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Λ-ABSOLUTE CONTINUITY

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ABSTRACT. A characterization of functions continuous in Λ-variation is given based on the decomposition of a local Λ-variation. A natural generalization of absolutely continuous functions is introduced, and several characterizations of it are stated. Important relationships between various classes and subclasses of functions of bounded Λ-variation are studied.

The concept of bounded harmonic variation arose naturally from the theory of Fourier series and has many applications to it. The most important among them is the following theorem that furnishes the same conclusions as the Dirichlet-Jordan theorem.

Waterman's Test. *If f is of bounded harmonic variation, then*

(i) $S[f](x)$ *converges to* $(f(x+) + f(x-))/2$ *pointwise*;

(ii) $S[f]$ *converges to* f *uniformly on every closed interval of points of continuity.*

The test was proven in [**18**] where also a general concept of Λ-variation of a function was introduced. Later Waterman found a direct proof of his test, not resting on the Lebesgue test, see [**21, 23**]. The Waterman test includes all other tests that yield Dirichlet-Jordan type conclusions: the classical Dirichlet-Jordan test, the Salem tests and the Garsia-Sawyer test. Waterman's test is much more convenient to use than the Lebesgue test since the second condition of the latter test corresponds to no simple property of a tested function.

There are numerous other applications of the concept of Λ-variation to the theory of Fourier series. The earlier results have been collected in [**1**].

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The aim of this paper is to study properties of a special subclass of functions of bounded Λ-variation – the so-called functions continuous in Λ-variation, first considered in [**19**] – and to answer an open question from [**20**, page 44]. We start with a collection of mostly classical definitions and specialized notation.

A sequence (λ_i) of positive numbers is said to be a Λ-sequence if it is nondecreasing and such that $\sum(1/\lambda_i) = \infty$. A Λ-sequence $\Lambda = (\lambda_i)$ is said to be proper if $\lim \lambda_i = +\infty$. Given a Λ-sequence $\Lambda = (\lambda_i)$ and a positive integer m, the omission of the first m terms supplies a new Λ-sequence $(\lambda_i)_{i=m+1}^{\infty}$ that will be denoted by $\Lambda_{(m)}$.

Throughout this paper we will be concerned primarily with real-valued functions defined on $[0, 1]$. A function is said to be regulated if it admits discontinuities of the first kind only. We do not assume that the value of a regulated function is always the arithmetic average of its one-sided limits. Note that regulated functions can be characterized as those which take monotone sequences into convergent ones.

A collection $\mathcal{I}$ of nonoverlapping closed intervals with endpoints in a set A will be called an A-family. In the most common case when $A = [0, 1]$, we will say shortly *a family* instead of a $[0, 1]$-family. Given a regulated function f and a point t, the lefthand and the righthand limits of f at t will be denoted by $f(t-)$ and $f(t+)$, respectively. We agree to write $f(0-) = f(0)$ and $f(1+) = f(1)$. Given a subinterval I, not necessarily closed, we write $f(I) = f(b) - f(a)$ and $|I| = b - a$ where a, b are the left and right endpoints of I, respectively. If $\mathcal{I} = \{I_i\}$ is a family such that $|f(I_{i+1})| \le |f(I_i)|$ for all indices, then $\mathcal{I}$ is said to be f-ordered. We write $\|\mathcal{I}\| := \sup\{|I| : I \in \mathcal{I}\}$ and $\|\mathcal{I}\|_f := \sup\{|f(I)| : I \in \mathcal{I}\}$.

Let Λ be a Λ-sequence or a finite set of positive numbers. Given a family $\mathcal{I}$ of cardinality not exceeding the cardinality of Λ, we set

$$\sigma_\Lambda(f, \mathcal{I}) \ := \sup \sum_{I \in \mathcal{I}} \frac{|f(I)|}{\beta(I)},$$

where the supremum is taken over all injective mappings $\beta : \mathcal{I} \to \Lambda$.

The value $V_\Lambda(f) := \sup \sigma_\Lambda(f, \mathcal{I})$, where the supremum is taken over all families $\mathcal{I}$, is called the Λ-variation of f. If $V_\Lambda(f)$ is finite, we say that f is of bounded Λ-variation and write $f \in \Lambda BV$. The purpose of introducing the symbol $\sigma_\Lambda(f, \mathcal{I})$ is to allow greater flexibility and brevity in the proof of our main decomposition theorem (Theorem 1) and related definitions. Definitely, the best way of thinking about $\sigma_\Lambda(f, \mathcal{I})$ is to perceive the family $\mathcal{I}$ as f-ordered $\mathcal{I} = \{I_1, I_2, \dots\}$ and then $\sigma_\Lambda(f, \mathcal{I})$ becomes simply $\sum |f(I_i)|/\lambda_i$, [**5**, Theorem 368].

To see that Λ-variation is a generalization of the classical variation of a function, it suffices to take the special Λ-sequence $\Lambda = (1)$, that is, the constant sequence of 1's. Then the Λ-variation of any function is exactly the ordinary variation. The best sources of basic properties of Λ-variation are papers [**9, 20**].

1. Wiener Λ-variation and its decomposition. We are going to alter the definition of Λ-variation slightly. On one hand the change will be small as our Proposition 1 shows, but on the other hand this definition will eventually provide us with a new characterization of an important subclass of ΛBV. A similar alteration of generalized variation has been used in the theory of functions of bounded ϕ-variation. It originates from an idea of Wiener [**24**, pages 72–73].

Definition 1. Given a Λ-sequence Λ and a positive number δ, we define $V_{\Lambda,\delta}(f) := \sup \sigma_\Lambda(f, \mathcal{I})$ where the supremum is taken over all families $\mathcal{I}$ with $\|\mathcal{I}\| \leq \delta$. The value

$$W_\Lambda(f) := \lim_{\delta \to 0+} V_{\Lambda,\delta}(f)$$

will be called the Wiener Λ-variation of f.

Note that if we replace the requirement $\|\mathcal{I}\| \leq \delta$ by $\|\mathcal{I}\| < \delta$ in the above definition, the final value W_Λ will not change.

Clearly, one always has $W_\Lambda(f) \leq V_\Lambda(f)$. Roughly speaking, the Wiener Λ-variation of f is the Λ-variation of f achieved on families of infinitely short intervals. At first glance this concept might seem artificial, especially because of the following equivalence.

Proposition 1. *For any Λ-sequence Λ and any function f, its generalized variation $V_\Lambda(f)$ is finite if and only if $W_\Lambda(f)$ is finite.*

Proof. Since $W_\Lambda(f) \leq V_\Lambda(f)$ for every function f, it suffices to prove that $W_\Lambda(f) < +\infty$ implies $V_\Lambda(f) < +\infty$.

Suppose f is not of bounded Λ-variation. Then there is a family (I_i) such that $\sum |f(I_i)|/\lambda_i = +\infty$, [**20**, Theorem 1]. Given $\delta > 0$, there is a positive integer N_δ such that $|I_i| \leq \delta$ for all $i \geq N_\delta$. Hence, $V_{\Lambda,\,\delta}(f) \geq \sigma_\Lambda(f, (I_i)_{i=N_\delta}^\infty) = +\infty$, and thus $W_\Lambda(f) = +\infty$. □

It is easy to see that the functional $\|f\|_\Lambda^W := |f(0)| + W_\Lambda(f)$ is a semi-norm on ΛBV. One could also consider a sequence of semi-norms

$$\|f\|_\Lambda^n := |f(0)| + V_{\Lambda,\,(1/n)}(f),$$

but if we endow ΛBV with the quasi-norm topology generated by this sequence [**25**, page 32], we obtain merely the usual norm topology on ΛBV given by $\|f\|_\Lambda := |f(0)| + V_\Lambda(f)$ as defined in [**20**].

We now define another kind of Λ-variation, a variation that is determined by infinitely small changes of f.

Definition 2. Given a Λ-sequence Λ and a positive number δ, we set $V_\Lambda^\delta(f) := \sup \sigma_\Lambda(f, \mathcal{I})$ where the supremum is taken over all families $\mathcal{I}$ with $\|\mathcal{I}\|_f \leq \delta$, and define

$$V_\Lambda^0(f) := \lim_{\delta \to 0+} V_\Lambda^\delta(f).$$

Clearly, $V_\Lambda^0(f) \leq V_\Lambda(f)$ for any function. It can be proven in a manner similar to that of the Proposition 1 that the above definition of generalized variation does not lead to any new class of functions.

Proposition 2. *Let Λ be a Λ-sequence, and let f be a regulated function. Then $V_\Lambda(f)$ is finite if and only if $V_\Lambda^0(f)$ is finite.*

Note that the assumption on f being regulated cannot be dropped as the example of the characteristic function of rationals shows. The last definition admits an equivalent formulation that however obscures the intuitive meaning of V_Λ^0 as of the Λ-variation determined by infinitely small changes of the underlying function.

Definition 3. Given a Λ-sequence Λ and a positive number δ, we define

$$^{\delta}V_{\Lambda}(f) := \sup \sum_i \frac{\min\{\delta, |f(I_i)|\}}{\lambda_i}$$

where the supremum is taken over all families $\mathcal{I}$, and further we define

$$^{0}V_{\Lambda}(f) := \lim_{\delta \to 0+} {}^{\delta}V_{\Lambda}(f).$$

In general, $^{0}V_{\Lambda}(f) \neq V_{\Lambda}^{0}(f)$. For instance, the characteristic function of the rationals $\chi_{\mathbf{Q}}$ yields $^{0}V_{\Lambda}(\chi_{\mathbf{Q}}) = +\infty$ and $V_{\Lambda}^{0}(\chi_{\mathbf{Q}}) = 0$. However, for regulated functions both definitions yield the same value.

Proposition 3. *The equality* $^{0}V_{\Lambda}(f) = V_{\Lambda}^{0}(f)$ *holds for every* Λ-*sequence* Λ *and every regulated function* f.

Proof. It is easy to see that $V_{\Lambda}^{\delta}(f) \leq {}^{\delta}V_{\Lambda}(f)$ for every $\delta > 0$. Hence, $V_{\Lambda}^{0}(f) \leq {}^{0}V_{\Lambda}(f)$.

We are now going to prove the opposite inequality. Given a positive number δ, let $K_{f,\delta}$ be the least upper bound of the cardinalities of families consisting entirely of intervals I with $|f(I)| > \delta$. Since f is regulated, $K_{f,\delta}$ is a nonnegative integer. Given now a positive number γ and a family (I_i), we get

$$\sum_i \frac{\min\{\gamma, |f(I_i)|\}}{\lambda_i} = \sum_{i:|f(I_i)|>\delta} \cdots + \sum_{i:|f(I_i)|\leq\delta} \cdots,$$

and hence

$$^{\gamma}V_{\Lambda}(f) \leq K_{f,\delta} \frac{\gamma}{\lambda_1} + V_{\Lambda}^{\delta}(f).$$

Passing to a limit with $\gamma \to 0+$, we obtain $^{0}V_{\Lambda}(f) \leq V_{\Lambda}^{\delta}(f)$ for every $\delta > 0$. Thus, $^{0}V_{\Lambda}(f) \leq V_{\Lambda}^{0}(f)$. □

The above proposition significantly simplifies the demonstration of the fact that

$$\|f\|_{\Lambda}^{0} := |f(0)| + V_{\Lambda}^{0}(f)$$

is a semi-norm on ΛBV.

Our next definition is motivated by the observation that, in the case of a regulated function that is constant on each interval of continuity, its change and thus its variation depends entirely on the behavior at discontinuities–the jumps.

Example 1. Given a decreasing sequence $u_n \searrow 0$, define

$$g(t) := \begin{cases} \sum_{i=n}^{\infty}(-1)^i u_i & \text{if } t \in ((1/2^n), (1/2^{n-1})],\ n \in \mathbf{N}; \\ 0 & \text{if } t = 0. \end{cases}$$

Then g is a regulated function on $[0, 1]$ and

$$V_\Lambda(g) = W_\Lambda(g) = \sum_{i=1}^{\infty} \frac{|g(2^{-i}+) - g(2^{-i})|}{\lambda_i} = \sum_{i=1}^{\infty} \frac{u_i}{\lambda_i}$$

for every Λ-sequence Λ. Thus, we can say that the Λ-variation of g (no matter what Λ is!) is obtained by arranging the jumps of g in the order of decreasing magnitude, say $(\eta_i^g)_{i=1}^{\infty}$, and then simply taking the sum $\sum(\eta_i^g/\lambda_i)$.

If we changed the value of g at $1/2$ from $\sum_{i=2}^{\infty}(-1)^i u_i$ to $u_2 + \sum_{i=2}^{\infty}(-1)^i u_i$, the modified function g_m would have a proper external saltus at $1/2$. This point contributes two terms to the series

$$\frac{|g_m(1/2+) - g_m(1/2)|}{\lambda_1} + \frac{|g_m(1/2) - g_m(1/2-)|}{\lambda_2} + \sum_{i=2}^{\infty} \frac{|g(2^{-i}+) - g(2^{-i})|}{\lambda_{i+1}}$$

that gives the Wiener Λ-variation of g_m. Thus, we can view g_m as having two jumps at the point $t = 1/2$ which makes it possible to describe the Wiener Λ-variation of g_m as obtained by arranging all jumps of g_m in the order of decreasing magnitude $(\eta_i^{g_m})_{i=1}^{\infty}$ and then simply taking $\sum(\eta_i^{g_m})/(\lambda_i)$. It provides motivation for the following definition of a sequence of jumps (η_i^f) of a regulated function f.

Definition 4. First, we will associate three indicators $\beta_{t,\,i}^f$, $i = 1, 2, 3$, with each discontinuity point t of the regulated function f.

Their purpose is to keep track of how many jumps the point t contributes to the sequence of jumps of f and what kind of jumps they are. We set

$$\beta^f_{t,1} = \beta^f_{t,2} := 0 \text{ and } \beta^f_{t,3} := |f(t+) - f(t-)|$$

if f has a proper internal saltus at t, that is, if f has an internal saltus at t but is not one-side continuous there, and we set

$$\beta^f_{t,1} := |f(t-) - f(t)|, \quad \beta^f_{t,2} := |f(t+) - f(t)|, \quad \beta^f_{t,3} := 0$$

otherwise.

Next we arrange all elements of the countable set

$$Z_f := \left\{\beta^f_{t,j} : \beta^f_{t,j} > 0,\ 0 \le t \le 1,\ j = 1,\ 2,\ 3\right\}$$

into a nonincreasing sequence $(\eta^f_i)^\infty_{i=1}$. If the set Z_f is finite, we put $\eta^f_i := 0$ for $i > \operatorname{card} Z_f$.

It is easy to see that $f \in \Lambda BV$ implies $\sum(\eta^f_i)/(\lambda_i) < +\infty$, but not conversely.

Proposition 4. *For every* Λ*-sequence* Λ*, the functional*

$$|||f|||_\Lambda := \sum_{i=1} \infty \frac{\eta^f_i}{\lambda_i}$$

is a semi-norm on ΛBV.

Proof. The least evident is subadditivity of the functional. However, given regulated functions f, g and a point $t \in [0,\ 1]$,

(i) if exactly one of the three indices $\beta^{f+g}_{t,j}$ is positive, then

$$\beta^{f+g}_{t,j} \le \max_{k=1,\,2,\,3} \beta^f_{t,k} + \max_{k=1,\,2,\,3} \beta^g_{t,k};$$

(ii) if both $\beta^{f+g}_{t,1}$ and $\beta^{f+g}_{t,2}$ are positive, then for arbitrary numbers $0 < \lambda_1 \le \lambda_2$,

(a) if $\beta^f_{t,3} = \beta^g_{t,3} = 0$, then

$$\frac{\beta^{f+g}_{t,1}}{\lambda_1} + \frac{\beta^{f+g}_{t,2}}{\lambda_2} \le \frac{\beta^f_{t,1}}{\lambda_1} + \frac{\beta^f_{t,2}}{\lambda_2} + \frac{\beta^g_{t,1}}{\lambda_1} + \frac{\beta^g_{t,2}}{\lambda_2};$$

(b) if $\beta^f_{t,3} > 0$ and $\beta^g_{t,3} = 0$, then

$$\frac{\beta^{f+g}_{t,1}}{\lambda_1} + \frac{\beta^{f+g}_{t,2}}{\lambda_2} \le \frac{\beta^f_{t,3}}{\lambda_1} + \frac{\beta^g_{t,1}}{\lambda_1} + \frac{\beta^g_{t,2}}{\lambda_2};$$

(c) if $\beta^f_{t,3} = 0$ and $\beta^g_{t,3} > 0$, then

$$\frac{\beta^{f+g}_{t,1}}{\lambda_1} + \frac{\beta^{f+g}_{t,2}}{\lambda_2} \le \frac{\beta^f_{t,1}}{\lambda_1} + \frac{\beta^f_{t,2}}{\lambda_2} + \frac{\beta^g_{t,3}}{\lambda_1};$$

(d) if $\beta^f_{t,3} > 0$ and $\beta^g_{t,3} > 0$, then

$$\frac{\beta^{f+g}_{t,1}}{\lambda_1} + \frac{\beta^{f+g}_{t,2}}{\lambda_2} \le \frac{\beta^f_{t,3}}{\lambda_1} + \frac{\beta^g_{t,3}}{\lambda_1}.$$

The rest is straightforward and will be omitted. □

We are now able to formulate our main decomposition theorem.

Theorem 1. *If Λ is a Λ-sequence and f is a regulated function, then*

$$\|f\|^W_\Lambda = |||f|||_\Lambda + \|f\|^0_\Lambda.$$

Before we prove it, let us make few comments. Following Young, we can define

$$\Lambda BV^* := \left\{ f \in \Lambda BV : W_\Lambda(f) = \sum \frac{\eta^f_i}{\lambda_i} \right\},$$

see [**26**, page 261]. An easy example of a function of the class ΛBV^* is provided by any step function of bounded Λ-variation. By our decomposition theorem, given a function f of bounded Λ-variation, $f \in \Lambda BV^*$ if and only if $V_\Lambda^0(f) = 0$. Therefore, $(\Lambda BV^*, \| \, \|_\Lambda)$ is a closed subspace of the Banach space ΛBV, because

$$V_\Lambda^0(f) \le V_\Lambda^0(f - f_n) + V_\Lambda^0(f_n) = V_\Lambda^0(f - f_n) \le \|f - f_n\|_\Lambda$$

for $f_n \in \Lambda BV^*$.

Denoting the ordinary variation of f by $V(f)$, we get $V(f) = V_\Lambda(f) = W_\Lambda(f)$ for the Λ-sequence $\Lambda = (1)_{i=1}^\infty$. Thus, our decomposition theorem is a generalization of the well-known fact that $V(f) = V(f_c) + V(f_s)$ where f_s is the saltus function of f and $f_c := f - f_s$ is the continuous part of f, see [**15**, page 308] or, for a modern and more general treatment, [**4**, Corollary 7.7, Theorem 5.3 and Theorem 5.5]. However, it is still unknown whether each function of bounded Λ-variation can be written as a sum of a continuous part and a "saltus-like" function.

Open problem. *Let Λ be a proper Λ-sequence, and let $f \in \Lambda BV$. Do there exist an $f_1 \in \Lambda BV^*$ and a continuous $f_2 \in \Lambda BV$ such that $f = f_1 + f_2$ and $W_\Lambda(f) = W_\Lambda(f_1) + W_\Lambda(f_2)$?*

The proof of Theorem 1 is rather elementary, but very lengthy. We start with some specialized notation and a number of lemmas that are necessary to handle the case of discontinuous f. From now on until the end of the proof of Lemma 8, we will always assume that f is a discontinuous regulated function. Furthermore, we will write η_i instead of η_i^f.

Definition 5. Given $\alpha > 0$ and a function f, we set $m(\alpha) := \operatorname{card}\{i : \eta_i \ge \alpha\}$. The integer-valued function m is nonincreasing on $(0, \infty)$, continuous on the left and $\lim_{\alpha \to 0+} m(\alpha) = \operatorname{card} Z_f$, see Definition 4. Further, given $\alpha > 0$, we set

$$C_1(\alpha) = \left\{t \in [0,\, 1] : \beta_{t,1}^f \ge \alpha \text{ and } \beta_{t,2}^f < \alpha\right\}$$
$$C_2(\alpha) = \left\{t \in [0,\, 1] : \beta_{t,1}^f < \alpha \text{ and } \beta_{t,2}^f \ge \alpha\right\}$$
$$C_3(\alpha) = \left\{t \in [0,\, 1] : \beta_{t,3}^f \ge \alpha\right\}$$
$$C_4(\alpha) = \left\{t \in [0,\, 1] : \beta_{t,1}^f \ge \alpha \text{ and } \beta_{t,2}^f \ge \alpha\right\}.$$

The sets $C_j(\alpha)$ are finite and pairwise disjoint. Denoting the set of all t satisfying $\max\{\beta^f_{t,j} : j = 1, 2, 3\} \geq \alpha$ by $C(a)$, we get

$$C(\alpha) = \bigcup_{j=1}^{4} C_j(\alpha) \quad \text{and} \quad m(\alpha) = \operatorname{card} C_4(\alpha) + \operatorname{card} C(\alpha).$$

Next, for $\alpha \leq \eta_1$, we set $\delta_1(\alpha) := (1/2)\min\{|t - s| : t, s \in C(\alpha) \cup \{0, 1\}, \ t \neq s\}$.

Let α, δ be positive numbers such that $\alpha \in (0, \eta_1]$ and $\delta \leq \delta_1(\alpha)$. We will say that a family $\mathcal{I}$ is a δ-cover of the set $C(\alpha)$ if $\mathcal{I}$ consists of all intervals of the following form

(i) $[t - \delta, t]$ for $t \in C_1(\alpha)$;

(ii) $[t, t + \delta]$ for $t \in C_2(\alpha)$;

(iii) $[t - (\delta/2), t + (\delta/2)]$ for $t \in C_3(\alpha)$;

(iv) $[t - \delta, t]$ and $[t, t + \delta]$ for $t \in C_4(\alpha)$.

Of course, by $[c, d]$ we mean the intersection $[c, d] \cap [0, 1]$. We will use the symbol $\mathcal{I}_{\alpha,\delta}$ to denote the δ-cover of the set $C(\alpha)$. For $t \in C(\alpha)$, we set $\mathcal{I}_{\alpha,t,\delta} := \{I \in \mathcal{I}_{\alpha,\delta} : \ t \in I\}$. It is easy to see that the number of intervals in $\mathcal{I}_{\alpha,\delta}$ is exactly $m(\alpha)$.

Lemma 1. *Let α and ε be positive numbers with $\alpha \leq \eta_1$. Then there exists a $\delta_2 = \delta_2(\alpha, \varepsilon) \in (0, \delta_1(\alpha)]$ such that for an arbitrary $\delta \in (0, \delta_2)$, if the variational δ-cover $\mathcal{I}_{\alpha,\delta} = \{I_i : \ i = 1, \dots, m(\alpha)\}$ is f-ordered, then $\big||f(I_i)| - \eta_i\big| < \varepsilon$ for $i = 1, \dots, m(\alpha)$.*

Proof. Given an $\alpha \in (0, \eta_1]$, let $(\mathcal{I}_{\alpha,\delta})_r$, $r = 1, \dots, m(\alpha)$, denote the rth interval from the left in $\mathcal{I}_{\alpha,\delta}$. Let $r \mapsto i_r$ be a bijection of $\{1, \dots, m(\alpha)\}$ onto itself such that $|f((\mathcal{I}_{\alpha,\delta})_r)| \underset{\delta \to 0+}{\longrightarrow} \eta_{i_r}$ for all $r = 1, \dots, m(\alpha)$. Then for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\big||f((\mathcal{I}_{\alpha,\delta})_r) - \eta_{i_r}\big| < \varepsilon$ for $\delta \leq \delta(\varepsilon)$, $r = 1, \dots, m(\alpha)$. Set

$$\varepsilon_\alpha := \tfrac{1}{2}\min\{1, |\eta_i - \eta_j| : i, j \leq m(\alpha) \text{ and } \eta_i \neq \eta_j\}$$

and, given $\varepsilon > 0$, set $\delta_2(\alpha, \varepsilon) := \min\{\delta(\varepsilon), \delta(\varepsilon_\alpha), \delta_1(\alpha)\}$.

Now let $\mathcal{I} = \{I_j : j = 1, \dots, m(\alpha)\}$ be an f-ordered variational δ-cover of the set $C(\alpha)$ with $\delta < \delta_2(\alpha, \varepsilon)$. For $r = 1, \dots, m(\alpha)$ we denote the index of the interval $(\mathcal{I})_r$ in the sequence $(I_j)_{j=1}^{m(\alpha)}$ by j_r. Given $r,\, s \in \{1, \dots , m(\alpha)\}$, if $\eta_{i_r} > \eta_{i_s}$, then

$$\max\big\{\big||f((\mathcal{I})_r)| - \eta_{i_r}\big|,\, \big||f((\mathcal{I})_s)| - \eta_{i_s}\big|\big\} < \varepsilon_\alpha \le \frac{\eta_{i_r} - \eta_{i_s}}{2},$$

since $\|\mathcal{I}\| \le \delta_2(\varepsilon,\, \alpha)$. Therefore $|f((\mathcal{I})_r| > |f((\mathcal{I})_s)|$ and $j_r < j_s$, because $\mathcal{I}$ is f-ordered. Hence, $\eta_{i_r} > \eta_{i_s}$ implies $\eta_{j_r} \ge \eta_{j_s}$, and thus $\eta_{i_r} = \eta_{j_r}$ for all $r = 1, \dots , m(\alpha)$. Since $\|\mathcal{I}\| \le \delta(\varepsilon)$, it follows that $\big||f(I_{j_r})| - \eta_{j_r}\big| = \big||f((\mathcal{I})_r)| - \eta_{i_r}\big| < \varepsilon$ for $r = 1, \dots , m(\alpha)$, that is, $\big||f(I_j)| - \eta_j\big| < \varepsilon$ for all j. □

The next lemma is a simple corollary of Lemma 1.

Lemma 2. *Let $\Lambda = (\lambda_1)$ be a Λ-sequence, and let $\alpha \in (0,\, \eta_1]$. Then*

$$\lim_{\delta \to 0+} \sigma_\Lambda(f,\, \mathcal{I}_{\alpha,\delta}) = \sum_{i=1}^{m(\alpha)} \frac{\eta_i}{\lambda_i}.$$

Definition 6. Given $\alpha \in (0,\, \eta_1]$, we will say that an interval $[s_1,\, s_2]$ touches the set $C(\alpha)$ at a point t if one of the following conditions holds:

(i) $t \in (s_1,\, s_2)$ for some $t \in C(\alpha)$;

(ii) $s_2 = t$ for some $t \in C_1(\alpha)$;

(iii) $s_1 = t$ for some $t \in C_2(\alpha)$;

(iv) $s_1 = t$ or $s_2 = t$ for some $t \in C_3(\alpha) \cup C_4(\alpha)$.

We will say that a family $\mathcal{I}$ does not touch the set $C(\alpha)$ if no interval from $\mathcal{I}$ touches the set $C(\alpha)$. Further, we set

$$\begin{aligned}
M_1(\alpha) &:= \sup\big\{|f(t-) - f(t)| : 0 \le t \le 1 \text{ and } |f(t-) - f(t)| < \alpha\big\},\\
M_2(\alpha) &:= \sup\big\{|f(t+) - f(t)| : 0 \le t \le 1 \text{ and } |f(t+) - f(t)| < \alpha\big\},\\
M_3(\alpha) &:= \sup\big\{|f(t-) - f(t+)| : 0 \le t \le 1 \text{ and } |f(t-) - f(t+)| < \alpha\big\},\\
M(\alpha) &:= \max\big\{M_i(\alpha) : i = 1,\, 2,\, 3\big\}
\end{aligned}$$

for $\alpha > 0$. Finally, we set

$$\mu(t) := \max\big\{|f(t) - f(t-)|,\, |f(t+) - f(t)|,\, |f(t+) - f(t-)|\big\}$$

for $t \in [0, 1]$, and

$$\theta(s_1, s_2) := \sup\{\mu(t) : t \in (s_1, s_2)\}$$

for any subinterval $[s_1, s_2]$.

We can now formulate two simple lemmas.

Lemma 3. *Let $[s, t]$ be a subinterval of $[0, 1]$. Then $\theta(s, t) = \mu(q)$ for some $q \in (s, t)$.*

Lemma 4. *Let $\alpha \in (0, \eta_1]$. Then $M(\alpha) < \alpha$.*

Both of these lemmas are based on the observation that for a regulated function f the sets $\{t \in [0, 1] : \mu(t) > \alpha\}$ are finite for $\alpha > 0$.

Lemma 5. *Let $\alpha \in (0, \eta_1]$. Then there is a $\delta_3(\alpha) > 0$ such that for every subinterval I with $|I| < \delta_3(\alpha)$ and with $|f(I)| > (\alpha + M(\alpha))/2$, the interval touches the set $C(\alpha)$.*

Proof. By virtue of Lemma 3 it suffices to show that there exists a $\delta_3(\alpha) > 0$ such that for an arbitrary interval $[s, t]$ if $t - s < \delta_3(\alpha)$ and $|f(t) - f(s)| > (\alpha + M(\alpha))/2$, then $\max\{|f(s+) - f(s)|, |f(t) - f(t-)|, \theta(s, t)\} \geq \alpha$.

Suppose this is not true. Then there exist intervals $[s_n, t_n]$, $n = 1, 2, \dots$, such that $t_n - s_n < 1/n$, $|f(t_n) - f(s_n)| > (\alpha + M(\alpha))/2$ and

$$\max\{|f(s_n+) - f(s_n)|, |f(t_n) - f(t_n)|, \theta(s_n, t_n)\} < \alpha.$$

Hence, by the definition of $M(\alpha)$ we get

$$\text{(1)} \quad \sup_n \max\{|f(s_n+) - f(s_n)|, |f(t_n) - f(t_n)|, \theta(s_n, t_n)\} \leq M(\alpha).$$

Passing, if necessary, to a subsequence, we conclude that there is a point $r \in [0, 1]$ such that exactly one of the following cases holds:

(i) $s_n \searrow r$;

(ii) $t_n \nearrow r$;

(iii) $s_n = r$ for all n;

(iv) $t_n = r$ for all n;

(v) $t_n \searrow r$ and $s_n \nearrow r$.

In case (i) (or (ii)) the right side (left side) limit of f at r does not exist and we get a contradiction, because f is regulated. In case (iii) we have $|f(r) - f(r+)| \geq (\alpha + M(\alpha))/2$. It follows that $|f(r) - f(r+)| \geq \alpha$ and thus $|f(s_n) - f(s_n+)| \geq \alpha$ for all n which contradicts (1). Case (iv) is analogous. In case (v), $|f(r+) - f(r-)| \geq (\alpha + M(\alpha))/2$, and hence $\theta(s_n, t_n) \geq \alpha$ for all n which contradicts (1). □

Lemma 6. *Given $\alpha \in (0, \eta_1]$, $c > 0$, $\varepsilon > 0$ and a point $t \in C(\alpha)$, there exists a $\delta_4 := \delta_4(\alpha, c, \varepsilon, t) \in (0, \delta_1)$ such that, if for a family $\mathcal{I}$,*

(a) *every interval $I \in \mathcal{I}$ touches the set $C(\alpha)$ at the point t;*

(b) $\|\mathcal{I}\| < \delta_4$;

then, for arbitrary numbers λ_1, $\lambda_2 \geq c$,

$$\sigma_{\{\lambda_1,\lambda_2\}}(f, \mathcal{I}) \leq \sigma_{\{\lambda_1,\lambda_2\}}(f, \mathcal{I}_{\alpha,t,\gamma}) + \varepsilon,$$

for every $\gamma \in (0, \delta_4)$.

Note that a family $\mathcal{I}$ satisfying (a) and (b) contains at most two intervals.

Proof. We will consider only the case $t \in C_4(\alpha)$, the other cases being similar. Let $\delta > 0$ be such that $|f(r) - f(\tilde{r})| < (1/2)\min\{c\varepsilon, \alpha\}$ for $r, \tilde{r} \in [t-\delta, t)$ and $|f(s) - f(\tilde{s})| < (1/2)\min\{c\varepsilon, \alpha\}$ for $s, \tilde{s} \in (t, t+\delta]$. Set $\delta_4 := \min\{\delta, \delta_1(\alpha)\}$. If $\mathcal{I}$ is a family satisfying conditions (a) and (b), then exactly one of the following cases holds:

(i) $\mathcal{I} = \{[r, s]\}$ for some $r < t < s$;

(ii) $\mathcal{I} = \{[r, t]\}$ for some $r < t$;

(iii) $\mathcal{I} = \{[t, s]\}$ for some $s > t$;

(iv) $\mathcal{I} = \{[r, t], [t, s]\}$ for some $r < t < s$.

In case (i), because f has an external saltus at t and

$$\max\{|f(r)-f(t-)|,\,|f(s)-f(t+)|\} \\ \leq \tfrac{1}{2}\min\{|f(t)-f(t+)|,\,|f(t)-f(t-)|\},$$

we have either $\min\{f(s),\,f(r)\} > f(t)$ or $\max\{f(r),\,f(s)\} < f(t)$. Hence, for any positive $\gamma < \delta_4$ the following inequalities are true

$$\begin{aligned}|f(r)-f(s)| &< \max\{|f(r)-f(t)|,\,|f(s)-f(t)|\,\} \\ &\leq \max\left\{|f(t-\gamma)-f(t)|+\frac{c\varepsilon}{2},\,|f(t+\gamma)-f(t)|+\frac{c\varepsilon}{2}\right\}.\end{aligned}$$

(Here we have assumed that $f(u)=f(0)$ for $u<0$ and that $f(u)=f(1)$ for $u>1$.) Thus, for any $\gamma<\delta_4$ and any $\lambda_2\geq\lambda_1\geq c$, we get

$$\begin{aligned}\sigma_{\{\lambda_1,\lambda_2\}}(f,\,\mathcal{I}) &= \frac{|f(r)-f(s)|}{\lambda_1} \\ &< \frac{\max\{|f(t-\gamma)-f(t)|,\,|f(t+\gamma)-f(t)|\}}{\lambda_1}+\varepsilon \\ &< \frac{\max\{|f(t-\gamma)-f(t)|,\,|f(t+\gamma)-f(t)|\}}{\lambda_1} \\ &\quad + \frac{\min\{|f(t-\gamma)-f(t)|,\,|f(t+\gamma)-f(t)|\}}{\lambda_2}+\varepsilon \\ &= \sigma_{\{\lambda_1,\lambda_2\}}(f,\,\mathcal{I}_{\alpha,t,\gamma})+\varepsilon.\end{aligned}$$

In case (ii) we get

$$\begin{aligned}\sigma_{\{\lambda_1,\,\lambda_2\}}(f,\,\mathcal{I}) &= \frac{|f(r)-f(s)|}{\lambda_1} \leq \frac{|f(t+\gamma)-f(t)|+(c\varepsilon)/2}{\lambda_1} \\ &< \frac{|f(t+\gamma)-f(t)|}{\lambda_1}+\frac{|f(t-\gamma)-f(t)|}{\lambda_2}+\varepsilon \\ &\leq \sigma_{\{\lambda_1,\,\lambda_2\}}(f,\,\mathcal{I}_{\alpha,t,\gamma})+\varepsilon.\end{aligned}$$

The remaining two cases are similar. □

Lemma 7. *Given $\alpha\in(0,\,\eta_1]$, $\varepsilon>0$ and a Λ-sequence $\Lambda=(\lambda_i)$, there exists a $\delta_5=\delta_5(\alpha,\,\Lambda,\,\varepsilon)>0$ such that for every family $\mathcal{I}$ satisfying*

(i) *every interval* $I \in \mathcal{I}$ *touches the set* $C(\alpha)$;

(ii) $\|\mathcal{I}\| < \delta_5$,

the following inequality holds:

$$\sigma_\Lambda(f, \mathcal{I}) \leq \sum_{i=1}^{m(\alpha)} \frac{\eta_i}{\lambda_i} + \varepsilon.$$

Proof. By Lemma 2 there exists a $\delta > 0$ such that

$$\sigma_\Lambda(f,\, \mathcal{I}_{\alpha,\gamma}) \leq \sum_{i=1}^{m(\alpha)} \frac{\eta_i}{\lambda_i} + \varepsilon$$

for all $\gamma \leq \delta$. Set $\varepsilon_1 := \varepsilon/\operatorname{card} C(\alpha)$ and $\delta_5 := \min\{\delta_4(\alpha,\, \lambda_1,\, \varepsilon_1,\, t) : t \in C(\alpha)\}$. Let $\mathcal{I} = \{I_i : i = 1, \dots, n\}$ be an f-ordered family satisfying (i) and (ii). For $t \in C(\alpha)$ we define $K_t := \{i : I_i \text{ touches the set } C(\alpha) \text{ at } t\}$. Without loss of generality we may assume that the sets K_t are nonempty for all $t \in C(\alpha)$, because $\delta_4 \leq \delta_1$. Furthermore, define for $t \in C(\alpha)$,

$$\Lambda_t := \{\lambda_i : i \in K_t\} \quad \text{and} \quad \mathcal{I}_t := \{I_i : i \in K_t\}.$$

Next, we set $\tilde{\Lambda}_t := \{\min \Lambda_t\}$ for $t \in \cup_{j=1}^3 C_j(\alpha)$. For $t \in C_4(\alpha)$ we set $\tilde{\Lambda}_t := \Lambda_t$ whenever K_t consists of two indices, and $\tilde{\Lambda}_t := \{\lambda_i,\, \lambda_{m(\alpha)}\}$ whenever $K_t = \{i\}$. Arranging the $m(\alpha)$ elements of all sets $\tilde{\Lambda}_t$ in a nondecreasing order without eliminating duplicates, we obtain a finite sequence $(\kappa_i)_{i=1}^{m(\alpha)}$ such that $\lambda_i \leq \kappa_i$ for $i = 1, \dots, m(\alpha)$. Thus, by Lemma 6, for any positive number $\gamma < \delta_5(\alpha,\, \Lambda,\, \varepsilon)$ we get

$$\begin{aligned}
\sigma_\Lambda(f, \mathcal{I}) &= \sum_{i=1}^{n} \frac{|f(I_i)|}{\lambda_i} = \sum_{t \in C(\alpha)} \sigma_{\Lambda_t}(f,\, \mathcal{I}_t) \\
&\leq \sum_{t \in C(\alpha)} \big(\sigma_{\tilde{\Lambda}_t}(f,\, \mathcal{I}_{\alpha,t,\gamma}) + \varepsilon_1\big) \\
&\leq \sigma_{(\kappa_i)}(f,\, \mathcal{I}_{\alpha,\gamma}) + \varepsilon \leq \sigma_\Lambda(f,\, \mathcal{I}_{\alpha,\,\gamma}) + \varepsilon
\end{aligned}$$

and finally, since $\gamma < \delta$,

$$\sigma_\Lambda(f,\mathcal{I}) \le \sum_{i=1}^{m(\alpha)} \frac{\eta_i}{\lambda_i} + 2\varepsilon. \qquad \square$$

Definition 7. Given $\alpha \in (0, \eta_1]$, $\delta > 0$, and a Λ-sequence Λ, we set $V_{\Lambda,\delta}(f)_\alpha := \sup \sigma_\Lambda(f,\mathcal{I})$ where the supremum is taken over all families $\mathcal{I}$ such that $\|\mathcal{I}\| < \delta$ and $\mathcal{I}$ does not touch the set $C(\alpha)$. Furthermore, we set $W_\Lambda(f)_\alpha := \lim_{\delta\to 0+} V_{\Lambda,\delta}(f)_\alpha$.

Lemma 8. *Let Λ be a Λ-sequence. Then for every $\alpha \in (0, \eta_1]$,*

$$V_\Lambda^0(f) \le W_{\Lambda_{(m(\alpha))}}(f)_\alpha.$$

Proof. Given positive numbers γ, δ and a family $\mathcal{I} = \{I_i : i = 1,\dots,n\}$ with $\|\mathcal{I}\|_f < \gamma$, we get

$$\sum_{i=1}^{n} \frac{|f(I_i)|}{\lambda_i} \le \sum_{(1)} \cdots + \sum_{(2)} \cdots + \sum_{(3)} \cdots + \sum_{(4)} \cdots ,$$

where the sum $\sum_{(1)} \cdots$ is taken over $i \le m(\alpha)$; the sum $\sum_{(2)} \cdots$ is taken over all i such that $|I_i| \ge \delta$; the sum $\sum_{(3)} \cdots$ extends over all i such that I_i touches the set $C(\alpha)$ and the sum $\sum_{(4)} \cdots$ is taken over all remaining i. Therefore,

$$V_\Lambda^\gamma(f) \le \frac{\gamma}{\lambda_1}\left(m(\alpha) + \frac{1}{\delta} + 2\,\mathrm{card}\, C(\alpha)\right) + V_{\Lambda_{(m(\alpha))},\delta}(f)_\alpha.$$

Passing to limits with $\gamma \to 0+$, we obtain $V_\Lambda^0(f) \le V_{\Lambda_{(m(\alpha))},\delta}(f)_\alpha$, and thus $V_\Lambda^0(f) \le W_{\Lambda_{(m(\alpha))}}(f)_\alpha$. $\square$

Proof of Theorem 1. First suppose that f is continuous. Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|I| < \delta$ implies $|f(I)| < \varepsilon$. Thus $V_{\Lambda,\,\delta}(f) \le V_\Lambda^\varepsilon(f)$, and therefore $W_\Lambda(f) \le V_\Lambda^0(f)$.

We are now going to prove the opposite inequality. Given ε, $\delta > 0$ and a family $\mathcal{I} = \{I_i : i = 1, \dots, n\}$ with $\|\mathcal{I}\|_f < \varepsilon$, we get

$$\sum_{i=1}^{n} \frac{|f(I_i)|}{\lambda_i} = \sum_{(1)} \cdots + \sum_{(2)} \cdots ,$$

where the first summation runs over all i such that $|I_i| \geq \delta$ and the second sum is taken over all remaining i. Thus,

$$V_\Lambda^\varepsilon(f) \leq \frac{\varepsilon}{\delta\lambda_1} + V_{\Lambda,\,\delta}(f).$$

Therefore, $V_\Lambda^0(f) \leq V_{\Lambda,\delta}(f)$ and finally $V_\Lambda^0(f) \leq W_\Lambda(f)$. Since $|||f|||_\Lambda = 0$, this finishes the proof for continuous f.

Suppose now that f is not continuous. Given an $\varepsilon > 0$ and an $\alpha \in (0, \eta_1]$, we set $\delta := \min\{\delta_3(\alpha), \delta_5(\alpha, \Lambda, \varepsilon)\}$. Given a family $\mathcal{I} = \{I_i : i = 1, \dots, n\}$ with $\|\mathcal{I}\| < \delta$, we get

$$\sum_{i=1}^{n} \frac{|f(I_i)|}{\lambda_i} = \sum_{(1)} \cdots + \sum_{(2)} \cdots ,$$

where the first summation runs over all i such that I_i touches the set $C(\alpha)$ and the second one runs over all other indices. By Lemma 7 we get

$$\sum_{(1)} \cdots \leq \sum_{i=1}^{m(\alpha)} \frac{\eta_1}{\lambda_i} + \varepsilon.$$

Next, by Lemmas 4 and 5, $|f(I_i)| \leq (\alpha + M(\alpha))/2 < \alpha$ for indices i such that I_i does not touch the set $C(\alpha)$. Hence,

$$W_\Lambda(f) \leq V_{\Lambda,\,\delta}(f) \leq \sum_{i=1}^{m(\alpha)} \frac{\eta_i}{\lambda_i} + \varepsilon + V_\Lambda^\alpha(f).$$

Since ε is arbitrary, it follows that for every $\alpha \in (0, \eta_1]$,

$$W_\Lambda(f) \leq \sum_{i=1}^{m(\alpha)} \frac{\eta_i}{\lambda_i} + V_\Lambda^\alpha(f).$$

Passing to limits as $\alpha \to 0+$, we obtain

$$W_\Lambda(f) \le \sum_{i=1}^{\infty} \frac{\eta_i}{\lambda_i} + V_\Lambda^0(f).$$

We will now prove the opposite inequality. Given an $\alpha \in (0,\, \eta_1]$ and a $\delta > 0$, let $\mathcal{I}$ be a family with $\|\mathcal{I}\| < \delta$ such that $\mathcal{I}$ does not touch the set $C(\alpha)$. Then

$$\sigma_\Lambda(f,\, \mathcal{I} \cup \mathcal{I}_{\alpha,\,\gamma}) \ge \sigma_\Lambda(f,\, \mathcal{I}_{\alpha,\,\gamma}) + \sigma_{\Lambda_{(m(\alpha))}}(f,\, \mathcal{I})$$

for γ sufficiently small, say $\gamma < \delta$, and such that the intervals of the γ-cover of $C(\alpha)$ do not overlap intervals of $\mathcal{I}$. Then, by Lemma 2,

$$V_{\Lambda,\,\delta}(f) \ge \sum_{i=1}^{m(\alpha)} \frac{\eta_i}{\lambda_i} + V_{\Lambda_{(m(\alpha))},\,\delta}(f)_\alpha.$$

Therefore, by Lemma 8 $V_{\Lambda,\,\delta}(f) \ge \sum(\eta_i/\lambda_i) + V_\Lambda^0(f)$, because α was arbitrarily small. Hence, $W_\Lambda(f) \ge \sum(\eta_i/\lambda_i) + V_\Lambda^0(f)$. □

2. Λ-absolute continuity. The notion of continuity in Λ-variation was introduced by Waterman in [**19**] to provide a sufficient condition for $(C,\, \beta)$-summability of Fourier series of a function.

Definition 8. A function f is said to be continuous in Λ-variation (in symbols $f \in \Lambda BV_c$) if $\lim_{m\to\infty} V_{\Lambda_{(m)}}(f) = 0$.

The above limit exists for every $f \in \Lambda BV$, because $\big(V_{\Lambda_{(m)}}(f)\big)_{m\in\mathbf{N}}$ then is a nonincreasing sequence of nonnegative numbers. It soon turned out that continuity in Λ-variation is very useful for estimating the order of magnitude of Fourier coefficients (for an overview see [**1**]). Clearly, $\Lambda BV_c \subset \Lambda BV$, but whether the two classes are equal has been an outstanding question since 1978 when Waterman formulated it in [**22**]. The only known characterization of functions continuous in Λ-variation was given by Wang in [**17**]. His theorem reads as follows.

Theorem [**17**]. *The necessary and sufficient condition for $f \in \Lambda BV_c$ is that there is a Λ-sequence $\Gamma = (\gamma_i)$ such that $\gamma_i = o(\lambda_i)$ and*

$f \in \Gamma BV$, *i.e.*,

$$\Lambda BV_c = \bigcup_{\Gamma = o(\Lambda)} \Gamma BV.$$

Our next proposition shows that the semi-norm $\| \; \|_\Lambda^0$ introduced in the previous section yields a new description of the class ΛBV_c.

Proposition 5. *Let Λ be a proper Λ-sequence, and let f be regulated. Then* $\lim_{m\to\infty} \|f\|_{\Lambda_{(m)}} = \|f\|_\Lambda^0$.

Proof. Given a positive number δ, a positive integer m and a family $\mathcal{I} = (I_i)_{i=1}^n$, we get

$$\sum_{i=1}^n \frac{|f(I_i)|}{\lambda_{m+i}} = \sum_{i:|f(I_i)|>\delta} \cdots + \sum_{i:|f(I_i)|<\delta} \cdots ,$$

and hence

$$V_{\Lambda_{(m)}}(f) \le K(f,\delta)\frac{\omega_f}{\lambda_{m+1}} + V_\Lambda^\delta(f),$$

where $K(f,\delta)$ denotes the least upper bound on the number of intervals in families consisting entirely of intervals I with $|f(I)| > \delta$, see [**11**, Lemma 2.1], and ω_f denotes the oscillation of f. Passing to limits as $m \to +\infty$ yields $\lim_{m\to\infty} V_{\Lambda_{(m)}}(f) \le V_\Lambda^\delta(f)$, because Λ is a proper Λ-sequence. Since $\delta > 0$ was arbitrary, we get $\lim_{m\to\infty} V_{\Lambda_{(m)}}(f) \le V_\Lambda^0(f)$.

We are now going to prove the opposite inequality. Given a positive number δ, a positive integer m and an f-ordered family $\mathcal{I} = \{I_i : i = 1, \dots, n\}$ such that $\|\mathcal{I}\|_f \le \delta$, we get

$$\sum_{i=1}^n \frac{|f(I_i)|}{\lambda_i} \le m\frac{\delta}{\lambda_1} + \sum_{n\ge i>m} \frac{|f(I_i)|}{\lambda_i}.$$

Thus,

$$V_\Lambda^\delta(f) \le m\frac{\delta}{\lambda_1} + V_{\Lambda_{(m)}}(f)$$

and now passing to limits, first with $\delta \to 0+$ and then with $m \to +\infty$, we conclude that

$$V_\Lambda^0(f) \le \lim_{m\to\infty} V_{\Lambda_{(m)}}(f). \qquad \square$$

By virtue of Proposition 5, a function f is continuous in the Λ-variation if and only if $V_\Lambda^0(f) = 0$. Thus, by Theorem 1, f is continuous in the Λ-variation if and only if $\|f\|_\Lambda^W = |f(0)| + |||f|||_\Lambda$, which leads to the following corollary.

Corollary 1. *$\Lambda BV_c = \Lambda BV^*$ for every proper Λ-sequence Λ.*

This yields a particularly nice characterization of the subspace of continuous functions that are continuous in the Λ-variation.

Theorem 2. *For every proper Λ-sequence Λ the following statements are equivalent*:

(i) $f \in C[0, 1] \cap \Lambda BV_c$;

(ii) f *is continuous and* $W_\Lambda(f) = 0$;

(iii) $W_\Lambda(f) = 0$.

The next proposition shows that the condition $W_\Lambda(f) = 0$ is equivalent to a generalization of absolute continuity.

Proposition 6. *Let Λ be a proper Λ-sequence. Then $f \in C[0, 1] \cap \Lambda BV_c$ if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\sigma_\Lambda(\mathrm{id}, \mathcal{I}) < \delta$ implies $\sigma_\Lambda(f, \mathcal{I}) < \varepsilon$ for every family $\mathcal{I}$ (where* id *denotes the identity mapping of* $[0, 1]$ *onto itself).*

Indeed the condition says that if $I_1, \dots, I_n$ are nonoverlapping subintervals of $[0, 1]$ such that

$$\sum_{i=1}^{n} \frac{|I_i|}{\lambda_i} < \delta,$$

then we must have

$$\sum_{i=1}^{n} \frac{|f(I_i)|}{\lambda_i} < \varepsilon.$$

For the special constant Λ-sequence $\lambda_i \equiv 1$, this becomes the definition of classical absolute continuity. Note that, for any proper Λ-sequence saying that $\sigma_\Lambda(\mathrm{id}, \mathcal{I})$ is small is actually equivalent to saying that $\|\mathcal{I}\|$ is small.

Proof of Proposition 6. If $f \in C[0, 1] \cap \Lambda BV_c$, then given $\varepsilon > 0$, there is by Theorem 2, a positive number δ_0 such that $\|\mathcal{I}\| \le \delta_0$ implies $\sigma_\Lambda(f, \mathcal{I}) < \varepsilon$. Set $\delta := \delta_0/\lambda_1$. Then the condition $\sigma_\Lambda(\mathrm{id}, \mathcal{I}) < \delta$ implies $\|\mathcal{I}\| < \delta_0$, and hence $\sigma_\Lambda(f, \mathcal{I}) < \varepsilon$.

Now suppose that, given $\varepsilon > 0$, there is a $\delta > 0$ such that for any family $\mathcal{I}$ with $\sigma_\Lambda(\mathrm{id}, \mathcal{I}) < \delta$, the inequality $\sigma_\Lambda(f, \mathcal{I}) < \varepsilon$ holds. Of course, $W_\Lambda(\mathrm{id}) = 0$ because Λ is a proper Λ-sequence. Thus, there is a $\gamma > 0$ such that $\|\mathcal{I}\| \le \gamma$ implies $\sigma_\Lambda(\mathrm{id}, \mathcal{I}) < \delta$ and hence $\sigma_\Lambda(f, \mathcal{I}) < \varepsilon$. Therefore, $V_{\Lambda,\,\eta}(f) \le \varepsilon$ for $\eta \le \gamma$. It follows that $W_\Lambda(f) = 0$, and thus $f \in C[0, 1] \cap \Lambda BV_c$ by Theorem 2. □

The above characterization motivates the following definition.

Definition 9. A function $f : [0, 1] \to \mathbf{R}$ is said to be Λ-absolutely continuous ($f \in \Lambda AC$) if $f \in C[0, 1] \cap \Lambda BV_c$.

Another description of Λ-absolutely continuous functions can be obtained in a manner analogous to one of characterizations of local p-variation given by Love and Young in [**7**, page 29].

Definition 10. Given a partition $\pi = (t_i)_{i=1}^n$ of $[0, 1]$, we define $\mathcal{I}_\pi := \{[t_{i-1}, t_i] : i = 1, \dots, n\}$, $\sigma_\Lambda(f, \pi) := \sigma_\Lambda(f, \mathcal{I}_\pi)$, and

$$V_\Lambda^{(\pi)}(f) := \sup_{\pi \subset \pi'} \sigma_\Lambda(f, \pi').$$

Proposition 7. *Let Λ be a proper Λ-sequence. Then a function f is Λ-absolutely continuous if and only if for every $\varepsilon > 0$ there is a partition π such that $V_\Lambda^{(\pi)}(f) < \varepsilon$.*

Proof. Necessity of the condition follows from Theorem 2 immediately.

On the other hand if, for every $\varepsilon > 0$, there is a partition π such that $V_\Lambda^{(\pi)}(f) < \varepsilon$, then f must be continuous, because

$$V_\Lambda^{(\pi)} \geq \frac{|f(t+) - f(t)|}{\lambda_1} + \frac{|f(t) - f(t-)|}{\lambda_2}$$

for any $t \in [0, 1]$. Given $\varepsilon > 0$, let $\pi_\varepsilon = (t_i)_{i=1}^n$ be a partition such that $V_\Lambda^{(\pi_\varepsilon)}(f) < \varepsilon$. Take $\delta > 0$ such that

$$\text{(2)} \qquad \delta < \min\{t_i - t_{i-1} : i = 1, \ldots, n\} \quad \text{and} \quad \omega_f(\delta) < \frac{\varepsilon}{\sum_{i=1}^{n-1} 1/\lambda_i},$$

where $\omega_f(\delta)$ denotes the modulus of continuity of f. Then, given a partition $\pi := (s_i)_{i=0}^m$ with $\|\mathcal{I}_\pi\| < \delta$, there is a bijection $\beta : \mathbf{N} \to \mathbf{N}$ such that

$$\sigma_\Lambda(f, \pi) = \sum_{i=1}^m \frac{|f(s_i) - f(s_{i-1})|}{\lambda_{\beta(i)}} = \sum_{i \in A} \cdots + \sum_{i \in B} \cdots,$$

where A is the set of all indices i such that none of the points $t_1, \ldots, t_{n-1}$ belongs to (s_{i-1}, s_i) and B is the set of all other indices. Thus,

$$\sigma_\Lambda(f, \pi) \leq V_\Lambda^{(\pi_\varepsilon)}(f) + \sum_{i \in B} \frac{\omega_f(\delta)}{\lambda_{\beta(i)}},$$

and therefore by our assumption about π_ε and, by (2), $\sigma_\Lambda(f, \pi) \leq 2\varepsilon$, which implies $V_{\Lambda, \delta}(f) \leq 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, it follows that $W_\Lambda(f) = 0$, and hence f is Λ-absolutely continuous by Theorem 2. □

Actually, we can introduce a quantity

$$V_\Lambda^*(f) := \inf_\pi V_\Lambda^{(\pi)}(f)$$

as Love and Young did in [**7**, page 29] for the p-variation. Then Proposition 7 says that f is in ΛAC if and only if $V_\Lambda^*(f) = 0$. It is easy to see that $V_\Lambda^*(f) \le W_\Lambda(f)$ for every regulated function f. Equality may not hold. Indeed, if we define

$$Z_f^* := \{\beta_{t,j}^f : \beta_{t,j}^f > 0,\ 0 \le t \le 1,\ j = 1, 2\}$$

(compare with Definition 4 of the set Z_f), then arrange the elements of Z_f^* into a nonincreasing sequence (η_i^{*f}) and set

$$|||f|||_\Lambda^* := \sum_{i=1}^\infty \frac{\eta_i^{*f}}{\lambda_i},$$

then it can be proven in a manner similar to that of the proof of Theorem 1 that

$$V_\Lambda^*(f) = V_\Lambda^0(f) + |||f|||_\Lambda^*.$$

It is worth noticing that for any increasing Λ-sequence Λ the following equivalence holds: $V_\Lambda^*(f) = W_\Lambda(f)$ for a regulated function f if and only if f has no proper internal saltus, that is, $f(t)$ does not lie in the open interval with endpoints $f(t+)$ and $f(t-)$ for any $t \in [0, 1]$. In particular, $V_\Lambda^*(f) = W_\Lambda(f)$ if f is one-sided continuous at every point.

Lemma 9. *Let $\pi = (t_i)_{i=0}^n$ be a partition of $[0, 1]$, and let f be such that $f(t_i) = f(t_1)$ for $i = 2, \dots, n-1$. Then, $V_\Lambda(f) \le 2V_\Lambda^{(\pi)}(f)$ for any Λ-sequence Λ.*

Proof. We assume, as we may, that $f(t_1) = 0$. Given a partition $\pi_1 = (x_i)_{i=0}^m$, let K be the set of integers such that, for $i \in K$, the open interval (x_{i-1}, x_i) contains a point of the partition π, say a point t_{j_i}. Then for any bijection β of positive integers onto themselves, we get

$$\sum_i \frac{|f(x_i) - f(x_{i-1})|}{\lambda_{\beta(i)}} \le \Bigg[\sum_{i\notin K} \frac{|f(x_i) - f(x_{i-1})|}{\lambda_{\beta(i)}} + \sum_{i\in K} \frac{|f(x_i) - f(t_{j_i})|}{\lambda_{\beta(i)}}\Bigg] + \sum_{i\in K} \frac{|f(x_{i-1}) - f(t_{j_i})|}{\lambda_{\beta(i)}} \le 2\, V_\Lambda^{(\pi)}(f). \qquad \square$$

Theorem 3. *ΛBV_c is the $\|\ \|_\Lambda$-closure of the set of all step functions of bounded Λ-variation.*

Proof. By virtue of Corollary 1, what we have said about the class ΛBV^* in the remarks following Theorem 1 is true for the class of functions continuous in Λ-variation. In particular, it contains all step functions and it is a closed subspace of ΛBV.

Given a function $f \in \Lambda BV_c$ and an $\varepsilon > 0$, we will construct a step function ϕ such that $\phi(0) = f(0)$ and $V_\Lambda(f - \phi) < 4\varepsilon$. We only discuss the case of discontinuous f, for the continuous case is much simpler.

By Corollary 1 we choose an $\alpha \in (0, \eta_1^f]$ such that $V_\Lambda^\alpha(f) < \varepsilon$. Let δ be the number $\delta_3(\alpha)$ whose existence is assured by Lemma 5. Now take a partition $\pi = (t_i)_{i=0}^n$ with $t_0 = 0$, $t_n = 1$, $\|\pi\| < \delta$ and containing all points of the set $C(\alpha)$. Next define a step function ϕ by setting

$$\phi(t) := \begin{cases} f((t_{i-1} + t_i)/2) & \text{if } t \in (t_{i-1}, t_i) \\ f(t_i) & \text{if } t = t_i. \end{cases}$$

From Lemma 9, we have

$$V_\Lambda(f - \phi) \le 2\, V_\Lambda^{(\pi)}(f - \phi). \tag{3}$$

Now take any partition $(s_i)_{i=0}^m$ finer than π. Then for any bijection $\beta : \{1, \dots, m\} \to \{1, \dots, m\}$, we get

$$\sum_{i=1}^m \frac{|(f-\phi)(s_{i-1}) - (f-\phi)(s_i)|}{\lambda_{\beta(i)}} = \sum_{(1)} \cdots + \sum_{(2)} \cdots ,$$

where the sum $\sum_{(1)} \cdots$ is taken over all indices i such that at least one of the endpoints of $[s_{i-1}, s_i]$ belongs to π and the sum $\sum_{(2)} \cdots$ is taken over all remaining i's.

If i is an index of the first sum, the functions f and ϕ coincide at at least one endpoint of $[s_{i-1}, s_i]$ and thus $|(f-\phi)(s_{i-1}) - (f-\phi)(s_i)|$ is equal either to 0 or to $\big|f(s_i) - f\big((t_{j+1} + t_j)/2\big)\big|$ if $s_{i-1} = t_j$ and $s_i < t_{j+1}$ for some j, or is equal to $\big|f(s_{i-1}) - f\big((t_{j-1} + t_j)/2\big)\big|$ if $s_i = t_j$ and $s_{i-1} > t_{j-1}$ for some j. Thus the first sum $\sum_{(1)} \dots$ does not exceed the variational sum $\sigma_\Lambda(f, \mathcal{I})$ for some family $\mathcal{I}$ with $\|\mathcal{I}\| < (\delta/2)$ such that no interval of $\mathcal{I}$ touches the set $C(\alpha)$. Thus, by Lemmas 4 and 5,

$$\sum_{(1)} \cdots \le V_\Lambda^\alpha(f) < \varepsilon.$$

For the indices i of the second sum, $\sum_{(2)} \cdots$, we have $\phi(s_{i-1}) = \phi(s_i)$, and thus the second sum does not exceed the variational sum $\sigma_\Lambda(f, \mathcal{I})$ for some family $\mathcal{I}$ with $\|\mathcal{I}\| < \delta$ such that no interval of $\mathcal{I}$ touches the set $C(\alpha)$. Thus, as previously,

$$\sum_{(2)} \cdots \leq V_\Lambda^\alpha(f) < \varepsilon.$$

Therefore, by (3), $V_\Lambda(f - \phi) < 4\varepsilon$, which completes the proof. $\square$

It is also possible to characterize Λ-absolute continuity in terms of translates of a function, generalizing the classical result of Plessner, Wiener and Young [**13**, Theorem III.11.2]. Our proof, though, closely follows the idea of Love [**6**].

Theorem 4. *Let Λ be a proper Λ-sequence, and let f be a measurable function. Then f is Λ-absolutely continuous if and only if $V_\Lambda(f_h - f, [0, 1-h]) \to 0$ as $h \to 0+$ where $f_h(x) := f(x+h)$.*

Proof of necessity. If f is Λ-absolutely continuous, then given an $\varepsilon > 0$, there is by Proposition 7, a partition $\pi = (t_i)_{i=0}^n$ of $[0, 1]$ with $V_\Lambda^{(\pi)}(f) < \varepsilon$. Define a step function g by $g(t) = f(0)$ for $t \leq 0$, $g(t) = f(t_i)$ for $t \in (t_{i-1}, t_i]$, $i = 1, \dots, n$. Clearly, $V_\Lambda^{(\pi)}(f) \geq V_\Lambda^{(\pi)}(g)$, and thus by Lemma 9,

$$V_\Lambda(f-g) \leq 2\, V_\Lambda^{(\pi)}(f-g) \leq 2V_\Lambda^{(\pi)}(f) + 2V_\Lambda^{(\pi)}(-g) \leq 4V_\Lambda^{(\pi)}(f) < 4\varepsilon.$$

For $h \in (0, \inf_{1\leq i\leq n}(t_i - t_{i-1}))$, we have $g(t) = g(t_i)$ for $t \in [t_i - h, t_i]$, $i = 1, \dots, n$, and hence $V_\Lambda^{(\pi)}(g) = V_\Lambda^{(\pi)}(g_{-h})$. This implies, again by Lemma 9,

$$V_\Lambda(g - g_{-h}; [h, 1]) \leq 2V_\Lambda^{(\pi)}(g - g_{-h}) \leq 2\big(V_\Lambda^{(\pi)}(g) + V_\Lambda^{(\pi)}(g_{-h})\big) < 4\varepsilon$$

for $0 < h < \|\mathcal{I}_\pi\|$. Finally,

$$\begin{aligned} V_\Lambda(f_h - f, [0, 1-h]) & \\ &\leq V_\Lambda(f_h - g_h, [0, 1-h]) + V_\Lambda(g_h - g, [0, 1-h]) \\ &\quad + V_\Lambda(g - f, [0, 1-h]) \\ &\leq V_\Lambda(f - g, [h, 1]) + V_\Lambda(g - g_{-h}, [h, 1]) \\ &\quad + V_\Lambda(g - f) < 4\varepsilon + 4\varepsilon + 4\varepsilon. \qquad \square \end{aligned}$$

We are now starting preparations for the proof of the sufficiency part in Theorem 4. Since the technique we are going to use rests on approximation by Fejér polynomials and on their convergence properties, until the end of the proof of Lemma 14, we will assume that f is a 2π-periodic function defined on the whole real line.

Lemma 10. *If*

$$V_\Lambda(f_h - f, [-\pi, \pi]) \longrightarrow 0 \textit{ with } h \to 0+,$$

then $V_\Lambda(f_h - f, [-2\pi, 2\pi])$ *is finite for* $h \in [-\pi, \pi]$ *and tends to* 0 *with* $h \to 0$.

Lemma 11. *If* (4) *holds, then* $v(t) := V_\Lambda(\phi(\cdot, t), [-\pi, \pi])$ *is finite and continuous as a function of* $t \in [0, \pi]$, *where* $\phi(x, t) := [f(x+t) + f(x-t)]/2 - f(x)$ *for* $x \in [\pi, \pi]$.

Proofs of both of these lemmas are virtually identical to those of Lemmas 6 and 7 of [**6**] and, therefore, are omitted.

Lemma 12. *If* f *is integrable and* σ_n *is its nth Fejér polynomial, then the following statements are equivalent*:

$$V_\Lambda(f_h - f, [-\pi, \pi]) \longrightarrow 0 \text{ as } h \to 0+; \tag{5}$$

$$V_\Lambda(\sigma_n - f, [-\pi, \pi]) \longrightarrow 0 \text{ as } n \to \infty. \tag{6}$$

Proof. Assume (6). The Fejér polynomials are absolutely continuous and therefore Λ-absolutely continuous by Theorem 1 of [**17**], or by Proposition 8 from the third section of this paper. Thus, f is a continuous function of bounded Λ-variation and so is f_h, for f is periodic. Hence, by the already proven necessity part of Theorem 4 and by [**20**, Theorem 3],

$$\begin{aligned} V_\Lambda(f_h - f, [-\pi, \pi]) \le V_\Lambda(f_h - f, [-\pi, \pi - h]) \\ + V_\Lambda(f_h - f, [\pi - h, \pi]) \longrightarrow 0 \end{aligned}$$

as $h \to 0+$, which proves (5).

We are now going to prove that (5) implies (6). Let $\phi(x, t)$ be the function defined in Lemma 11. Then

$$(\sigma_n - f)(x) = \frac{2}{\pi}\int_0^\pi \phi(x, t)K_n(t)\,dt$$

where K_n is the nth Fejér kernel [**2**, page 134],

$$K_n(t) = \frac{1}{2(n+1)}\left(\frac{\sin((n+1)/t)2}{\sin(t/2)}\right)^2.$$

For a partition $\pi = (s_i)_{i=1}^m$ of $[-\pi, \pi]$ and for any bijection β of $\{1, \dots, m\}$ onto itself, we get

$$\sum_{i=1}^m \frac{|(\sigma_n - f)(s_i) - (\sigma_n - f)(s_{i-1})|}{\lambda_{\beta(i)}} \leq \frac{2}{\pi}\int_0^\pi \sum_{i=1}^m \frac{|\phi(s_i, t) - \phi(s_{i-1}, t)|}{\lambda_{\beta(i)}} K_n(t)\,dt.$$

Hence,

$$V_\Lambda(\sigma_n - f, [-\pi, \pi]) \leq \frac{2}{\pi}\int_0^\pi v(t)K_n(t)\,dt,$$

the integral being not only finite by Lemma 11, but also converging to $v(0)$ as $n \to \infty$ by Fejér's theorem, since v is continuous. This establishes (6). □

Lemma 13. *If f is measurable and $V_\Lambda(f_h - f, [0, \pi - h]) \to 0$ as $h \to 0+$, then f is continuous on $[0, \pi]$, and the set*

$$\{V_\Lambda(f_h - f, [0, \pi - h]) : h \in (0, \pi)\}$$

is bounded.

Proof. Let m be a positive integer such that $V_\Lambda(f_\theta - f, [0, \pi - \theta]) < 1$ for $\theta \in (0, (\pi/m))$. Given an $h \in (0, \pi)$, let $h_\nu = (\nu h)/m$. Then

$$\begin{aligned} V_\Lambda(f_h - f, [0, \pi - h]) &\leq \sum_{\nu=1}^m V_\Lambda(f_{h_\nu} - f_{h_{\nu-1}}, [0, \pi - h]) \\ &\leq \sum_{\nu=1}^m V_\Lambda\left(f_{h/m} - f, \left[0, \pi - \frac{h}{m}\right]\right) < m. \end{aligned}$$

We are now going to show that the measurable function f is, in fact, regulated. We will do this along the lines of Ursell's proof of Lemma 1 from [**16**, page 411–412]. Suppose that f is not regulated. Then there are a positive number k and a point $t_0 \in [0, \pi]$ such that

$$\lim_{t\to t_0^+} \omega_f(t_0, t] > 2k \quad \text{or} \quad \lim_{t\to t_0^-} \omega_f[t, t_0) > 2k,$$

where $\omega_f I$ denotes the oscillation of f on an interval I. We restrict our attention to the second case, because the first one is analogous.

We have then $t_0 \in (0, \pi]$. Fix a positive integer N, and let ε be a positive number such that

$$\varepsilon < \frac{k}{6} \tag{7}$$

and

$$\varepsilon < \frac{t_0}{N(2N+1)}. \tag{8}$$

Further, let ϕ be a continuous function such that $|f(t) - \phi(t)| < \varepsilon$ on $[0, \pi]$ except in a set of measure at most ε. Let δ be a positive number such that $|\phi(t) - \phi(s)| < \varepsilon$ if $|t - s| < \delta$, t, $s \in [0, \pi]$. We can choose δ arbitrarily small, and we shall suppose it so chosen that $\delta < \varepsilon$.

Next, we pick an increasing sequence (a_j) of points of $(t_0 - \delta, t_0)$ convergent to t_0 and such that

$$|f(a_{2n} - f(a_{2n-1})| > \tfrac{3}{2}k$$

for all $n \in \mathbf{N}$. Then, for any positive integer n, one has

$$\begin{aligned}
|f(a_2n - h) - f(a_{2n-1} - h)| \\
&\le |f(a_{2n} - h) - \phi(a_{2n} - h)| + |\phi(a_{2n} - h) - \phi(a_{2n-1} - h)| \\
&\quad + |\phi(a_{2n-1} - h) - f(a_{2n-1} - h)| \\
&< 3\varepsilon
\end{aligned}$$

for all $h \in [0, t_0 - \varepsilon]$ except for a set E_n of measure at most 2ε. Thus, one has

$$|f(a_{2n} - h) - f(a_{2n-1} - h)| < 3\varepsilon \text{ for } n = 1, 2, \ldots, N, \tag{9}$$

for all $h \in [0,\, t_0 - \varepsilon]$ except for a set $E = \cup_{n=1}^N E_n$ of measure at most $2N\varepsilon$.

The measure of all $h \in [0,\, t_0]$ for which (9) holds is at least

$$t_0 - \varepsilon - 2N\varepsilon \overset{(8)}{>} t_0\left(1 - \frac{1}{N}\right).$$

Therefore, there is a number $h_N \in (0,\, (t_0)/N]$ such that (9) holds for h_N. Since

$$\begin{aligned} V_\Lambda(f_{h_N} - f,\, [0,\, \pi - h_N]) &> \sigma_\Lambda(f_{h_N} - f,\, \{[a_{2n-1} - h_N,\, a_{2n} - h_N]\}_{n=1}^N) \\ &\geq \sum_{n=1}^N \frac{|f(a_{2n}) - f(a_{2n} - h_N) - f(a_{2n-1}) + f(a_{2n-1} - h_N)|}{\lambda_n} \\ &\geq \sum_{n=1}^N \frac{|f(a_{2n}) - f(a_{2n-1})| - |f(a_{2n} - h_N) - f(a_{2n-1} - h_N)|}{\lambda_n} \\ &\geq \sum_{n=1}^N \frac{(3/2)k - 3\varepsilon}{\lambda_n} \overset{(7)}{>} k \sum_{n=1}^N \frac{1}{\lambda_n}, \end{aligned}$$

we have

$$\limsup_{h \to 0^+} V_\Lambda(f_h - f,\, [0,\, \pi - h]) = +\infty,$$

a contradiction. Thus, f cannot have a discontinuity of the second kind which means that f is regulated.

Continuity of f can be now proven just as in Love's [**6**, Lemma 8]. □

Lemma 14. *If f is continuous and $V_\Lambda(f_h - f,\, [0,\, \pi - h]) \to 0$ as $h \to 0+$, then both $V_\Lambda(f, [0,\, h])$ and $V_\Lambda(f,\, [\pi - h,\, \pi])$ tend to 0 as $h \to 0+$.*

Proof. It suffices to prove that $V_\Lambda(f,\, [0,\, h])$ and $V_\Lambda(f,\, [\pi - h, \pi])$ are finite for some $h > 0$ [**20**, Theorem 3].

Suppose that $V_\Lambda(f,\, [0,\, h]) = +\infty$. Then there is a strictly decreasing sequence (x_k) with $x_0 = \pi/2$ such that

$$\sum_{k=1}^{\infty} \frac{|f(x_k) - f(x_{k-1})|}{\lambda_{\beta(k)}} = \infty \tag{10}$$

for some bijection β of positive integers onto themselves. Observe that

$$N := \sum_{k=1}^{\infty} \frac{x_{k-1} - x_k}{\lambda_{\beta(k)}} < \infty.$$

Now, by Lemma 13, for $t \in [0, \pi/2]$,

$$\sum_{k=1}^{n} \frac{|f(x_k + t) - f(x_k) - f(x_{k-1} + t) + f(x_{k-1})|}{\lambda_{\beta(k)}} \leq V_\Lambda\left(f_t - f, \left[0, \frac{\pi}{2}\right]\right) < m$$

for some constant m and for every n. Thus,

$$\sum_{k=1}^{n} \left| \int_0^{\pi/2} \frac{(f(x_k + t) - f(x_k)) - (f(x_{k-1} + t) - f(x_{k-1}))}{\lambda_{\beta(k)}} \, dt \right| \leq \frac{\pi m}{2}.$$

Next, letting $M := \sup |f|$, we get

$$\sum_{k=1}^{n} \frac{\left| \int_0^{\pi/2} (f(x_k + t) - f(x_{k-1} + t)) \, dt \right|}{\lambda_{\beta(k)}} \leq 2M \sum_{k=1}^{n} \frac{x_{k-1} - x_k}{\lambda_{\beta(k)}} < 2MN,$$

and finally,

$$\begin{aligned} \sum_{k=1}^{n} \frac{|f(x_k) - f(x_{k-1})|}{\lambda_{\beta(k)}} &\leq \sum_{k=1}^{n} \frac{\left| \int_0^{\pi/2} (f(x_k) - f(x_{k-1})) - (f(x_k + t) - f(x_{k-1} + t)) \, dt \right|}{\lambda_{\beta(k)}} \\ &\quad + \sum_{k=1}^{n} \frac{\left| \int_0^{\frac{\pi}{2}} (f(x_k + t) - f(x_{k-1} + t)) \, dt \right|}{\lambda_{\beta(k)}} \\ &\leq \frac{\pi m}{2} + 2MN, \end{aligned}$$

which contradicts (10). □

Proof of Theorem 4. *Sufficiency.* We are going to prove that if f is defined on $[0, \pi]$ and $V_\Lambda(f_h - f, [0, \pi - h]) \to 0$, then f is Λ-absolutely continuous. Given a function f with the first two above properties, we extend it to the whole real line as an even function of period 2π. Suppose $0 < h < \pi$. Then

$$V_\Lambda(f_h - f, [-h, 0]) \leq V_\Lambda(f_h, [-h, 0]) + V_\Lambda(f, [-h, 0]) = 2V_\Lambda(f, [0, h]).$$

Similarly, $V_\Lambda(f_h - f, [\pi - h, \pi]) \leq 2V_\Lambda(f, [\pi - h, \pi])$. Also, $V_\Lambda(f_h - f, [-\pi, -h]) = V_\Lambda(f_h - f, [0, \pi - h])$. Thus,

$$\begin{aligned} &V_\Lambda(f_h - f, [-\pi, \pi]) \\ &\quad \leq V_\Lambda(f_h - f, [-\pi, -h]) + V_\Lambda(f_h - f, [-h, 0]) \\ &\qquad + V_\Lambda(f_h - f, [0, \pi - h]) + V_\Lambda(f_h - f, [\pi - h, \pi]) \\ &\quad \leq 2V_\Lambda(f_h - f, [0, \pi - h]) + 2V_\Lambda(f, [\pi - h, \pi]) + 2V_\Lambda(f, [0, h]), \end{aligned}$$

which tends to 0 as $h \to 0+$ by our hypothesis and by Lemmas 13 and 14. Hence, by Lemma 12, $V_\Lambda(f - \sigma_n, [-\pi, \pi]) \to 0$, which proves that $f \in \Lambda AC$ on $[-\pi, \pi]$, and thus on $[0, \pi]$. □

In light of Theorem 4, Lemma 12 becomes a statement about approximating Λ-absolutely continuous functions by their Fejér polynomials. Namely, if f is a measurable 2π-periodic function, then the following statements are equivalent:

(i) $f \in \Lambda AC[\pi, \pi]$;

(ii) $\|\sigma_n - f\|_\Lambda \to 0$ as $n \to \infty$.

3. Inclusions and representations. We are now going to characterize inclusions among classes ΛBV, ΛBV_c and ΛAC for different Λ-sequences. The function h described in the next lemma is the tool for handling these issues.

Lemma 15. *Let* $(u_n)_{n=1}^\infty$ *be a nonincreasing sequence of positive numbers converging to* 0, *and let a function* $h : [0, 1] \to \mathbf{R}$ *be defined by the conditions*:

(a) $h(0) = 0$ *and* $h(2^{1-k}) := \sum_{i=k}^\infty (-1)^{i+1} u_i$ *for all* $k \in \mathbf{N}$;

(b) *h is linear on each closed interval of the form $[2^{-k},\, 2^{1-k}]$ for all $k \in \mathbf{N}$.*

Then for any Λ-sequence Λ one has $V_\Lambda(h) = \sum(u_i/\lambda_i)$. Moreover, for any proper Λ-sequence Λ, $h \in \Lambda BV$ if and only if $h \in \Lambda AC$.

Proof. The set K_h of points of varying monotonicity of the function h is $K_h = \{0\} \cup \{2^{-i} : i \in \mathbf{N}\}$. Given any h-ordered K_h-family $\mathcal{I} = \{I_1, \dots , I_k\}$, one has

$$|h(I_i)| \leq |h([2^{-i},\, 2^{-i+1}])| \quad \text{for } i = 1, 2, \dots , k.$$

Thus,

$$\sigma_\Lambda(h,\, \mathcal{I}) \leq \sum_{i=1}^{k} \frac{u_i}{\lambda_i}.$$

Hence, the Λ-variation of the function h on the set K_h satisfies the inequality

$$V_\Lambda(h,\, K_h) \leq \sum_{i=1}^{\infty} \frac{u_i}{\lambda_i}.$$

Since the opposite inequality is obvious, one has

$$V_\Lambda(h,\, K_h) = \sum_{i=1}^{\infty} \frac{u_i}{\lambda_i},$$

and it follows from [**12**, Corollary 1.5] that

$$V_\Lambda(h) = \sum_{i=1}^{\infty} \frac{u_i}{\lambda_i}.$$

The function h is continuous and, hence, if $V_\Lambda(h) < +\infty$, then $h \in \Lambda AC$ because

$$\lim_{m\to\infty} V_{\Lambda_{(m)}}(h) = \lim_{m\to\infty} \sum_{i=1}^{\infty} \frac{u_i}{\lambda_{i+m}} = 0. \qquad \square$$

The following proposition adds four more equivalent statements to the fundamental Theorem 3 from [**10**].

Proposition 8. *Let* $\Lambda = (\lambda_i)$ *and* $\Gamma = (\gamma_i)$ *be* λ*-sequences. Then the following statements are equivalent*:

(i) $\Lambda BV \subset \Gamma BV$;

(ii) $\Lambda BV_c \subset \Gamma BV$;

(iii) $\Lambda BV_c \subset \Gamma BV_c$;

(iv) $\Lambda AC \subset \Gamma AC$;

(v) $\Lambda AC \subset \Gamma BV$;

(vi) $\sum_{i=1}^{n}(1/\gamma_i) = O\left(\sum_{i=1}^{n}(1/\lambda_i)\right)$ *as* $n \to \infty$.

Proof. In the case when both Λ and Γ are proper Λ-sequences, the equivalence of (i) and (vi) has been established in [**10**, Theorem 3]. It has also been shown in the course of the proof of Theorem 3 that condition (vi) implies $V_\Gamma^\delta = O(V_\Lambda^\delta)$ as $\delta \to 0^+$. Thus, condition (vi) implies (iii) by Corollary 1 and by the equivalence : $f \in \Lambda BV^* \Leftrightarrow V_\Lambda^0(f) = 0$. The implications (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) are straightforward for proper Λ-sequences.

If (vi) does not hold, then there is a nonincreasing sequence (u_i) of positive numbers converging to 0 such that $\sum(u_i/\lambda_i) < +\infty$ and $\sum(u_i/\gamma_i) = +\infty$ (as shown in the proof of Theorem 3 of [**10**]). Thus, by our Lemma 15 there is a function $h \in \Lambda AC \setminus \Gamma BV$, that is, (v) does not hold. Therefore (v) $\Rightarrow$ (vi).

The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (v) are obvious.

The easy proof in the remaining case when at least one of the Λ-sequences is improper has been omitted. □

It is known that the union of all ΛBV classes yields the class of all regulated functions [**9**, Theorem 10]. However, the classes ΛBV are relatively small compared to the class of regulated functions, because for every countable collection of classes ΛBV there is always a regulated function that does not belong to any class from that collection [**9**, Theorem 12]. The mutual relationship between ΛBV classes is rather chaotic. It is easy to see that for every proper Λ-sequence Λ there is a proper Λ-sequence Γ such that ΓBV is a proper subset of ΛBV and that for every Λ-sequence Λ there is a proper Λ-sequence Γ such that ΓBV contains ΛBV properly. On the other hand, $\Lambda BV = BV$ for any improper Λ-sequence Λ, and it is the smallest ΛBV class [**9**, Theorem 5]. Moreover, it is not difficult to show that for any proper Λ-sequence Λ, there exists a proper Λ-sequence Γ such that neither $\Lambda BV \subseteq \Gamma BV$ nor $\Gamma BV \subseteq \Lambda BV$.

Nevertheless, the number of nice relationships between classes ΛBV, ΛBV_c and ΛAC is not small. We start with a description of intersection of finite families of classes ΛBV_c.

Lemma 16. *Let* $\Lambda^{(k)} = (\lambda_i^k)_{i=1}^{\infty}$, $k = 1, \dots, n$, *be* Λ*-sequences. Then*

$$\bigcap_{k=1}^{n} \Lambda^{(k)} BV_c = \Lambda BV_c,$$

where the Λ*-sequence* $\Lambda = (\lambda_i)$ *is defined by*

$$\frac{1}{\lambda_i} = \sum_{k=1}^{n} \frac{1}{\lambda_i^k} \quad \text{for all } i \in \mathbf{N}.$$

Also,

$$\bigcap_{k=1}^{n} \Lambda^{(k)} BV_c = \Gamma BV_c,$$

where the Λ*-sequence* $\Gamma = (\gamma_i)$ *is defined recursively by*

$$\sum_{i=1}^{m} \frac{1}{\gamma_i} = \max_{1 \le k \le n} \sum_{i=1}^{m} \frac{1}{\lambda_i^k} \quad \text{for all } m \in \mathbf{N}.$$

Proof. Since for any family $\mathcal{I} = (I_i)_{i=1}^{m}$ and any function f

$$\sum_{i=1}^{m} \frac{|f(I_i)|}{\lambda_i} = \sum_{k=1}^{n} \sum_{i=1}^{m} \frac{|f(I_i)|}{\lambda_i^k},$$

it follows that for any $\delta > 0$ and any $l \in \{1, \dots, n\}$,

$$V_{\Lambda^{(l)}}^{\delta}(f) \le V_{\Lambda}^{\delta}(f) \le \sum_{k=1}^{n} V_{\Lambda^{(k)}}^{\delta}(f).$$

Thus, $V^0_\Lambda(f) = 0$ if and only if $V^0_{\Lambda^{(k)}}(f) = 0$ for all $k \in \{1, \dots, n\}$, and hence, by Corollary 1,

$$\bigcap_{k=1}^{n} \Lambda^{(k)} BV_c = \Lambda BV_c.$$

The second conclusion in the lemma follows from the equality $\Lambda BV_c = \Gamma BV_c$, which in turn follows from Proposition 8, because the Λ-sequences Λ and Γ are equivalent. Indeed,

$$\sum_{i=1}^{m} \frac{1}{\gamma_i} \le \sum_{i=1}^{m} \frac{1}{\lambda_i} \le n \sum_{i=1}^{m} \frac{1}{\gamma_i}$$

for every $m \in \mathbf{N}$. □

Thus, the intersection of every finite collection of ΛBV_c classes always is a ΛBV_c class. An analogous result for ΛBV classes was proven by Perlman [**9**, Theorem 6]. On the other hand, every ΛBV class can be represented as an intersection of a suitable family of much larger ΛBV classes, as the next proposition shows.

Proposition 9. *For any Λ-sequence $\Lambda = (\lambda_i)$*

$$\Lambda BV = \bigcap_{\Lambda = o(\Gamma)} \Gamma BV,$$

where the intersection is taken over all Λ-sequences $\Gamma = (\gamma_i)$ such that $\lambda_i = o(\gamma_i)$.

Proof. Since $\lambda_i = o(\gamma_i)$ implies that $\sum_1^n (1/\gamma_i) = o(\sum_1^n (1/\lambda_i))$, Proposition 8 implies that $\Lambda BV \subset \Gamma BV$ whenever $\Lambda = o(\Gamma)$.

Thus, it remains to show that $\cap_{\Lambda=o(\Gamma)} \Gamma BV \subset \Lambda BV$. Suppose that a function f is not of bounded Λ-variation. There exists a family $\{I_i\}$ such that $\sum |f(I_i)|/\lambda_i = +\infty$. Set $n_0 := 0$, and let (n_k) be an increasing sequence of positive integers such that

$$\sum_{i=1+n_{k-1}}^{n_k} \frac{|f(I_i)|}{\lambda_i} \ge 2^k \quad \text{and} \quad \sum_{i=1+n_{k-1}}^{n_k} \frac{1}{\lambda_i} \ge 2^k \quad \text{for } k \in \mathbf{N}.$$

Now define a Λ-sequence $\Gamma = (\gamma_i)$ by $\gamma_i = 2^k \lambda_i$ for $i = n_{k-1}+1, \dots, n_k$, $k \in \mathbf{N}$. Then $\Lambda = o(\Gamma)$ and

$$\sum_{i=1}^{\infty} \frac{|f(I_i)|}{\gamma_i} = \sum_{k=1}^{\infty} \left(\frac{1}{2^k} \sum_{i=1+n_{k-1}}^{n_k} \frac{|f(I_i)|}{\lambda_i} \right) \geq \sum_{k=1}^{\infty} 1 = +\infty.$$

Therefore, $f \notin \cap_{\Lambda = o(\Gamma)} \Gamma BV$. □

The above proposition generalizes the well-known result that the class of all functions of bounded variation is the intersection of all ΛBV classes with Λ's being proper Λ-sequences [**9**, Theorem 5]. Further, since $\Lambda BV \subset \Gamma BV_c$ for any Λ-sequences satisfying $\Lambda = o(\Gamma)$ [**17**, Theorem 1], the following corollary is trivial.

Corollary 2. *For any Λ-sequence Λ,*

$$\Lambda BV = \bigcap_{\Lambda = o(\Gamma)} \Gamma BV_c.$$

However, the behavior of intersections of countable collections of ΛBV_c classes is somewhat surprising. The following proposition is related to Theorem 1 from [**3**].

Proposition 10. *Let $\Lambda = (\lambda_i)$ and $\Lambda^{(n)} = (\lambda_i^n)_{i \in \mathbf{N}}$, $n = 1, 2, \dots$, be Λ-sequences. Then the following statements are equivalent*:

(i) $\cap_{n=1}^{\infty} \Lambda^{(n)} BV_c \subset \Lambda BV$;

(ii) $\cap_{n=1}^{N} \Lambda^{(n)} BV_c \subset \Lambda BV$ *for some positive integer* N;

(iii) $\sum_{i=1}^{k} (1/\lambda_i) = O(\max_{1 \leq n \leq N} \sum_{i=1}^{k} (1/\lambda_i^n))$ *for some positive integer N as $k \to \infty$.*

Proof. The equivalence of conditions (ii) and (iii) is an immediate consequence of Proposition 8 and Lemma 16. Since the implication (ii) $\Rightarrow$ (i) is obvious, the only part of Proposition 10 that requires hard work is the implication (i) $\Rightarrow$ (ii).

We are going to prove it by contraposition. Thus, we assume that the sets $\cap_{n=1}^{N} \Lambda^{(n)} BV_c \setminus \Lambda BV$ are nonempty for all positive integers N.

We will construct a function $f \in \cap_{n=1}^{\infty} \Lambda^{(n)} BV_c \setminus \Lambda BV$. In light of Lemma 16, we may assume that $\Lambda^{(n+1)} BV_c \subset \Lambda^{(n)} BV_c$ for all n. Even more, defining, if necessary,

$$\frac{1}{\tilde{\lambda}_i^n} := \sum_{j=1}^{n} \frac{1}{\lambda_i^j}$$

for i, $n \in \mathbf{N}$, we can require that

$$\lambda_i^n \geq \lambda_i^{n+1} \text{ for all } i,\, n \in \mathbf{N}. \tag{11}$$

Our assumption implies that $\Lambda^{(n)} BV_c \setminus \Lambda BV \neq \varnothing$ for every positive integer n. Thus, condition (vi) of Proposition 8 does not hold which in turn implies that there is a nonincreasing sequence $(u_i^n)_{i \in \mathbf{N}}$ of positive numbers converging to 0 and such that

$$\sum_{i=1}^{\infty} \frac{u_i^n}{\lambda_i^n} < +\infty \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{u_i^n}{\lambda_i} = +\infty.$$

Actually, given arbitrary positive numbers α_n and β_n, we may assume without loss of generality that

$$\sum_i \frac{u_i^n}{\lambda_i^n} \leq \alpha_n \quad \text{and} \quad u_1^n \leq \beta_n.$$

Now we are going to define inductively a special nondecreasing sequence $(u_i)_{i \in \mathbf{N}}$ of positive numbers converging to 0. We start with a sequence $(u_i^1)_{i \in \mathbf{N}}$ such that

$$\sum_{i=1}^{\infty} \frac{u_i^1}{\lambda_i^1} \leq 1 \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{u_i^1}{\lambda_i} = +\infty.$$

Choose an index k_1 such that $\sum_{i=1}^{k_1} (u_i^1/\lambda_i) \geq 1$, and define $u_i := u_i^1$ for $1 \leq i \leq k_1$.

Suppose now that n is a positive integer such that an index k_n and numbers $(u_i)_{i=1}^{k_n}$ have been defined and that they satisfy the following properties:

$$\sum_{i=1}^{k_n} \frac{u_i}{\lambda_i^n} \leq \sum_{i=1}^{n} \frac{1}{i^2} \quad \text{and} \quad \sum_{i=1}^{k_n} \frac{u_i}{\lambda_i} \geq n.$$

Taking a sequence $(u_i^{n+1})_{i\in\mathbf{N}}$ such that

$$\sum_{i=1}^{\infty} \frac{u_i^{n+1}}{\lambda_i^{n+1}} \leq \frac{1}{(n+1)^2} \quad \text{and} \quad u_1^{n+1} < \frac{u_{k_n}}{2},$$

and

$$\sum_{i=1}^{\infty} \frac{u_i^{n+1}}{\lambda_i} = +\infty,$$

we choose a positive integer $k_{n+1} > k_n$ so that

$$\sum_{i=1+k_n}^{k_{n+1}} \frac{u_i^{n+1}}{\lambda_i} \geq 1.$$

Then define $u_i := u_i^{n+1}$ for $k_n < i \leq k_{n+1}$. Hence,

$$\sum_{i=1}^{k_{n+1}} \frac{u_i}{\lambda_i} \geq n+1. \tag{12}$$

In this way we define inductively a nonincreasing sequence (u_i) of positive numbers converging to 0 such that $\sum_i u_i/\lambda_i = +\infty$, because of (12), and such that

$$\sum_{i=1}^{\infty} \frac{u_i}{\lambda_i^n} < +\infty \quad \text{for every positive integer } n,$$

since

$$\begin{aligned} \sum_{i=1}^{\infty} \frac{u_i}{\lambda_i^n} &= \sum_{i=1}^{k_{n-1}} \frac{u_i}{\lambda_i^n} + \sum_{s=n}^{\infty} \sum_{i=1+k_{s-1}}^{k_s} \frac{u_i^{(s)}}{\lambda_i^n} \\ &\overset{(11)}{\leq} \sum_{i=1}^{k_{n-1}} \frac{u_i}{\lambda_i^n} + \sum_{s=n}^{\infty} \sum_{i=1+k_{s-1}}^{k_s} \frac{u_i^{(s)}}{\lambda_i^s} \leq \sum_{i=1}^{k_{n-1}} \frac{u_i}{\lambda_i^n} + \sum_{s=n}^{\infty} \frac{1}{s^2} < +\infty. \end{aligned}$$

Thus, by Lemma 15, there is a function $h \in \cap\Lambda^{(n)}BV_c \setminus \Lambda BV$. □

The above proposition shows that Proposition 9 and Corollary 2 are, in a sense, best possible. Namely, they cannot be improved by taking countable intersections.

Corollary 3. *Let* $\Lambda = (\lambda_i)$ *and* $\Lambda^{(n)} = (\lambda_i^n)_{i\in\mathbf{N}}$, $i = 1, 2\dots,$ *be* Λ*-sequences such that* $\Lambda = o(\Lambda^{(n)})$ *for all* n*. Then the inclusions*

$$\Lambda BV \subset \bigcap_{n=1}^{\infty} \Lambda^{(n)} BV \quad \textit{and} \quad \Lambda BV \subset \bigcap_{n=1}^{\infty} \Lambda^{(n)} BV_c$$

are proper.

We will now discuss some analogous results for classes of Λ-absolutely continuous functions. The next fact is a simple consequence of Corollaries 2 and 3.

Proposition 11. *For any* Λ*-sequence* Λ*,*

$$C\Lambda BV = \bigcap_{\Lambda = o(\Gamma)} \Gamma AC,$$

and this equality cannot be achieved by an intersection of any countable subfamily.

Perlman and Waterman have shown in the course of the proof of Theorem 3 from [**10**] that, if there is a constant $C > 0$ such that

$$\sum_{k=1}^{n} \frac{1}{\lambda_k} < C \sum_{k=1}^{n} \frac{1}{\gamma_k} \quad \text{for all } n,$$

then, given any nonincreasing sequence (a_k) of nonnegative numbers, we get

$$\sum_{k=1}^{n} \frac{a_k}{\lambda_k} \le C \sum_{k=1}^{n} \frac{a_k}{\gamma_k}.$$

It follows that $\|f\|_\Lambda \le (C+1)\|f\|_\Gamma$ for any $f \in \Gamma BV$. Thus, if $\Gamma AC \subsetneq \Lambda AC$, then the inclusion $A : \Gamma AC \to \Lambda AC$ is a continuous linear operator between Banach spaces, and hence $A(\Gamma AC) = \Gamma AC$ is σ-strongly porous in ΛAC by the Banach-Steinhaus-Olevskii theorem [**8**], since we have assumed that $\Gamma AC \neq \Lambda AC$. In particular, since no Banach space is σ-strongly porous in itself, we obtain the following equivalent conditions:

Proposition 12. *Let* $\Lambda = (\lambda_i)$ *and* $\Lambda^{(n)} = (\lambda_i^n)_{i\in\mathbf{N}}$, $n = 1, 2, \dots,$ *be* Λ*-sequences. Then the following statements are equivalent*:

(i) $\Lambda AC \subset \cup_{n=1}^{\infty} \Lambda^{(n)} AC$;

(ii) $\Lambda AC \subset \Lambda^{(N)} AC$ *for some positive integer* N;

(iii) *there is an index* N *such that*

$$\sum_{i=1}^{k} \frac{1}{\lambda_i^N} = O\bigg(\sum_{i=1}^{k} \frac{1}{\lambda_i} \bigg) \quad \text{as } k \to \infty.$$

Analogous equivalences hold for the ΛBV classes [**3**, Theorem 2].

We are now able to improve Wang's Theorem 1 [**17**].

Proposition 13. *Let* Λ *be a proper* Λ*-sequence. Then*

$$\Lambda AC = \bigcup_{\Gamma = o(\Lambda)} \Gamma AC,$$

and this equality cannot be achieved by a union of any countable subfamily.

Proof. If $\gamma_i = o(\lambda_i)$, then $\sum_{i=1}^{k}(1/\lambda_i) = o(\sum_{i=1}^{k}(1/\gamma_i))$, and hence $\Gamma AC \subseteq \Lambda AC$ by Proposition 8, which shows that $\cup_{\Gamma=o(\Lambda)} \Gamma AC \subseteq \Lambda AC$.

On the other hand, given $f \in \Lambda AC$, let (m_k) be an increasing sequence of positive integers such that

$$V_{\Lambda_{(m_k)}}(f) < \frac{V_\Lambda(f)}{2^k}; \tag{13}$$

$$\frac{k+1}{\lambda_{m_{k+1}}} < \frac{k}{\lambda_{m_k}}. \tag{13a}$$

Since, given a positive integer k, one has $(k+1)/\lambda_i \to 0$ as $i \to \infty$, we have $(k+1)/\lambda_i < (k/\lambda_{m_k})$ for all sufficiently large i.

The series $\sum_{i=1}^{\infty}((k+1)/\lambda_i)$ diverges, and hence we may assume additionally that

$$\sum_{i=1+m_k}^{m_{k+1}} \min\left\{\frac{k+1}{\lambda_i}, \frac{k}{\lambda_{m_k}}\right\} > 1. \tag{13b}$$

Now define a Λ-sequence Γ by $\gamma_i = \lambda_i$ for $i = 1, \dots, m_1$, and

$$\frac{1}{\gamma_i} := \min\left\{\frac{k+1}{\lambda_i}, \frac{k}{\lambda_{m_k}}\right\} \quad \text{for } m_k < i \le m_{k+1}, \quad k = 1, 2, \dots . \tag{14}$$

Clearly, $\gamma_i \le \gamma_{i+1}$ if $i \ne m_k$. Further, by (13a),

$$\frac{1}{\gamma_{1+m_k}} \le \frac{k}{\lambda_{m_k}} = \frac{1}{\gamma_{m_k}}.$$

Because of (14), we get $\gamma_i \le 1/(k+1)\,\lambda_i$ for $m_k < i \le m_{k+1}$ which implies that $\Gamma = o(\Lambda)$. Moreover, the condition (13b) implies that $\sum(1/\gamma_i) = +\infty$. Thus, Γ is a Λ-sequence.

Now, given a family $\mathcal{I}$ and a positive integer s, we get

$$\sum_{k=s}^{\infty}\sum_{i=1+m_k}^{m_{k+1}} \frac{|f(I_i)|}{\gamma_i} \le \sum_{k=s}^{\infty}(k+1)\sum_{i=1+m_k}^{m_{k+1}} \frac{|f(I_i)|}{\lambda_i} \le \sum_{k=s}^{\infty}(k+1)V_{\Lambda_{(m_k)}}(f).$$

Thus,

$$V_{\Gamma_{(m_s)}}(f) \le V_\Lambda(f)\sum_{k=s}^{\infty}\frac{k+1}{2^k} \longrightarrow 0$$

as $s \to \infty$, and, since f is continuous, this means that $f \in \Gamma AC$, which completes the proof of the inclusion $\Lambda AC \subseteq \cup_{\Gamma=o(\Lambda)}\Gamma AC$.

That ΛAC cannot be written as a union of any countable family of ΓAC classes with $\Gamma = o(\Lambda)$ follows from Proposition 12. □

Wang has shown that $\Gamma = o(\Lambda)$ implies $\Gamma BV \subset \Lambda BV_c$. Our next proposition yields a slightly more general sufficient condition for the inclusion.

Proposition 14. *If* $\sum_{i=1}^{n}(1/\lambda_i) = o(\sum_{i=1}^{n}(1/\gamma_i))$, *then* $\Gamma BV \subset \Lambda BV_c$.

Proof. Because of Corollary 1, it suffices to show that $f \in \Gamma BV$ implies $V_\Lambda^0(f) = 0$. Given an $\varepsilon > 0$, let N be a positive integer such that

$$\sum_{i=1}^{n} \frac{1}{\lambda_i} \le \varepsilon \sum_{i=1}^{n} \frac{1}{\gamma_i} \quad \text{for } n \ge N.$$

Next, let $\delta > 0$ be such that

$$\delta N \sum_{i=1}^{N} \frac{1}{\lambda_i} < \varepsilon \quad \text{and} \quad \delta \sum_{i=1}^{N+1} \frac{1}{\gamma_i} < 1.$$

Then, given an f-ordered family $\mathcal{I} = (I_k)_{k=1}^{\infty}$ with $\|\mathcal{I}\|_f < \delta$ and a positive integer $m > N$, one has

$$\begin{aligned}
\sum_{k=1}^{m} \frac{|f(I_k)|}{\lambda_k} &= \sum_{\substack{k \le N \\ k \le m-1}} \left(\sum_{i=1}^{k} \frac{1}{\lambda_i} \right) \big(|f(I_k)| - |f(I_{k+1})|\big) \\
&\quad + \sum_{N < k \le m-1} \left(\sum_{i=1}^{k} \frac{1}{\lambda_i} \right) \big(|f(I_k)| - |f(I_{k+1})|\big) \\
&\quad + |f(I_m)| \sum_{k=1}^{m} \frac{1}{\lambda_k} \le \delta \sum_{\substack{k \le N \\ k \le m-1}} \sum_{i=1}^{k} \frac{1}{\lambda_i} \\
&\quad + \varepsilon \sum_{N < k \le m-1} \left(\sum_{i=1}^{k} \frac{1}{\gamma_i} \right) \big(|f(I_k)| - |f(I_{k+1})|\big) \\
&\quad + \varepsilon |f(I_m)| \sum_{k=1}^{m} \frac{1}{\gamma_k} \\
&\le \delta N \sum_{i=1}^{N} \frac{1}{\lambda_i} + \varepsilon \left(|f(I_{N+1})| \sum_{i=1}^{N+1} \frac{1}{\gamma_i} + \sum_{N+1 < k \le m} \frac{|f(I_k)|}{\gamma_k} \right) \\
&\le \varepsilon + \varepsilon(1 + V_\Gamma(f)),
\end{aligned}$$

which completes the proof of Proposition 14. □

It is known that the equality $\Lambda BV = \Lambda BV_c$ can occur [**14**, Theorem 6], and hence the condition in Proposition 14 is only sufficient. Finding a characterization of the inclusion $\Gamma BV \subset \Lambda BV_c$ remains an open problem, as it has been for 20 years.

On the other hand, it is easy to see that for every Λ-sequence Γ there exists a Λ-sequence Λ such that $\Gamma = o(\Lambda)$, and therefore, since the union of all ΛBV classes is the class of regulated functions [**9**, Theorem 10], we obtain the following fact concerning the structure of continuous functions.

Proposition 15. *The union of all classes ΛAC is the class of all continuous functions on* $[0, 1]$.

In other words, for every continuous function $f : [0, 1] \to \mathbf{R}$, there is a Λ-sequence Λ such that f is Λ-absolutely continuous. However, no countable union of ΛAC classes yields all continuous functions, which can be proven either by direct construction as in [**9**, Theorem 12] or by another application of the Banach-Steinhaus-Olevskii theorem, since convergence in Λ-variation implies uniform convergence.

REFERENCES

1. M. Avdispahič, *Concepts of generalized bounded variation and the theory of Fourier series*, Internat. J. Math. Math. Sci. **9** (1986), 223–244.

2. N. Bari, *A treatise on trigonometric series*, Vol.1, The MacMillan Company, New York, 1964.

3. J. Ciemnoczołowski, *Inclusion theorems for classes of functions of bounded λ-variation*, Funct. Approx. **15** (1986), 77–79.

4. K.M. Garg, *Relativization of some aspects of the theory of functions of bounded variation*, Dissert. Math. **320** (1992), 1–123.

5. G. Hardy, J.E. Littlewood and G. Pólya, *Inequalities*, 2nd ed., Cambridge University Press, Cambridge, 1952.

6. E.R. Love, *A generalization of absolute continuity*, J. London Math. Soc. **26** (1951), 1–13.

7. E.R. Love and L.C. Young, *On fractional integration by parts*, Proc. London Math. Soc. **44** (1938), 1–35.

8. V. Olevskii, *A note on the Banach-Steinhaus theorem*, Real Anal. Exch. **17** (1991/92), 399–401.

9. S. Perlman, *Functions of generalized variation*, Fund. Math. **105** (1980), 199–211.

10. S. Perlman and D. Waterman, *Some remarks on functions of* Λ*-bounded variation*, Proc. Amer. Math. Soc. **74** (1979), 113–118.

11. F. Prus-Wiśniowski, *Some remarks on functions of bounded* ϕ*-variation*, Comment. Math. **30** (1990), 147–166.

12. ———, λ*-variation and Hausdorff distance*, Math. Nachr. **158** (1992), 283–297.

13. S. Saks, *Theory of the integral*, 2nd. ed., Dover, New York, 1964.

14. G. Shao, *A note on the functions of* Λ*-bounded variation*, Chinese Ann. Math. **6** (1985), 311–316.

15. E. Study, *Über eine besondere Classe von Functionen einer reellen Veränderlichen*, Math. Ann. **47** (1896), 298–316.

16. H.D. Ursell, *On the total variation of* $\{f(t+\tau)-f(t)\}$, Proc. London Math. Soc. **37** (1932), 402–415.

17. S. Wang, *Some properties of functions of* Λ*-bounded variation*, Sci. Sinica **25** (1982), 149–160.

18. D. Waterman, *On convergence of Fourier series of functions of generalized bounded variation*, Stud. Math. **44** (1972), 107–117.

19. ———, *On the summability of Fourier series of functions of* Λ*-bounded variation*, Stud. Math. **55** (1976), 97–109.

20. ———, *On* Λ*-bounded variation*, Stud. Math. **57** (1976), 33–45.

21. ———, *Bounded variation and Fourier series*, Real Anal. Exch. **3** (1977–78), 61–85.

22. ———, Λ*-bounded variation*: *Recent results and unsolved problems*, Real Anal. Exch. **4** (1978–79), 69–75.

23. ———, *Fourier series of functions of* Λ*-bounded variation*, Proc. Amer. Math. Soc. **74** (1979), 119–123.

24. N. Wiener, *The quadratic variation of a function and its Fourier coefficients*, J. Math. Phys. **3** (1924), 72–94.

25. K. Yosida, *Functional analysis*, Springer Verlag, Heidelberg, 1963.

26. L.C. Young, *An inequality of the Hölder type, connected with Stieltjes integration*, Acta Math. **67** (1936), 251–282

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