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NORM-PRESERVING SURJECTIONS ON ALGEBRAS OF CONTINUOUS FUNCTIONS

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ABSTRACT. Let X, Y be two compact Hausdorff spaces, and let $C(X)$ denote the Banach algebra of all complex-valued, continuous functions on X endowed with the supremum norm $\|f\|_X$. It is shown that if T is a surjective map of $C(X)$ onto $C(Y)$ such that $T\lambda = \lambda$ for $\lambda \in \{\pm 1, \pm i\}$ and satisfying $\|(Tf)\overline{(Tg)} - 1\|_Y = \|f\bar{g} - 1\|_X$ for every pair f and g in $C(X)$, then T is given by $Tf = f \circ \phi$ for some homeomorphic map ϕ of Y onto X; in particular, T is an isometric algebra *-isomorphism.

1. Introduction. There are many papers that deal with spectrum-preserving maps between Banach algebras. Molnár [**8**] initiated the study of multiplicatively spectrum-preserving maps and showed that a unit preserving surjective map $T : C(X) \to C(X)$, not necessarily linear, on a first-countable compact Hausdorff space X is an algebra isomorphism if

$$\sigma(TfTg) = \sigma(fg)$$

holds for every pair f and g in $C(X)$. Rao and Roy [**11**] dealt with uniform algebras on compact Hausdorff spaces which are regarded as the maximal ideal space and generalized the result of Molnár. Hatori, Miura and Takagi [**1**] extended the result of Molnár by replacing the spectrum with the range and showed that a unit preserving surjective map $T : A \to B$ between two uniform algebras is an algebra isomorphism if

$$\mathrm{Ran}\,(TfTg) = \mathrm{Ran}\,(fg)$$

holds for every pair f and g in A, where $\mathrm{Ran}\,(h)$ denotes the range of h. Recall that the *peripheral range*, $\mathrm{Ran}_\pi(f)$, of f in a uniform algebra is defined by $\mathrm{Ran}_\pi(f) = \{z \in \mathrm{Ran}\,(f) : |z| = \|f\|\}$, where $\|f\|$ is the supremum norm of f. Luttman and Tonev [**7**] extended the result of Hatori, Miura and Takagi by replacing the ranges with the peripheral

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ranges and showed that a unit preserving surjective map $T : A \to B$ is an algebra isomorphism if T is Ran_π-*multiplicative* (or, *peripherally-multiplicative*), i.e.,

$$\mathrm{Ran}_\pi(TfTg) = \mathrm{Ran}_\pi(fg), \quad f, g \in A.$$

Lambert, Luttman and Tonev [**6**] considered mappings $T : A \to B$ that satisfy the condition $\mathrm{Ran}_\pi((Tf)(Tg)) \cap \mathrm{Ran}_\pi(fg) \neq \varnothing$ for all $f, g \in A$, in general nonlinear, and called them *weakly peripherally-multiplicative*. They have shown that a mapping $T\colon A \to B$ between uniform algebras is an isometric algebra isomorphism if and only if it is weakly peripherally-multiplicative and preserves the class of peak functions. In particular, they proved that if a map $T : A \to B$ is *norm-multiplicative*, i.e.,

$$\|TfTg\| = \|fg\|, \quad f, g \in A,$$

and preserves the class of peak functions, then there exists a homeomorphism $\phi : \delta A \to \delta B$ so that the equality $|(Tf)(\phi(x))| = |f(x)|$ holds for every $f \in A$ and all x in the Choquet boundary δA of A.

Molnár [**8**] also gave a characterization on algebra *-isomorphisms: a unit preserving surjective map $T : C(X) \to C(X)$ for a first-countable compact Hausdorff space X is an algebra *-isomorphism if

$$\sigma(Tf\overline{Tg}) = \sigma(f\overline{g})$$

holds for every pair f and g in $C(X)$. Hatori, Miura and Takagi [**2**] generalized this result for certain semi-simple commutative Banach *-algebras. The author [**5**] extended the result of Molnár by considering the peripheral ranges instead of the spectra and, in particular, it was proved that a unit preserving surjective map $T : C(X) \to C(Y)$ for compact Hausdorff spaces X and Y (not necessarily first-countable) is an algebra *-isomorphism if

$$\mathrm{Ran}_\pi(Tf\overline{Tg}) = \mathrm{Ran}_\pi(f\overline{g})$$

holds for every pair f and g in $C(X)$.

In this paper we consider a further extension: multiplicatively norm-preserving maps. First of all, one can modify an example of Lambert, Luttman and Tonev [**6**, Example 1] and exhibit a map which is not linear such that the equality

$$\|Tf\overline{Tg}\|_Y = \|f\overline{g}\|_X$$

holds for every pair f and g in $C(X)$. In the following we mainly consider the condition

$$\|Tf\overline{Tg} - 1\|_Y = \|f\overline{g} - 1\|_X, \quad f, g \in C(X)$$

on a map T from $C(X)$ onto $C(Y)$. If T satisfies the hypothesis

$$\sigma(Tf\overline{Tg}) = \sigma(f\overline{g}), \quad f, g \in C(X),$$

then T satisfies the above condition.

Our main result is the following.

Theorem 1.1. *Let X, Y be two compact Hausdorff spaces. If a surjective map $T : C(X) \to C(Y)$ satisfies the conditions*

(a) $T\lambda = \lambda$ *for* $\lambda \in \{\pm 1, \pm i\}$ *and*

(b) $\|(Tf)\overline{(Tg)} - 1\|_Y = \|f\overline{g} - 1\|_X$ *for all* $f, g \in C(X)$,

*then there exists a homeomorphism ϕ of Y onto X such that $Tf = f \circ \phi$ for every f in $C(X)$; in particular, T is an isometric algebra *-isomorphism.*

2. Preliminaries. Let X be a compact Hausdorff space. We denote by $\sigma_\pi(f)$ the *peripheral spectrum* of an element $f \in C(X)$:

$$\sigma_\pi(f) = \{z \in \sigma(f) : |z| = r(f)\},$$

where $r(f)$ denotes the spectral radius of f. Clearly, $\sigma_\pi(f) = \mathrm{Ran}_\pi(f)$. We denote by P_X° the set of all peak functions in $C(X)^{-1}$, that is, $P_X^\circ = \{u \in C(X)^{-1} : \sigma_\pi(u) = \{1\}\}$. If $x \in X$, then $P_X^\circ(x) = \{u \in P_X^\circ : u(x) = 1\}$.[1]

Lemma 2.1. *Let $x_0 \in X$, and let F be a closed subset of X with $x_0 \notin F$. Then for each $\varepsilon > 0$ there exists a $u \in P_X^\circ(x_0)$ such that $|u(x)| < \varepsilon$ for $x \in F$.*

Proof. Let $\varepsilon > 0$ be given. Since $\{x_0\}$ and F are disjoint closed subsets of X, by Urysohn's lemma there exists a continuous function $v_1 : X \to [0,1]$ such that $v_1(x_0) = 1$ and $v_1 = 0$ on F. Let $v_2 = (1+v_1)/2$. Then $v_2 \in P_X^\circ(x_0)$ and $v_2 = 1/2$ on F. We see that $u = v_2^n$ has the required properties for some sufficiently large n. ◻

In the following lemma, the same result for a not necessarily invertible, peak function is proved in [**6**] for arbitrary uniform algebra.

Lemma 2.2. *Let $f \in C(X)$ and $x_0 \in X$. If $\lambda = f(x_0)$ and $\lambda \neq 0$, then there exists a $u \in P_X^\circ(x_0)$ such that $\sigma_\pi((1/\lambda)fu) = \{1\}$.*

Proof. Suppose $\lambda \neq 0$. Let $F_0 = \{x \in X : |f(x) - \lambda| \geq 2^{-1}|\lambda|\}$ and

$$F_n = \left\{x \in X : 2^{-n-1}|\lambda| \leq |f(x) - \lambda| \leq 2^{-n}|\lambda|\right\}$$

for $n = 1, 2, \ldots$. Clearly, $F_0, F_1, \ldots$ are closed subsets of X that do not contain x_0; so by Urysohn's lemma, there exist continuous functions $v_0, v_1, \ldots$ such that $0 \leq v_j \leq 1$, $v_j(x_0) = 1$ and $v_j = 0$ on F_j for $j = 0, 1, \ldots$. For each j we take a positive integer n_j so that $u_j = (1 + v_j/2)^{n_j}$ may have the property: If $j = 0$,

$$|u_0(x)| < \frac{|\lambda|}{\|f\|_X}, \quad x \in F_0$$

and, if $j > 0$,

$$|u_j(x)| < \frac{1}{2^j + 1}, \quad x \in F_j.$$

Now put $u = u_0 \sum_{n=1}^\infty 2^{-n} u_n$. This series is majorized by the convergent series $\sum 2^{-n}$, so u is well defined and $u \in C(X)$. Moreover, u is easily seen to be a function in $P_X^\circ(x_0)$.

Put $g = (1/\lambda)fu$. To verify $\sigma_\pi(g) = \{1\}$, pick $x \in X$. If $x \in F_0$, then we see

$$|g(x)| = \frac{1}{|\lambda|}|f(x)||u_0(x)| \sum_{n=1}^\infty \frac{|u_n(x)|}{2^n} < \frac{1}{|\lambda|}\|f\|_X \frac{|\lambda|}{\|f\|_X} \sum_{n=1}^\infty \frac{1}{2^n} = 1.$$

If $x \in F_n$ for some positive integer n, then

$$\begin{aligned}|g(x)| &= \frac{1}{|\lambda|}|f(x)||u_0(x)|\left(\frac{|u_n(x)|}{2^n} + \sum_{j\neq n}\frac{|u_j(x)|}{2^j}\right)\\ &\leq \frac{1}{|\lambda|}(|f(x)-\lambda| + |\lambda|)\left(\frac{|u_n(x)|}{2^n} + \sum_{j\neq n}\frac{1}{2^j}\right)\\ &< \frac{1}{|\lambda|}\left(\frac{|\lambda|}{2^n} + |\lambda|\right)\left(\frac{1}{2^n}\frac{1}{2^n+1} + 1 - \frac{1}{2^n}\right) = 1.\end{aligned}$$

If $x \in X \setminus \cup_{j=0}^{\infty} F_j$, then $f(x) = \lambda$ and $g(x) = u(x) \in D \cup \{1\}$, where $D = \{z \in \mathbf{C} : |z| < 1\}$. Thus, $g(X) \subset D \cup \{1\}$. In particular, $g(x_0) = u(x_0) = 1$. Hence, $\sigma_\pi(g) = \{1\}$. This completes the proof. □

3. A proof of Theorem 1.1. Throughout this section, T denotes a surjective map which satisfies the hypotheses of Theorem 1.1:

(a) $T\lambda = \lambda$ for $\lambda \in \{\pm 1, \pm i\}$ and

(b) $\|(Tf)\overline{(Tg)} - 1\|_Y = \|f\bar{g} - 1\|_X$ for all $f, g \in C(X)$.

Lemma 3.1. $T(C(X)^{-1}) = C(Y)^{-1}$.

Proof. Let $f \in C(X)^{-1}$. Then we see that $\|Tf\overline{T(\overline{f^{-1}})} - 1\|_Y = \|ff^{-1} - 1\|_X = 0$, thus $Tf\overline{T(\overline{f^{-1}})} = 1$. Hence, $Tf \in C(Y)^{-1}$. Let $F \in C(Y)^{-1}$. Since T is surjective, there are f and g in $C(X)$ such that $Tf = F$ and $Tg = \overline{F^{-1}}$. Then we see that $\|f\bar{g} - 1\|_X = \|Tf\overline{Tg} - 1\|_Y = \|FF^{-1} - 1\|_Y = 0$. Thus, we have that $f\bar{g} = 1$, and $f \in C(X)^{-1}$. □

Lemma 3.2. *T is injective.*

Proof. Suppose $Tf = Tg$ for $f, g \in C(X)$. We will show that $f = g$. Let $x \in X$. First we consider the case where $f(x) \neq 0$ and $g(x) \neq 0$. By Lemma 2.2, there exist u_f, $u_g \in P_X^{\circ}(x)$ such that $\sigma_\pi(fu_f) = \{f(x)\}$ and $\sigma_\pi(gu_g) = \{g(x)\}$. Let $u = u_f u_g$. Then u is an element in $P_X^{\circ}(x)$

such that $\sigma_\pi(fu) = \{f(x)\}$ and $\sigma_\pi(gu) = \{g(x)\}$. Then we see that

$$\begin{aligned} 2 &= \left\| f\frac{-1}{f(x)}u - 1\right\|_X = \left\| Tf\overline{T\left(\overline{\frac{-u}{f(x)}}\right)} - 1\right\|_Y \\ &= \left\| Tg\overline{T\left(\overline{\frac{-u}{f(x)}}\right)} - 1\right\|_Y = \left\| g\frac{-1}{f(x)}u - 1\right\|_X \\ &\le \frac{1}{|f(x)|}\|gu\|_X + 1 = \frac{|g(x)|}{|f(x)|} + 1. \end{aligned}$$

Therefore, $|f(x)| \le |g(x)|$ holds. In a similar way, we see that $|g(x)| \le |f(x)|$. So we have $|f(x)| = |g(x)|$. This fact implies that $\|g(-1/f(x))u\|_X = 1$. It follows that $-1 \in \sigma_\pi(g(-1/f(x))u)$, since $\|g(-1/f(x))u - 1\|_X = 2$. Thus, $f(x) \in \sigma_\pi(gu) = \{g(x)\}$, and $f(x) = g(x)$.

Next we consider the case where $f(x) = 0$ or $g(x) = 0$. Without loss of generality we may assume $f(x) = 0$. We will show that $g(x) = 0$. Suppose not. Let ε be a positive number with $\varepsilon < |g(x)|$. Let $F = \{x' \in X : |f(x')| \ge \varepsilon\}$. Since F is a closed subset of X with $x \notin F$, by Lemma 2.1 there exists a $u_f \in P_X^\circ(x)$ such that $|u_f(x')| < \varepsilon/(\|f\|_X + 1)$ for all $x' \in F$. We have that $|fu_f| < \varepsilon$ on X. Since $g(x) \neq 0$, by Lemma 2.2 there exists a $u_g \in P_X^\circ(x)$ such that $\sigma_\pi(gu_g) = \{g(x)\}$. Let $u = u_f u_g$. Then u is an element of $P_X^\circ(x)$ such that $|fu| < \varepsilon$ on X and $\sigma_\pi(gu) = \{g(x)\}$. Choose $\alpha \in \mathbf{C}$ such that $|\alpha| = 1$ and $(\alpha gu)(x) = -|g(x)|$. Then we see that

$$\begin{aligned} |g(x)| + 1 &= \|\alpha gu - 1\|_X = \|(Tg)(\overline{T(\overline{\alpha u})}) - 1\|_Y \\ &= \|(Tf)(\overline{T(\overline{\alpha u})}) - 1\|_Y = \|\alpha fu - 1\|_X \\ &\le 1 + \varepsilon. \end{aligned}$$

Thus we have $|g(x)| \le \varepsilon$. This contradicts $\varepsilon < |g(x)|$. Hence, $g(x) = 0$. This completes the proof. □

By Lemma 3.2, we can consider the inverse T^{-1} of $C(Y)$ onto $C(X)$. Clearly T^{-1} has the same properties as T:

(a′) $T^{-1}\lambda = \lambda$, $(\lambda \in \{\pm 1, \pm i\})$,

(b′) $\|(T^{-1}F)\overline{(T^{-1}G)} - 1\|_X = \|F\overline{G} - 1\|_Y$, $F, G \in C(Y)$.

Lemma 3.3. *If $f, g \in C(X)^{-1}$, then the equation $\|(Tf)\overline{(Tg)}\|_Y = \|f\bar{g}\|_X$ holds; in particular, $\|Tf\|_Y = \|f\|_X$ holds for every $f \in C(X)^{-1}$. Since $|z\overline{w}| = |zw|$ for any $z, w \in \mathbf{C}$, as a consequence of it T is norm-multiplicative on $C(X)^{-1}$, that is, $\|TfTg\|_Y = \|fg\|_X$ for every pair f and g in $C(X)^{-1}$.*

Proof. Let $f, g \in C(X)^{-1}$. We will show that $\|f\overline{g}\|_X \leq \|(Tf)(\overline{Tg})\|_Y$. From Lemma 3.1, $Tf, Tg \in C(Y)^{-1}$. Let $K_n = T(nf)(Tf)^{-1}$ for $n = 1, 2, \dots$. In the proof of Lemma 3.1, we have shown that $(Tf)^{-1} = \overline{T(\overline{f^{-1}})}$. It follows that $K_n = T(nf)\overline{T(\overline{f^{-1}})}$. Then we have that $\|K_n\|_Y \leq n$ for each n, since $\|K_n\|_Y - 1 \leq \|K_n - 1\|_Y = \|T(nf)\overline{T(\overline{f^{-1}})} - 1\|_Y = \|(nf)f^{-1} - 1\|_X = n - 1$. For each n we have

$$\begin{aligned} n\|f\overline{g}\|_X - 1 &\leq \|(nf)\overline{g} - 1\|_X = \|T(nf)\overline{Tg} - 1\|_Y \\ &\leq \|T(nf)\overline{Tg}\|_Y + 1 \leq \|K_n\|_Y \|(Tf)(\overline{Tg})\|_Y + 1. \end{aligned}$$

Since $\|K_n\|_Y \leq n$ for every n, it follows that

$$\|f\overline{g}\|_X - \frac{1}{n} \leq \|(Tf)(\overline{Tg})\|_Y + \frac{1}{n}.$$

Letting n tend to ∞, gives $\|f\bar{g}\|_X \leq \|(Tf)(\overline{Tg})\|_Y$. Applying a similar argument to T^{-1}, yields $\|(Tf)(\overline{Tg})\|_Y \leq \|f\bar{g}\|_X$. Hence, $\|(Tf)(\overline{Tg})\|_Y = \|f\bar{g}\|_X$ holds for every pair f and g in $C(X)^{-1}$. $\square$

Lemma 3.4. *If f and g are elements in $C(X)^{-1}$ such that $\|Tf\|_Y = 1 = \|Tg\|_Y$ and $|Tf|^{-1}(1) \subset |Tg|^{-1}(1)$, then $|f|^{-1}(1) \subset |g|^{-1}(1)$.*

Proof. We assume that $|Tf|^{-1}(1) \subset |Tg|^{-1}(1)$. We will show that $|f|^{-1}(1) \subset |g|^{-1}(1)$. Suppose not. Then there is an $x \in X$ such that $x \in |f|^{-1}(1) \setminus |g|^{-1}(1)$. By Lemma 3.3 we have that $\|f\|_X = \|Tf\|_Y = 1$ and $\|g\|_X = \|Tg\|_Y = 1$. $|g|^{-1}(1)$ is a closed subset of X which does not contain x. By Lemma 2.1 there exists a $u \in P_X^{\circ}(x)$ such that $|u(x')| < 1$ for $x' \in |g|^{-1}(1)$. Then we have that $|ug| < 1$ on X, that is, $\|ug\|_X < 1$. Then we have

$$\|(Tu)(\overline{Tg})\|_Y = \|u\bar{g}\|_X = \|ug\|_X < 1. \tag{1}$$

On the other hand, since $|(u\bar{f})(x)| = 1$ and $\|f\|_X = 1 = \|u\|_X$, we have $\|(Tu)(\overline{Tf})\|_Y = \|u\bar{f}\|_X = 1$. Hence, there is a $y \in Y$ such that

$|Tu(y)||Tf(y)| = 1$. Since by Lemma 3.3 $\|Tu\|_Y = \|u\|_X = 1 = \|Tf\|_Y$, we have $|Tu(y)| = 1 = |Tf(y)|$. This implies $|(Tu\overline{Tg})(y)| = |Tu(y)||Tg(y)| = 1$ because $|Tf|^{-1}(1) \subset |Tg|^{-1}(1)$. Hence we have that $\|(Tu)(\overline{Tg})\|_Y = 1$. This contradicts the inequality (1). $\square$

We will construct a homeomorphism of Y onto X which satisfies the resulting conditions of Theorem 1.1. Lambert, Luttman and Tonev [**6**] proved that if a map $T : A \to B$ between two uniform algebras preserves the class of peak functions and is norm-multiplicative, then there exists a homeomorphism $\phi : \delta A \to \delta B$ so that the equality $|(Tf)(\phi(x))| = |f(x)|$ holds for every $f \in A$ and all x in the Choquet boundary δA of A. A similar argument was used in [**7, 11**]. The idea of our construction of a homeomorphism has the same vein. First, we will show that there exists a homeomorphism ϕ of Y onto X such that $|Tf(y)| = |f(\phi(y))|$ holds for every $f \in C(X)^{-1}$ and $y \in Y$. Next, we will show that the indicated map has the desired properties.

For $x \in X$, set $\mathfrak{F}_X(x) = C(X)^{-1} \cap \{f \in C(X) : |f(x)| = 1 = \|f\|_X\}$. Clearly, $\mathfrak{F}_X(x)$ properly contains $P_X^\circ(x)$.

Lemma 3.5. *For $x_1, x_2 \in X$, $x_1 = x_2$ if and only if $\mathfrak{F}_X(x_1) \subset \mathfrak{F}_X(x_2)$.*

Proof. The 'only if' part is trivial. We will show the 'if' part. Assume that $x_1 \neq x_2$. Then, by Lemma 2.1, there exists a $u \in P_X^\circ(x_1)$ such that $|u(x_2)| < 1$. Thus, we see that $u \in \mathfrak{F}_X(x_1) \setminus \mathfrak{F}_X(x_2)$, that is, $\mathfrak{F}_X(x_1) \not\subset \mathfrak{F}_X(x_2)$. $\square$

Lemma 3.6. *For each $y \in Y$, there corresponds a unique $x \in X$ such that $T[\mathfrak{F}_X(x)] = \mathfrak{F}_Y(y)$.*

Proof. Let $y \in Y$. First, we will show that there exists an $x \in X$ such that $T^{-1}[\mathfrak{F}_Y(y)] \subset \mathfrak{F}_X(x)$. Set

$$L_y = \bigcap_{f \in T^{-1}[\mathfrak{F}_Y(y)]} |f|^{-1}(1).$$

Let $f_1, \dots, f_n \in T^{-1}[\mathfrak{F}_Y(y)]$. Let $F = Tf_1 \cdots Tf_n$ and $f = T^{-1}F$. Then we see that $F \in C(Y)^{-1}$ and $|F(y)| = 1$. Since $\|Tf_j\|_Y = 1$

for every j, we have that $\|F\|_Y = 1$. Thus by Lemma 3.3, we have that $\|f\|_X = \|Tf\|_Y = \|F\|_Y = 1$. Thus, $|f|^{-1}(1)$ is nonempty. Since $\|Tf_j\|_Y = 1$, we have that $|Tf|^{-1}(1) \subset |Tf_j|^{-1}(1)$ for every j. By Lemma 3.4 we see that $|f|^{-1}(1) \subset |f_j|^{-1}(1)$ for every j, that is, the intersection of all $|f_j|^{-1}(1)$ contains the nonempty set $|f|^{-1}(1)$. Thus, the class $\{|f|^{-1}(1) : f \in T^{-1}[\mathfrak{F}_Y(y)]\}$ has the finite intersection property. Since $|g|^{-1}(1)$ is compact for every $g \in T^{-1}[\mathfrak{F}_Y(y)]$, L_y is nonempty. Take an element, say x, from L_y; then we see that $T^{-1}[\mathfrak{F}_Y(y)] \subset \mathfrak{F}_X(x)$.

Secondly, we show that $T[\mathfrak{F}_X(x)] = \mathfrak{F}_Y(y)$. Applying the above argument to T^{-1}, there exists a $y' \in Y$ such that $T[\mathfrak{F}_X(x)] \subset \mathfrak{F}_Y(y')$ because T^{-1} has the same properties as T. Since T is surjective, we have that $\mathfrak{F}_Y(y) = T[T^{-1}[\mathfrak{F}_Y(y)]] \subset T[\mathfrak{F}_X(x)] \subset \mathfrak{F}_Y(y')$. By Lemma 3.5, we see that $y = y'$ and hence $T[\mathfrak{F}_X(x)] = \mathfrak{F}_Y(y)$.

Finally, we show the uniqueness of x. Suppose that there is an $x' \in X$ such that $T[\mathfrak{F}_X(x')] = \mathfrak{F}_Y(y)$. Then the injectivity of T implies that $\mathfrak{F}_X(x) = T^{-1}[\mathfrak{F}_Y(y)] = \mathfrak{F}_X(x')$. Lemma 3.5 yields that $x = x'$. This completes the proof. □

By Lemma 3.6, we can consider a map ϕ of Y into X such that $T[\mathfrak{F}_X(\phi(y))] = \mathfrak{F}_Y(y)$ for every $y \in Y$.

Lemma 3.7. *ϕ is a homeomorphism of Y onto X such that $|Tf(y)| = |f(\phi(y))|$ holds for every $f \in C(X)^{-1}$ and $y \in Y$.*

Proof. First, we will show that $|Tf(y)| = |f(\phi(y))|$ holds for every $f \in C(X)^{-1}$ and $y \in Y$. Let $f \in C(X)^{-1}$ and $y \in Y$. Then $f(\phi(y)) \neq 0$. By Lemma 2.2, there exists an $h \in P_X^{\circ}(\phi(y))$ such that $\sigma_\pi(fh) = \{f(\phi(y))\}$. Thus,

$$\|fh\|_X = |f(\phi(y))|. \tag{2}$$

Since $P_X^{\circ}(\phi(y)) \subset \mathfrak{F}_X(\phi(y))$, by the definition of ϕ, we see that $|Th(y)| = 1 = \|Th\|_Y$. Since $f, h \in C(X)^{-1}$ and $|Th(y)| = 1$, by Lemma 3.3 we have that

$$\begin{aligned}\|fh\|_X &= \|(Tf)(Th)\|_Y \geq |Tf(y)||Th(y)| \\ &= |Tf(y)|.\end{aligned}$$

This fact and equality (2) imply that

$$|Tf(y)| \leq |f(\phi(y))|. \tag{3}$$

Since $Tf \in C(X)^{-1}$, $Tf(y) \neq 0$, by Lemma 2.2 there exists a $G \in P_Y^\circ(y)$ such that $\sigma_\pi((Tf)G) = \{Tf(y)\}$. Let $g = T^{-1}G$. Then G is also in $\mathfrak{F}_Y(y)$; thus, $|g(\phi(y))| = 1$ from the definition of ϕ. Since $G \in C(Y)^{-1}$, we have that $g \in C(X)^{-1}$ by Lemma 3.1. Thus, by Lemma 3.3 we have that $|Tf(y)| = \|(Tf)G\|_Y = \|fg\|_X \geq |f(\phi(y))|$. This fact and inequality (3) imply that $|Tf(y)| = |f(\phi(y))|$.

Secondly, we will show that ϕ is continuous. Let $\mathbf{T}_1$ be the given topology on X, and let $\{y_\alpha\}$ be a convergent net in Y with $\lim y_\alpha = y$. Then the first part shows that $\lim |f(\phi(y_\alpha))| = \lim |Tf(y_\alpha)| = |Tf(y)| = |f(\phi(y))|$ for every $f \in C(X)^{-1}$. Thus, $\phi(y_\alpha)$ converges to $\phi(y)$ with respect to the weak topology $\mathbf{T}_2$ on X generated by $|C(X)^{-1}| = \{|f| : f \in C(X)^{-1}\}$. The identity map of $(X, \mathbf{T}_1)$ onto $(X, \mathbf{T}_2)$ is continuous, and $(X, \mathbf{T}_2)$ is Hausdorff because $|C(X)^{-1}|$ separates the points of X; since $(X, \mathbf{T}_1)$ is compact, the map is a homeomorphism. Hence, ϕ is continuous.

Finally, we will show that ϕ is a homeomorphism of Y onto X. Since T^{-1} has the same properties as T, there exists a continuous map ψ of X into Y such that $T^{-1}[\mathfrak{F}_Y(\psi(x))] = \mathfrak{F}_X(x)$ and $|T^{-1}F(x)| = |F(\psi(x))|$ for every $F \in C(Y)^{-1}$. Let $y \in Y$ and $f \in C(X)^{-1}$ with $F = Tf$. Then we have that

$$\begin{aligned} |Tf(y)| &= |f(\phi(y))| = |T^{-1}F(\phi(y))| \\ &= |F(\psi(\phi(y)))| = |Tf(\psi(\phi(y)))|. \end{aligned}$$

Thus $y = \psi(\phi(y))$, since $|C(Y)^{-1}| = |T(C(X)^{-1})|$ separates the points of Y. In a similar way, we have $x = \phi(\psi(x))$ for every $x \in X$. Hence, ϕ is a bijective map of Y onto X with $\phi^{-1} = \psi$. Since ψ is continuous, ϕ is a homeomorphism of Y onto X. □

Lemma 3.8. *$T\lambda = \lambda$ holds for every $\lambda \in S^1$, where $S^1 = \{z \in \mathbf{C} : |z| = 1\}$.*

Proof. Let $\lambda \in S^1$. We may assume that $\lambda \notin \{\pm 1, \pm i\}$. From condition (b), we have that $\||T\lambda|^2 - 1\|_Y = \|T\lambda\overline{T\lambda} - 1\|_Y = \|\lambda\overline{\lambda} - 1\|_X =$

0, thus $|T\lambda| = 1$, or equivalently, $(T\lambda)(Y) \subset S^1$. Since $T1 = 1$, we have

$$\|T\lambda - 1\|_Y = \|T\lambda\overline{T1} - 1\|_Y = \|l \cdot \bar{1} - 1\|_X = |\lambda - 1|. \tag{4}$$

Since $T(-1) = -1$, we have

$$\|T\lambda + 1\|_Y = \|T\lambda\overline{T(-1)} - 1\|_Y = \|\lambda \cdot (\overline{-1}) - 1\|_X = |\lambda + 1|. \tag{5}$$

Since $(T\lambda)(Y) \subset S^1$, (4) and (5) imply that $(T\lambda)(Y) \subset \{\lambda, \bar{\lambda}\}$. If $\operatorname{Im}\lambda > 0$, the condition $Ti = i$ gives

$$\|T\lambda - i\|_Y = \|T\lambda\overline{Ti} - 1\|_Y = \|\lambda \cdot \bar{i} - 1\|_X = |\lambda - i|.$$

This implies that $(T\lambda)(Y) = \{\lambda\}$, that is, $T\lambda = \lambda$. If $\operatorname{Im}\lambda < 0$, in a similar way, we have that $\|T\lambda + i\|_Y = |\lambda + i|$ because $T(-i) = -i$. It follows that $T\lambda = \lambda$. Thus the proof is complete. $\square$

Lemma 3.9. *$T(\alpha P_X^\circ(\phi(y))) = \alpha P_Y^\circ(y)$ holds for every $\alpha \in S^1$ and $y \in Y$.*

Proof. Let $\alpha \in S^1$. First, we will show that $T(\alpha P_X^\circ) = \alpha P_Y^\circ$. Let $f \in P_X^\circ$. From Lemma 3.8 we have

$$\begin{aligned} 2 &= \|-\overline{\alpha}\alpha f - 1\|_X = \|T(\alpha f)\overline{T(-\alpha)} - 1\|_Y \\ &= \|-\overline{\alpha}T(\alpha f) - 1\|_Y. \end{aligned}$$

Thus, $\alpha \in \sigma_\pi(T(\alpha f))$ since $\|T(\alpha f)\|_Y = \|\alpha f\|_X = 1$ by Lemma 3.3. Let $\beta \in \sigma_\pi(T(\alpha f))$. Lemma 3.8 gives

$$\begin{aligned} 2 &= \|-\overline{\beta}T(\alpha f) - 1\|_Y = \|\overline{T(-\beta)}T(\alpha f) - 1\|_Y \\ &= \|-\overline{\beta}\alpha f - 1\|_X. \end{aligned}$$

Thus, $\beta \in \sigma_\pi(\alpha f)$ since $\|\alpha f\|_X = 1$. Since $\sigma_\pi(\alpha f) = \{\alpha\}$, we have that $\beta = \alpha$ and $\sigma_\pi(T(\alpha f)) = \{\alpha\}$. Thus, $T(\alpha P_X^\circ) \subset \alpha P_Y^\circ$. In a similar way, we have that $T^{-1}(\alpha P_Y^\circ) \subset \alpha P_X^\circ$. Hence, $T(\alpha P_X^\circ) = \alpha P_Y^\circ$.

Let $y \in Y$ and $f \in P_X^\circ(\phi(y))$. From the above argument, we have that $T(\alpha f) \in \alpha P_Y^\circ$. We show that $T(\alpha f)(y) = \alpha$. Since $f \in C(X)^{-1}$, by Lemmas 3.3 and 3.7 we see that

$$\begin{aligned} |T(\alpha f)(y)| &= |\alpha f(\phi(y))| = 1 = \|\alpha f\|_X \\ &= \|T(\alpha f)\|_Y. \end{aligned}$$

Thus, we have $T(\alpha f)(y) = \alpha$. Since $f \in P_X^\circ(\phi(y))$ is arbitrary, we have that $T(\alpha P_X^\circ(\phi(y))) \subset \alpha P_Y^\circ(y)$. In a similar way, it holds for T^{-1} that $T^{-1}(\alpha P_Y^\circ(\phi^{-1}(x))) \subset \alpha P_X^\circ(x)$ for every $x \in X$. Let $x = \phi(y)$. Then $T^{-1}(\alpha P_Y^\circ(y)) \subset \alpha P_X^\circ(\phi(y))$, so we see that

$$\begin{aligned}\alpha P_Y^\circ(y) &= T(T^{-1}(\alpha P_Y^\circ(y))) \subset T(\alpha P_X^\circ(\phi(y))) \\ &\subset \alpha P_Y^\circ(y).\end{aligned}$$

Thus, we have that $T(\alpha P_X^\circ(\phi(y))) = \alpha P_Y^\circ(y)$. □

Lemma 3.10. *If $f \in C(X)^{-1}$, then $Tf(y) = f(\phi(y))$ holds for every $y \in Y$.*

Proof. Let $f \in C(X)^{-1}$ and $y \in Y$. From Lemma 3.7, we have $|Tf(y)| = |f(\phi(y))|$. Suppose $Tf(y) \neq f(\phi(y))$. Since $Tf(y) \neq 0$, there exists an $H \in P_Y^\circ(y)$ such that $\sigma_\pi((Tf)H) = \{Tf(y)\}$ by Lemma 2.2. Since T is surjective, there exists an $h \in C(X)$ such that $Th = \overline{H}$. Since $\overline{H}$ is also in $P_Y^\circ(y)$, Lemma 3.9 implies that $h \in P_X^\circ(\phi(y))$ and $\sigma_\pi((Tf)(\overline{Th})) = \sigma_\pi((Tf)H) = \{Tf(y)\}$. Let $\alpha = -|f(\phi(y))|^{-1}\overline{f(\phi(y))}$. Then $\sigma_\pi(\alpha(Tf)(\overline{Th})) = \{\alpha(Tf(y))\}$. Our assumption $Tf(y) \neq f(\phi(y))$ implies that $\alpha(Tf(y)) \neq -|Tf(y)|$. Thus, we see that $-|Tf(y)| \notin [\alpha(Tf)(\overline{Th})](Y)$. It follows that

$$\|\alpha(Tf)(\overline{Th}) - 1\|_Y < |Tf(y)| + 1. \tag{6}$$

Since $\overline{\alpha}Th \in \overline{\alpha}P_Y^\circ(y)$, from Lemma 3.9 there exists an $h' \in P_X^\circ(\phi(y))$ such that $T(\overline{\alpha}h') = \overline{\alpha}Th$. Then we see that

$$\begin{aligned}\|\alpha(Tf)(\overline{Th}) - 1\|_Y &= \|(Tf)(\overline{T(\bar{\alpha}h')}) - 1\|_Y = \|\alpha f\bar{h}' - 1\|_X \\ &\geq |\alpha f(\phi(y))\overline{h'(\phi(y))} - 1| \\ &= |-|f(\phi(y))| - 1| = |-|Tf(y)| - 1| \\ &= |Tf(y)| + 1.\end{aligned}$$

Thus, $\|\alpha(Tf)(\overline{Th}) - 1\|_Y \geq |Tf(y)| + 1$. This contradicts inequality (6). Thus, we have that $Tf(y) = f(\phi(y))$. □

Lemma 3.11. *If $g \in C(X)$, then $Tg(y) = g(\phi(y))$ holds for every $y \in Y$. In particular, T is an isometric algebra *-isomorphism of $C(X)$ onto $C(Y)$.*

Proof. First, we consider the case where $g(\phi(y)) \neq 0$ and $Tg(y) \neq 0$. Using Lemma 2.2 gives $u_1 \in P_X^\circ(\phi(y))$ such that $\sigma_\pi(gu_1) = \{g(\phi(y))\}$. Let $\alpha = -(\overline{g(\phi(y))})/|g(\phi(y))|$. Then we see that $\sigma_\pi(\alpha g u_1) = \{-|g(\phi(y))|\}$. Also, there exists a $U_2 \in P_Y^\circ(y)$ such that $\sigma_\pi((Tg)U_2) = \{Tg(y)\}$. Let $u_2 = T^{-1}U_2$. Since $U_2 \in P_Y^\circ(y)$, by Lemma 3.9, $u_2 \in P_X^\circ(\phi(y))$. Thus, we have that $u_2(\phi(y)) = 1 = \|u_2\|_X$. It follows that $\sigma_\pi(\alpha g u_1 u_2) = \{-|g(\phi(y))|\}$. Thus, we have

$$\|(Tg)\overline{T(\overline{\alpha u_1 u_2})} - 1\|_Y = \|\alpha g u_1 u_2 - 1\|_X = |g(\phi(y))| + 1. \tag{7}$$

Since $u_1, u_2 \in C(X)^{-1}$, Lemma 3.10 shows that $T(\overline{\alpha u_1 u_2}) = (\overline{\alpha u_1 u_2}) \circ \phi$ and $u_2 \circ \phi = Tu_2 = U_2$. So we see that

$$\begin{aligned} \|(Tg)\overline{T(\overline{\alpha u_1 u_2})} - 1\|_Y &= \|(Tg)((\alpha u_1 u_2) \circ \phi) - 1\|_Y \\ &\leq |\alpha| \|(Tg)U_2\|_Y \|u_1 \circ \phi\|_Y + 1 \\ &= \|(Tg)U_2\|_Y + 1 = |Tg(y)| + 1. \end{aligned}$$

Combining this inequality and equality (7) gives that $|g(\phi(y))| \leq |Tg(y))|$. In a similar way, we see $|G(\psi(x))| \leq |T^{-1}G(x)|$, where $G = Tg$ and $x = \phi(y)$. Thus, $|Tg(y))| = |g(\phi(y))|$. Since $\sigma_\pi(\alpha(Tg)U_2) = \{\alpha Tg(y)\}$ and $u_1(\phi(y)) = 1 = \|u_1\|_X$, we have that $\sigma_\pi((Tg)\overline{T(\overline{\alpha u_1 u_2})}) = \sigma_\pi((Tg)((\alpha u_1 u_2) \circ \phi))) = \sigma_\pi(\alpha(Tg)U_2(u_1 \circ \phi)) = \{\alpha Tg(y)\}$. Thus, $\|(Tg)\overline{T(\overline{\alpha u_1 u_2})}\|_Y = |\alpha(Tg(y))| = |g(\phi(y))|$. Equality (7) gives that $-|g(\phi(y))| \in \sigma_\pi((Tg)(\overline{T(\overline{\alpha u_1 u_2})}))$. Hence, $-|g(\phi(y))| = \alpha Tg(y)$ because $\sigma_\pi((Tg)\overline{T(\overline{\alpha u_1 u_2})}) = \{\alpha Tg(y)\}$. Since $-|g(\phi(y))| = \alpha g(\phi(y))$ holds from the definition of α, the equality $Tg(y) = g(\phi(y))$ holds.

Next, we consider the case where $Tg(y) = 0$ and suppose $g(\phi(y)) \neq 0$. Let ε be a positive number with $\varepsilon < |g(\phi(y))|$. Then there exists a $U_1 \in P_Y^\circ(y)$ such that $|(Tg)U_1| < \varepsilon$ on Y. Also, there exists a $u_2 \in P_X^\circ(\phi(y))$ such that $\sigma_\pi(gu_2) = \{g(\phi(y))\}$. Choose a complex number α with $|\alpha| = 1$ such that $\alpha g(\phi(y)) u_2(\phi(y)) = -|g(\phi(y))|$. Let $u_1 = T^{-1}U_1$; then, since $u_1 \in P_X^\circ(\phi(y))$, we see that $\sigma_\pi(\alpha g u_1 u_2) = \{-|g(\phi(y))|\}$. It follows that

$$\|(Tg)\overline{T(\overline{\alpha u_1 u_2})} - 1\|_Y = \|\alpha g u_1 u_2 - 1\|_X = |g(\phi(y))| + 1. \tag{8}$$

Applying Lemma 3.10 to $\overline{\alpha u_1 u_2}$ and u_1,

$$\begin{aligned} \|(Tg)\overline{T(\overline{\alpha u_1 u_2})} - 1\|_Y &= \|\alpha(Tg)(u_1 \circ \phi)(u_2 \circ \phi) - 1\|_Y \\ &= \|\alpha(Tg)U_1(u_2 \circ \phi) - 1\|_Y \\ &\leq \|(Tg)U_1\|_Y \|u_2 \circ \phi\|_Y + 1 < \varepsilon + 1. \end{aligned}$$

This inequality and (8) show that $|g(\phi(y))| < \varepsilon$, and this is a contradiction, so $g(\phi(y)) = 0 = Tg(y)$.

Finally, the case $g(\phi(y)) = 0$. Let $g = T^{-1}G$, $G \in C(Y)$ and $y = \psi(x)$, $x \in X$. Then the hypothesis implies that $T^{-1}G(x) = 0$. Noticing that $\phi = \psi^{-1}$, we can conclude from the argument in the previous paragraph that $Tg(y) = G(\psi(x))$. This completes the proof. $\square$

ENDNOTES

1. An element in $P_X^{\circ}(x)$ may have other points where it takes the value one, not merely at the point x.

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