$ by $\Delta_c(\lambda)$ in the sampling expansion. Simple calculations yield

$$\Delta_c'(\lambda_k) = \begin{cases} (-1)^{k+1}(N-1/2)\big(2\cos\theta_k - 1\big)/(\cos(\theta_k/2)) & k = 0,1,\dots,N-2. \\ -2\cos\big(\{N-1/2\}\pi/3\big)\mathbf{Q} & k = N-1, \end{cases} \tag{5.20}$$

where $\theta_k = (2k+1)\pi/(2N+1)$, $k = 0, \dots, N-2$, $\theta_{N-1} = \pi/3$.

In this case the sampling representation of the transform

$$F(\lambda) = \sum_{n=1}^{N} \overline{g}(n)\, \Phi(n,\lambda), \quad \Phi(n,\lambda) = \omega(\lambda)\, G_0(n,\lambda), \tag{5.21}$$

takes the form

(5.22)

$$\begin{aligned} F(\lambda) = F(1) &\frac{-\big(2\cos\theta - 1\big)\cos\big(\{N-1/2\}\theta\big)}{2\big(\lambda - 2\cos\theta_{N-1}\big)\cos\big(\{N-1/2\}\pi/3\big)\cos(\theta/2)} \\ &+ \sum_{k=0}^{N-2} F(\lambda_k) \frac{(-1)^{k+1}\big(2\cos\theta - 1\big)\cos\big(\{N-1/2\}\theta\big)\cos(\theta_k/2)}{(N-1/2)\big(\lambda - 2\cos\theta_k\big)\big(2\cos\theta_k - 1\big)\cos(\theta/2)}. \end{aligned}$$

Case 2. If $N-2$, is divisible by 3, i.e., the eigenvalue $\lambda_{N-1} = \lambda_{(N-2)/3}$ is double, and the other eigenvalues are simple. Hence, the term $-(1-2\cos\theta) = -(1-\lambda)$ is repeated in Δ_c, so there is some constant β such that

(5.23)

$$\omega(\lambda) = \beta\,\frac{\cos(n-1/2)\theta}{\cos(\theta/2)}, \quad \text{thus} \quad \omega'(\lambda_k) = \beta\,\frac{(-1)^{k+1}(N-1/2)}{\cos(\theta_k/2)},$$
$$k = 0, 1, \ldots, N-2.$$

Therefore, we have the sampling formula

$$F(\lambda) = \sum_{k=0}^{N-2} F(\lambda_k) \frac{(-1)^{k+1}\cos\big(\{N-1/2\}\theta\big)\cos(\theta_k/2)}{(N-1/2)\big(\lambda - 2\cos\theta_k\big)\cos(\theta/2)}, \tag{5.24}$$

for the associated transform of the type (4.12).

Example 5.3. Consider the problem (5.1)–(5.2) with $(r(n)) = (1, 1, \ldots, 1)$. We have φ_1, φ_2 as in the previous examples and

$$P(n,\lambda) = \frac{\sin(n-1)\theta - \sin n\theta + \sin\theta}{4\sin^2(\theta/2)\sin\theta}, \tag{5.25}$$

$$C(\lambda) = 1 + \frac{N\sin\theta - \sin N\theta}{4\sin^2(\theta/2)\sin\theta}. \tag{5.26}$$

It is easy to see that when $\theta = 0$ or π, i.e., $\lambda = 2$ or -2, $C(\lambda) \neq 0$. For $0 < \theta < \pi$, equating (5.25) to zero, one gets

$$U_{N-1} + U_1 = N + 2, \qquad U_n := \frac{\sin(n+1)\theta}{\sin\theta}. \tag{5.27}$$

Since $|U_n| < n + 1$, for $0 < \theta < \pi$, we deduce that $C(\lambda) \neq 0$ for any real λ. Here we have

$$\begin{aligned}
\phi_1(n,\lambda) &= -\frac{\sin(n-1)\theta}{\sin\theta} + \frac{\cos(\theta/2) - \cos(N-1/2)\theta}{4\sin^2(\theta/2)\sin\theta + N\sin\theta - \sin N\theta} \\
&\quad \times \frac{\sin(n-1)\theta - \sin n\theta + \sin\theta}{2\sin(\theta/2)\sin\theta} \\
\phi_2(n,\lambda) &= \frac{\sin n\theta}{\sin\theta} - \frac{\cos(\theta/2) - \cos(N+1/2)\theta}{4\sin^2(\theta/2)\sin\theta + N\sin\theta - \sin N\theta} \\
&\quad \times \frac{\sin(n-1)\theta - \sin n\theta + \sin\theta}{2\sin(\theta/2)\sin\theta}.
\end{aligned}$$

Since $\Theta_1(n,\lambda) = -\phi_2(n,\lambda)$, the eigenvalues are the zeros of

$$\begin{aligned}
\omega_1(\lambda) &= -\frac{\sin\,(N+1)\theta}{\sin\theta} + \frac{\cos(\theta/2) - \cos(N+1/2)\theta}{4\sin^2(\theta/2)\sin\theta + N\sin\theta - \sin N\theta} \\
&\quad \times \frac{\sin N\theta - \sin(N+1)\theta + \sin\theta}{2\sin(\theta/2)\sin\theta} = 0.
\end{aligned} \tag{5.28}$$

Therefore, if $g(\cdot) \in \ell^2(\mathbf{Z}_N)$, and

$$F(\lambda) = \sum_{n=1}^{N} \overline{g}(n)\Phi(n,\lambda), \quad \lambda \in \mathbf{C}, \tag{5.29}$$

where Φ is given in (4.11) above, then $F(\lambda)$ has the sampling expansion

$$F(\lambda) = \sum_{k=1}^{s} F(\lambda_k) \frac{\omega_1(\lambda)}{(\lambda - \lambda_k)\,\omega_1'(\lambda_k)}, \tag{5.30}$$

where $\{\lambda_k\}_{k=1}^{N}$ are the zeros of (5.28).

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