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GENERALIZED HARMONIC MAPS ON NORMAL ALMOST CONTACT MANIFOLDS

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ABSTRACT. We define the φ-harmonic maps by using adapted connections on normal almost contact manifolds. A number of conditions for a map to be φ-harmonic or generalized φ-pluriharmonic are given and some examples related to these results are found.

1. Introduction. In 1993 Jost and Yau (see [**7**]) introduced the notion of Hermitian harmonic maps as follows. If $f : M \to N$ is a smooth map between a Hermitian manifold (M, J, g) with complex dimension $\dim_{\mathbf{C}} M = m$, equipped with local complex coordinates $(z_\alpha)_{\alpha=1}^m$, and a Riemannian manifold (N, h) of (real) dimension $\dim_{\mathbf{R}} N = n$, with local real coordinates $(x_i)_{i=1}^n$, then f is called a Hermitian harmonic map if

$$g^{\alpha\overline{\beta}}\left[\frac{\partial^2 f^k}{\partial z^\alpha \partial z^{\overline{\beta}}} + {}^N\Gamma_{ij}^k \frac{\partial f^i}{\partial z^\alpha}\frac{\partial f^j}{\partial z^{\overline{\beta}}}\right] = 0, \quad k = \overline{1, n}, \tag{1.1}$$

where ${}^N\Gamma_{ij}^k$ are the Riemann-Christoffel symbols of the Levi-Civita connection on N. This equation does not arise from a variational problem nor has it a divergence form.

Obviously, when (M, J, g) is a Kähler manifold, then (1.1) is the equation of a harmonic map. Also, when M is a Hermitian manifold and N is a Kähler manifold, it is easy to see that any $\pm$holomorphic map $f : M \to N$ is a Hermitian harmonic map.

Since the paper of Jost and Yau, the Hermitian harmonic map was studied by a number of authors, such as Chen [**2**], Grunau and Kühnel [**5**] and Ni [**15**]. Their papers are devoted to the study of the rigidity of Hermitian manifolds by using the notion of Hermitian harmonicity.

In 1999 the Hermitian harmonic morphisms were defined and studied by Loubeau, see [**9**].

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Hermitian harmonic maps can be viewed as a special class of generalized harmonic maps, which are defined and studied by Lu [**10**]. In the following, let us recall some aspects of Lu's work. First let ∇^M and ∇^N be arbitrary smooth connections on the Riemannian manifolds M and N, respectively. Let $f : M \to N$ be a smooth map. If ∇^M and ∇^N are torsion-free, an $f^{-1}TN$-valued bilinear symmetric form, ∇df, can be defined by

$$(\nabla df)(X,Y) = \widetilde{\nabla}^N_X df Y - df(\nabla^M_X Y), \tag{1.2}$$

for any $X, Y \in \chi(M)$, where $\widetilde{\nabla}^N$ is the induced connection on the induced bundle $f^{-1}TN$ over M of TN, defined by $\widetilde{\nabla}^N_X V = \nabla^N_{df X} V$, for any C^∞-section V of $f^{-1}TN$.

Definition 1.1. Let $f : M \to N$ be a smooth map between Riemannian manifolds M and N. Denote $\sigma(f) = \operatorname{trace}(\nabla df)$. If $\sigma(f) = 0$, then f is called a generalized harmonic map.

Remark 1.2. If (M, J, g) is a Hermitian manifold, ∇^M is a holomorphic torsion-free connection on M, and N is a Riemannian manifold with Levi-Civita connection ∇^N; then it is easy to prove that a Hermitian harmonic map $f : M \to N$ is a generalized harmonic map with respect to ∇^M and ∇^N, see [**10, 15**].

Concerning the existence of generalized harmonic maps, Lu claims the following

Theorem 1.3. *Let M and N be compact manifolds. Assume N is strictly negatively curved $(\langle\mu^2,\ \mu\rangle 0)$. Let ∇^N be the Levi-Civita connection on N, and let ∇^M be an arbitrary smooth torsion-free connection on M. Then there exists a generalized harmonic map in each homotopy class of maps from M to N, and it is unique if it doesn't map to a point or a closed geodesic.*

As we have seen, Hermitian harmonic maps is a well-studied subject. It seems interesting to define a similar notion on almost contact manifolds.

In this paper we consider generalized harmonic maps between a normal almost contact manifold and a Riemannian manifold by using on the domain an adapted connection defined by Matzeu and Oproiu in [**11, 12**]. Working this way was suggested by the fact that, as in the case of Hermitian harmonic maps, this adapted connection is compatible with the almost contact structure instead of the Riemannian structure. We also define a notion analogous to φ-pluriharmonic maps, (see [**6**]), in terms of generalized harmonicity.

2. Basic definitions. In this section we briefly recall some definitions and results concerning the almost contact manifolds as they are presented in [**1, 11, 12**]. Let (φ, ξ, η) define an almost contact structure on $(2n+1)$-dimensional manifold M. Then φ is a tensor field of type (1,1), ξ is a vector field on M and η is a 1-form, satisfying the conditions:

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{2.1}$$

From (2.1), one obtains by some algebraic computations

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad \varphi^3 + \varphi = 0. \tag{2.2}$$

Let us consider the manifold $M \times \mathbf{R}$. Denote a vector field on $M \times \mathbf{R}$ by $(X, f(\partial/\partial t))$ where X is tangent to M, t is the usual coordinate on $\mathbf{R}$ and f is a function on $M \times \mathbf{R}$. Define an almost complex structure on $M \times \mathbf{R}$ by

$$J\left(X, f\frac{\partial}{\partial t}\right) = \left(\varphi X - f\xi, \eta(X)\frac{\partial}{\partial t}\right). \tag{2.3}$$

It is easy to check that $J^2 = -I$. If J is integrable we say that the almost contact structure on M is normal. It is proved that the almost contact structure is normal if the tensor field S of type (1,2), defined by

$$S = N_\varphi + d\eta \otimes \xi, \tag{2.4}$$

vanishes, see [**1**], where

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X,Y],$$

for $X, Y \in \chi(M)$, is the Nijenhuis tensor field of φ.

If g is a (semi)-Riemannian metric on M such that

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y),$$

for any $X, Y \in \chi(M)$, then we say that (φ, ξ, η, g) is a metric almost contact structure and M is called a metric almost contact manifold. The metric g is called an associated (semi)-Riemannian metric. The existence of associated (semi)-Riemannian metrics to any almost contact structure is proved in [**1**].

Consider the vector subbundles in the tangent bundle of M defined by $\mathcal{H} = \ker\eta$ and $\mathcal{V} = \operatorname{span}\{\xi\}$. Then $TM = \mathcal{H} \oplus \mathcal{V}$, and let h, v be the projectors corresponding to this direct sum decomposition. That is,

$$v = \eta \otimes \xi, \quad h = I - v, \quad h^2 = h, \quad v^2 = v, \quad hv = vh = 0.$$

Then

$$\varphi^2 = -h, \quad h\varphi = \varphi h = \varphi, \quad \varphi v = v\varphi = 0.$$

The almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is called a cosymplectic manifold if it is normal, $d\Omega = 0$ and $d\eta = 0$; (1,2)-symplectic manifold if

$$d\Omega(X, \varphi X, Y) = 0, \quad X \in \Gamma(\mathcal{H}), \quad Y \in \chi(M),$$

where Ω is the fundamental 2-form associated to metric g, which is defined by $\Omega(X,Y) = g(X, \varphi Y)$, for any $X, Y \in \chi(M)$. If $\Omega = d\eta$ then M is called a contact metric manifold. A contact metric manifold which is normal is called a Sasakian manifold, see [**1**].

Definition 2.1. An adapted connection to the almost contact structure (φ, ξ, η) is a linear connection ∇^M on M, such that, for any $X, Y \in \chi(M)$,

$$\begin{cases} (\nabla^M_X \varphi)Y = \eta(Y)hX + (1/4)[d\eta(\varphi X, hY) - d\eta(X, \varphi Y)]\xi, \\ (\nabla^M_X \eta)(Y) = (1/4)[d\eta(X,Y) + d\eta(\varphi X, \varphi Y)], \\ \nabla_X \xi = \varphi X - (1/4)\, d\eta(X, \xi)\xi. \end{cases} \tag{2.5}$$

Note that if M is a Sasakian manifold, then the Levi-Civita connection on M is an adapted connection.

In [**11**] it is proved that at least one adapted connection exists on an almost contact manifold (M, φ, ξ, η).

Finally, we must recall the following result related to the adapted connections, ([**11**]).

Theorem 2.2. *The almost contact structure (φ, ξ, η) is normal if and only if a torsion-free adapted connection exists.*

Remark 2.3. If ∇^M is a torsion-free adapted connection to the normal almost contact structure (φ, ξ, η), then the conditions (2.5) becomes

$$\begin{cases} (\nabla^M_X \varphi)Y = \eta(Y)hX + (1/2)\, d\eta(\varphi X, Y)\xi, \\ (\nabla^M_X \eta)(Y) = (1/2)\, d\eta(X, Y),\ \nabla_X \xi = \varphi X. \end{cases} \tag{2.6}$$

3. A class of generalized harmonic maps.

Definition 3.1. We call a smooth map $f : M \to N$, between a normal almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ and a Riemannian manifold (N, h) a φ-harmonic map if f is a generalized harmonic map with respect to an adapted torsion-free connection ∇^M on M and Levi-Civita connection ∇^N on N.

The notion of φ-harmonicity is well defined since we have the following.

Proposition 3.2. *The definition of φ-harmonic maps does not depend on the chosen adapted torsion-free connection on M.*

Proof. Let ∇^M be an adapted torsion-free connection to the normal almost contact structure on M. If we consider the decomposition $TM = \mathcal{H} \oplus \mathcal{V}$ as in the previous section, one obtains, from Definition 3.1, for any $X \in \Gamma(\mathcal{H})$,

$$(\nabla^M_X \varphi)\varphi X = 0, \qquad (\nabla^M_{\varphi X} \varphi)X = 0.$$

It follows that

$$\nabla^M_X X + \nabla^M_{\varphi X}\varphi X = \varphi(\nabla^M_{\varphi X} X - \nabla^M_X \varphi X) = \varphi[\varphi X, X]. \tag{3.1}$$

Let $\{e_i, \varphi e_i, \xi\}$ be an orthonormal (with respect to associated metric g) φ-basis in M, and let ${}^1\nabla^M$, ${}^2\nabla^M$ be two adapted torsion-free connections on M. If $f : M \to N$ is a smooth map between M and a Riemannian manifold (N, h) with Levi-Civita connection ∇^N, then

$$\begin{aligned}\sigma_1(f) = \operatorname{trace}(\nabla_1\, df) = \sum_{i=1}^{n}[(\widetilde{\nabla}^N_{e_i} df e_i + \widetilde{\nabla}^N_{\varphi e_i} df \varphi e_i + \widetilde{\nabla}^N_{\xi} df \xi) \\ - df({}^1\nabla^M_{e_i} e_i + {}^1\nabla^M_{\varphi e_i}\varphi e_i + {}^1\nabla^M_{\xi}\xi)],\end{aligned}$$

and

$$\begin{aligned}\sigma_2(f) = \operatorname{trace}(\nabla_2\, df) = \sum_{i=1}^{n}[(\widetilde{\nabla}^N_{e_i} df e_i + \widetilde{\nabla}^N_{\varphi e_i} df \varphi e_i + \widetilde{\nabla}^N_{\xi} df \xi) \\ - df({}^2\nabla^M_{e_i} e_i +{}^2 \nabla^M_{\varphi e_i}\varphi e_i +{}^2 \nabla^M_{\xi}\xi)],\end{aligned}$$

where $\dim M = 2n+1$, and $df : TM \to TN$ is the tangent map induced by f. Since ${}^1\nabla^M_{\xi}\xi ={}^2 \nabla^M_{\xi}\xi = 0$, from (3.1) one obtains $\sigma_1(f) = \sigma_2(f)$. Thus, f is a φ-harmonic map with respect to ${}^1\nabla^M$ if and only if it is φ-harmonic with respect to ${}^2\nabla^M$. □

Remark 3.3. The associated metric g does not interfere in the definition of φ-harmonic maps. Hence, this notion is strictly related to the normal almost contact structure on M. We still continue to consider an associated (semi)-Riemannian metric on the manifold M because in the proofs of some results we will use the orthonormal (with respect to this metric) φ-basis in M. However this metric will be arbitrarily chosen.

Definition 3.4. Let $f : M \to N$ be a smooth map between an almost contact manifold (M, φ, ξ, η) and an almost complex manifold (N, J). If $df\varphi = \pm J df$, we call f a $\pm(\varphi, J)$-holomorphic map.

Definition 3.5. Let $f : M \to N$ be a smooth map between two almost contact manifolds (M, φ, ξ, η) and (N, ψ, ζ, θ). If $df\varphi = \pm\psi df$ we call f a $\pm(\varphi, \psi)$-holomorphic map.

Now, we can state the following

Theorem 3.6. *Let $(M, \varphi, \xi, \eta, g)$ be a normal almost contact metric manifold and (N, J, h) a Hermitian manifold with Levi-Civita connection ∇^N satisfying*

$$(\nabla^N_X J)X = 0, \tag{3.2}$$

for any $X \in \chi(N)$. Then a $\pm(\varphi, J)$-holomorphic map $f : M \to N$ is a φ-harmonic map.

Proof. It is sufficient to study the case when f is a (φ, J)-holomorphic map. Consider an orthonormal φ-basis $\{e_i, \varphi e_i, \xi\}$ in M, $i = \overline{1, n}$, $dim\ M = 2n + 1$. Since $df\varphi = Jdf$, using (3.2) and the fact that the Levi-Civita connection on N is torsion-free, one obtains

(3.3)

$$\begin{aligned}
\nabla^N_{df\varphi e_i} df\varphi e_i &= \nabla^N_{Jdfe_i} Jdfe_i = J\nabla^N_{Jdfe_i} dfe_i \\
&= J(\nabla^N_{dfe_i} Jdfe_i + [Jdfe_i, dfe_i]) \\
&= -\nabla^N_{dfe_i} dfe_i + Jdf[\varphi e_i, e_i] = -\nabla^N_{dfe_i} dfe_i + df\varphi[\varphi e_i, e_i],
\end{aligned}$$

for any $i = \overline{1, n}$, and

$$\nabla^N_{df\xi} df\xi = -J^2\nabla^N_{df\xi} df\xi = -J\nabla^N_{df\xi} Jdf\xi = -J\nabla^N_{df\xi} df\varphi\xi = 0. \tag{3.4}$$

From (3.1) it follows that

$$\nabla^M_{e_i} e_i + \nabla^M_{\varphi e_i} \varphi e_i = \varphi[\varphi e_i, e_i], \quad i = \overline{1, n}. \tag{3.5}$$

Taking into account (3.3), (3.4), (3.5) and $\nabla^M_\xi \xi = 0$, we have

$$\begin{aligned}
\sigma(f) = \operatorname{trace}(\nabla df) &= \sum_{i=1}^n (\nabla^N_{dfe_i} dfe_i + \nabla^N_{df\varphi e_i} df\varphi e_i) + \nabla^N_{df\xi}\, df\xi \\
&\quad - df\left(\sum_{i=1}^n (\nabla^M_{e_i} e_i + \nabla^M_{\varphi e_i} \varphi e_i) + \nabla^M_\xi \xi\right) = 0.
\end{aligned}$$

Hence f is a φ-harmonic map. $\square$

Proposition 3.7. *Let $(M, \varphi, \xi, \eta, g)$ be a normal almost contact metric manifold with $(d\eta)_x \neq 0$, for any $x \in M$, and $(N, \psi, \zeta, \theta, h)$ an almost contact metric manifold. Let $f : M \to N$ be a $\pm(\varphi, \psi)$-holomorphic map such that*

$$d\theta(dfX, df\xi) = 0, \tag{3.6}$$

for any $X \in \chi(M)$. Then $df\xi = a\zeta$, where $a \in \mathbf{R}$ is a constant.

Proof. As above, it is sufficient to consider the case $df\varphi = \psi df$. One obtains that $df\xi = a\zeta$, where $a : M \to \mathbf{R}$ is a function on M. Define the 1-form $f^*\theta$ on M, by

$$f^*\theta(X) = \theta(dfX),$$

for any $X \in \chi(M)$.

Then it is easy to see that $f^*\theta = a\eta$. By differentiating, one obtains

$$f^*d\theta = da \wedge \eta + ad\eta. \tag{3.7}$$

From the normality of M, we have $d\eta(X, Y) = \eta([\varphi X, \varphi Y])$, for any $X, Y \in \chi(M)$. Hence, from (3.6), by computing (3.7) in (X, ξ), with $X \in \chi(M)$, it follows that $da(X) = L_\xi a \cdot \eta(X)$, where L denotes the Lie derivative. One obtains $da \wedge \eta = 0$, and then $da \wedge d\eta = 0$. Thus, $L_\xi a \cdot \eta \wedge d\eta = 0$. In (X, Y, ξ), with $X, Y \in M$, we have $L_\xi a \cdot d\eta(X, Y) = 0$. Since $(d\eta)_x \neq 0$, for any $x \in M$, we conclude that $L_\xi a = 0$, that is, $da = 0$. Thus, a is a constant. □

It is easy to see that, if the target manifold, (N, ψ, ζ, θ), is a contact manifold or a normal almost contact manifold, then for a $\pm(\varphi, \psi)$-holomorphic map, $f : M \to N$, equation (3.6) holds. We can state the following

Corollary 3.8. *Let M be a manifold as above and $(N, \psi, \zeta, \theta, h)$ a contact manifold or a normal almost contact manifold. If $f : M \to N$ is a $\pm(\varphi, \psi)$-holomorphic map, then $df\xi = a\zeta$, where $a \in \mathbf{R}$ is a constant.*

Theorem 3.9. *Let $(M, \varphi, \xi, \eta, g)$ be a normal almost contact metric manifold with $(d\eta)_x \neq 0$, for any $x \in M$, and $(N, \psi, \zeta, \theta, h)$ a $(1, 2)$-symplectic manifold. If $f : M \to N$ is a $\pm(\varphi, \psi)$-holomorphic map satisfying* (3.6)*, then f is a φ-harmonic map.*

Proof. In [**1**] the author proves the following formula for the covariant derivative of ψ for a general almost contact metric structure (ψ, ζ, θ, h),

(3.8)
$$\begin{aligned} 2h((\nabla_X \psi)Y, Z) = {} & 3d\Omega(X, \varphi Y, \varphi Z) - 3d\Omega(X, Y, Z) + h(N_\psi(Y, Z), \psi X) \\ & + ((L_{\psi Y}\theta)(Z) - (L_{\psi Z}\theta)(Y))\theta(X) \\ & + 2d\theta(\psi Y, X)\theta(Z) - 2d\theta(\psi Z, X)\theta(Y), \end{aligned}$$

where L denotes the Lie derivative and Ω is the fundamental 2-form associated to the structure.

Consider $\mathcal{H}' = \ker\theta \subset TN$. Then, for $\widetilde{X} \in \Gamma(\mathcal{H}')$, one obtains, from (3.8), since $(N, \psi, \zeta, \theta, h)$ is a $(1,2)$-symplectic manifold,

$$2h((\nabla^N_{\widetilde{X}}\psi)\psi\widetilde{X}, \widetilde{Z}) = h(N_\psi(\psi\widetilde{X}, \widetilde{Z}), \psi\widetilde{X})$$

and

$$2h((\nabla^N_{\psi\widetilde{X}}\psi)\widetilde{X}, \widetilde{Z}) = -h(N_\psi(\widetilde{X}, \widetilde{Z}), \widetilde{X}),$$

for any $\widetilde{Z} \in \chi(N)$. Hence, $2h((\nabla^N_{\widetilde{X}}\psi)\psi\widetilde{X} - (\nabla^N_{\psi\widetilde{X}}\psi)\widetilde{X}, \widetilde{Z}) = 0$, for any $\widetilde{Z} \in \chi(N)$. That means $(\nabla^N_{\widetilde{X}}\psi)\psi\widetilde{X} - (\nabla^N_{\psi\widetilde{X}}\psi)\widetilde{X} = 0$. Thus,

$$\nabla^N_{\widetilde{X}}\widetilde{X} + \nabla^N_{\psi\widetilde{X}}\psi\widetilde{X} = \psi[\psi\widetilde{X}, \widetilde{X}]. \tag{3.9}$$

It can easily be verified that $X \in \Gamma(\mathcal{H})$, $\mathcal{H} = \ker\eta$, implies $dfX \in \Gamma(\mathcal{H}')$. Since from Proposition 3.7 we have that $\nabla^N_{df\xi}df\xi = 0$, by choosing an orthonormal φ-basis in M, as in the proof of Proposition 3.6 one obtains $\sigma(f) = 0$, which means that f is a φ-harmonic map. □

From Proposition 3.9 and Corollary 3.8, one obtains

Corollary 3.10. *Let M be a manifold as above and $(N, \psi, \zeta, \theta, h)$ a contact manifold or a normal almost contact manifold. If $f : M \to N$ is a $\pm(\varphi, \psi)$-holomorphic map, then f is a φ-harmonic map.*

Concerning the possibility that a φ-harmonic map between a normal almost contact manifold and a Hermitian (or a $(1,2)$-symplectic) manifold is $\pm(\varphi, J)$-holomorphic (or $\pm(\varphi, \psi)$-holomorphic), by Theorem 1.3, Theorem 1.6 and Theorem 3.9, we can state:

Theorem 3.11. *Let M be a compact normal almost contact manifold. Assume (N, J, h) is a compact Hermitian manifold strictly negatively curved ($\langle\mu^2, \mu\rangle 0$) such that the Levi-Civita connection on N verifies* (3.2). *If $f : M \to N$ is a φ-harmonic map homotopic with a $\pm(\varphi, J)$-holomorphic map, $f_0 : M \to N$, such that neither f nor f_0 map to a point or a closed geodesic, then $f = f_0$.*

Theorem 3.12. *Let $(M, \varphi, \xi, \eta, g)$ be a compact normal almost contact manifold with $(d\eta)_x \neq 0$, for any $x \in M$. Assume $(N, \psi, \zeta, \theta, h)$ is a compact* (1,2)*-symplectic manifold strictly negatively curved ($\langle\mu^2, \mu\rangle 0$). If $f : M \to N$ is a φ-harmonic map homotopic with a $\pm(\varphi, \psi)$-holomorphic map, $f_0 : M \to N$, satisfying condition* (3.6)*, such that neither f nor f_0 map to a point or a closed geodesic, then $f = f_0$.*

Remark 3.13. All results in this section remain valid in the case when the metric on the target manifold is not Riemannian but is a semi-Riemannian metric.

4. Hermitian harmonic maps on tangent bundle of a normal almost contact manifold. In order to find relations between Hermitian harmonic maps and φ-harmonic maps, let us consider M to be a differentiable manifold of dimension m and $\pi : TM \to M$ its tangent bundle. Then TM can be organized as a $2m$-dimensional manifold as follows. A local coordinate neighborhood $(U; x^i)$, $i = \overline{1, 2m}$, in M, induces a local coordinate neighborhood $(\pi^{-1}(U); x^i, y^j)$, $i, j = \overline{1, m}$, on TM, where we denote $x^i \circ \pi$ by x^i and y^j are the coordinates of the vectors on $\pi^{-1}(U)$ in natural basis $\{\partial/\partial x^i\}_{i=1}^m$.

If ω is a differentiable 1-form on M, then it can be regarded as a function on TM which we denote by $\iota\omega$.

If f is a function on M, we define the vertical lift f^V of f by $f^V = f \circ \pi$ and the complete lift f^C of f by $f^C = \iota(df)$. We have $f^C = y^i(\partial f/\partial x^i) = y^i \partial_i f = \partial f$ with respect to the induced coordinates in TM, where ∂_i denote $(\partial/\partial x^i)$ and ∂ denote $y^i\partial_i$. The vertical lift $(df)^V$ of the 1-form df is defined by $(df)^V = df^V$. For two functions f and g on M we have $(gdf)^V = g^V(df)^V = g^V(df^V)$.

Let $X = X^i(\partial/\partial x^i)$ be a vector field on M. We define the vertical lift X^V of X by $X^V(\iota\omega) = (\omega(X))^V$, ω being an arbitrary 1-form on M, and the complete lift X^C of X by $X^C f^C = (Xf)^C$, f being an arbitrary function on M. With respect to the induced coordinates in TM, one obtains

$$X^V = X^i \frac{\partial}{\partial y^i}, \qquad X^C = X^i \frac{\partial}{\partial x^i} + \partial X^i \frac{\partial}{\partial y^i}.$$

Let $\eta = \eta_i dx^i$ be a differentiable 1-form on M. We define the vertical lift η^V of η by $\eta^V = \eta_i^V (dx_i)^V$ and the complete lift η^C of η by $\eta^C(X^C) = (\eta(X))^C$, X being an arbitrary vector field on M. Then we have, with respect to the induced coordinates in TM,

$$\eta^V = \eta_i dx^i, \qquad \eta^C = \partial\eta_i dx^i + \eta_i dy^i.$$

The vertical and the complete lifts of a tensor field on M can be defined, using the conditions

$$\begin{aligned} (P+Q)^V &= P^V + Q^V, & (P\otimes Q)^V &= P^V \otimes Q^V, \\ (P+Q)^C &= P^C + Q^C, & (P\otimes Q)^C &= P^C \otimes Q^V + P^V \otimes Q^C, \end{aligned}$$

where P, Q are tensor fields on M. Let $\varphi = \varphi_i^h(\partial/\partial x^h) \otimes dx^i$ be a tensor field of type (1,1) on M. Then one obtains, for the complete lift φ^C of φ,

$$\varphi^C = \varphi_i^h \frac{\partial}{\partial x^h} \otimes dx^i + \varphi_i^h \frac{\partial}{\partial y^h} \otimes dy^i + \partial\varphi_i^h \frac{\partial}{\partial y^h} \otimes dx^i,$$

with respect to the induced coordinates in TM.

Let $g = g_{ij} dx^i \otimes dx^j$ be a tensor field of type $(0,2)$ on M. Then one obtains, for the complete lift g^C of g,

$$g^C = \partial g_{ij} dx^i \otimes dx^j + g_{ij} dx^i \otimes dy^j + g_{ij} dy^i \otimes dx^j$$

with respect to the induced coordinates in TM.

Let (M, φ, ξ, η) be a $(2n+1)$-dimensional almost contact manifold, and let TM be its tangent bundle. Define the tensor field J^M, of type $(1,1)$, on TM, by

$$J^M = \varphi^C - \eta^C \otimes \xi^C + \eta^V \otimes \xi^V. \tag{4.1}$$

It is easy to check that $(J^M)^2 = -I$, see [**17**]. Thus, (TM, J^M) is a $(4n+2)$-dimensional almost complex manifold. After a straightforward computation it can be proved that if the almost contact structure on M is normal then J^M is integrable, (see also Proposition 4.1 in [**3**], together with [**4**]). We just proved

Proposition 4.1. *If (M, φ, ξ, η) is a normal almost contact manifold, then (TM, J^M, G) is a Hermitian manifold, where G is an arbitrarily chosen Hermitian metric on TM.*

Note that every almost complex manifold admits a Hermitian metric, see, for example, [**1**].

Now, let us consider a Kähler manifold (N, J, h), and let (TN, J^N, H) be its tangent bundle endowed with the tensor field $J^N = J^C$, of type $(1,1)$ and with semi-Riemannian metric $H = h^C$. From $N_{J^C} = (N_J)^C$ and $\nabla^C J^C = (\nabla^N J)^C$, where ∇^C is the torsion-free torsion corresponding to semi-Riemannian structure on TN, see [**17**], it follows easily that

Proposition 4.2. *If (N, J, h) is a Kähler manifold, then (TN, J^N, H) is a complex manifold such that $\nabla^C J^N = 0$.*

Next, let $f : M \to N$ be a smooth map between an almost contact manifold (M, φ, ξ, η) and an almost complex manifold (N, J). Denote by $F = df : TM \to TN$ the tangent map induced by f and by $dF : TTM \to TTN$ the tangent map induced by F. Assume that $dFJ^M = \pm J^N dF$. Obviously, it will be sufficient to study only the case when $dFJ^M = J^N dF$. For a vector field $X \in \Gamma(\mathcal{H})$, $\mathcal{H} = \ker \eta \subset TM$, we have $dFJ^M X^V = J^N dFX^V$. But $dFJ^M X^V = dF(\varphi X)^V = (df\varphi X)^V$ and $J^N dFX^V = J^N (dfX)^V = (JdfX)^V$, see [**17**]. Hence, $df\varphi X = JdfX$, for any $X \in \Gamma(\mathcal{H})$.

Since $dFJ^M \xi^C = J^N dF\xi^C$, $dFJ^M \xi^C = dF\xi^V = (df\xi)^V$ and $J^N dF\xi^C = J^N (df\xi)^C = (Jdf\xi)^C$, it follows that $(df\xi)^V = (Jdf\xi)^C$. Taking into account the expressions for vertical and complete lifts of vector fields in local coordinates, one obtains $Jdf\xi = 0$ and that means $df\varphi\xi = Jdf\xi$.

Conversely, if we suppose $df\varphi = Jdf$, it is easy to verify that $dFJ^M X^C = J^N dFX^C$ and $dFJ^M X^V = J^N dFX^V$, for any vector field $X \in \chi(M)$. Together, all these results give

Proposition 4.3. *If $f : M \to N$ is a smooth map between two manifolds as above, then $df\varphi = \pm Jdf$ if and only if $dFJ^M = \pm J^N dF$.*

From Proposition 3.6, Proposition 4.1, Proposition 4.2 and Proposition 4.3, one obtains

Proposition 4.4. *Let $f : M \to N$ be a smooth map between a normal almost contact manifold (M, φ, ξ, η) and a Kähler manifold (N, J, h), and let $F = df : TM \to TN$ be the tangent map induced by f, where TM and TN are endowed with the Hermitian structures (J^M, G) and (J^N, H) as above. If $df\varphi = \pm Jdf$, then F is a Hermitian harmonic map and if $dFJ^M = \pm J^N dF$, then f is a φ-harmonic map.*

In the following, let us recall some results from [**3**]. Consider an almost complex metric manifold $(P, \psi, \zeta, \theta, h)$. On the tangent bundle TP we define

$$H = h^C + (\theta^V - \theta^C) \otimes (\theta^V - \theta^C).$$

It is easy to verify that H is a semi-Riemannian Hermitian metric with respect to complex structure

$$J^P = \psi^C - \theta^C \otimes \zeta^C + \theta^V \otimes \zeta^V.$$

If P is a cosymplectic manifold, then $\nabla^H J^P = 0$, where ∇^H is the Levi-Civita connection corresponding to H.

If $f : M \to P$ is a smooth map between two almost contact manifolds and $F = df : (TM, J^M) \to (TP, J^P)$, then it is proved that $dfJ^M = \pm J^P dF$ if and only if $df\varphi = \psi df$ and $df\xi = a\zeta$, $a \in \mathbf{R}$, see [**3**].

Using these results and Proposition 3.7, one obtains

Proposition 4.5. *Let $f : M \to P$ be a smooth map between a normal almost contact manifold, (M, φ, ξ, η), such that $(d\eta)_x \neq 0$,* *for any $x \in M$, and a cosymplectic manifold, $(P, \psi, \zeta, \theta, h)$, and let $F = df : (TM, J^M, G) \to (P, J^P, H)$ be the induced tangent map. If $df\varphi = \pm\psi df$, then F is a Hermitian harmonic map.*

Proposition 4.6. *If in Proposition 4.5 we take M to be only a normal almost contact manifold, then $dFJ^M = \pm J^P dF$ implies that f is a φ-harmonic map.*

5. Generalized φ-pluriharmonic maps. In [**10**] the concept is introduced of a φ-pluriharmonic map from an almost contact manifold into a Riemannian manifold as follows.

Let $(M, \varphi, \xi, \eta, N)$ be an almost contact metric manifold, and let N be a Riemannian manifold. If ∇ is the Levi-Civita connection on M and ∇^N is the Levi-Civita connection on N, then the second fundamental form, α, of a smooth map $f : M \to N$ is defined by

$$\alpha(X, Y) = \widetilde{\nabla}^N_X df Y - df(\nabla_X Y), \quad X, Y \in \chi(M).$$

The map f is called φ-pluriharmonic if α satisfies

$$\alpha(X, Y) + \alpha(\varphi X, \varphi Y) = 0, \quad X, Y \in \chi(M).$$

In this section we consider an analogous notion by using an adapted connection on M instead of Levi-Civita connection.

Definition 5.1. Let (M, φ, ξ, η) be a normal almost contact metric manifold, and let N be a Riemannian manifold. A smooth map $f : M \to N$ is said to be generalized φ-pluriharmonic if

$$(\nabla df)(X, Y) + (\nabla df)(\varphi X, \varphi Y) = 0, \quad X, Y \in \chi(M), \tag{5.1}$$

where $(\nabla df)(X, Y) = \widetilde{\nabla}^N_X df Y - df(\nabla^M_X Y)$, $X, Y \in \chi(M)$, ∇^N the Levi-Civita connection on N and ∇^M a torsion-free adapted connection on M. If (5.1) holds only for $X, Y \in \Gamma(\mathcal{H})$, $\mathcal{H} = \ker \eta \subset \chi(M)$, then f is called generalized $\mathcal{H}$-pluriharmonic.

Proposition 5.2. *The definition of a generalized φ-pluriharmonic map does not depend on the chosen torsion-free adapted connection on M.*

Proof. Let ∇^M be any torsion-free adapted connection on a normal almost contact manifold (M, φ, ξ, η). Due to the definition of adapted connections, we get

$$(\nabla^M_X \varphi)(Y) = \frac{1}{2}\, d\eta(\varphi X, \varphi Y)\xi$$

and

$$(\nabla^M_{\varphi X} \varphi)Y = \eta(Y)\varphi X - \frac{1}{2}\, d\eta(X, Y)\xi.$$

Hence, since ∇^M is torsion-free,

$$\nabla^M_X Y + \nabla^M_{\varphi X}\varphi Y = \varphi[\varphi X, Y] + [X, Y] + Y(\eta(X))\xi. \tag{5.2}$$

Thus, the expression of $(\nabla df)(X, Y) + (\nabla df)(\varphi X, \varphi Y)$ remains unchanged for any torsion-free adapted connection on M. □

Proposition 5.3. *Let $f : M \to N$ be a generalized φ-pluriharmonic map from a normal almost contact manifold into a Riemannian manifold. Then f is a φ-harmonic map.*

Proof. Let g be an associated metric on M, and let $\{e_i, \varphi e_i, \xi\}$ be an orthonormal φ-basis in M. Since f is generalized φ-pluriharmonic we have

$$(\nabla df)(e_i, e_i) + (\nabla df)(\varphi e_i, \varphi e_i) = 0, \quad (\nabla df)(\xi, \xi) = 0,$$

for $i = \overline{1, n}$, where $\dim M = 2n + 1$. One obtains

$$\operatorname{trace}(\nabla df) = \sum_{i=1}^{n}[(\nabla df)(e_i, e_i) + (\nabla df)(\varphi e_i, \varphi e_i)] + (\nabla df)(\xi, \xi) = 0,$$

and that means f is a φ-harmonic map. □

Next, we have

Proposition 5.4. *Let (M, φ, ξ, η) be a normal almost contact metric manifold and (N, J, h) a Kähler manifold. Then a $\pm(\varphi, J)$-holomorphic* *map $f : M \to N$ is generalized $\mathcal{H}$-pluriharmonic but it is not generalized φ-pluriharmonic.*

Proof. Due to (5.2),

$$\nabla^M_X Y + \nabla^M_{\varphi X}\varphi Y = \varphi[\varphi X, Y] + [X, Y],$$

for any $X, Y \in \Gamma(\mathcal{H})$. On the other hand, since N is Kähler and f is a $\pm(\varphi, J)$-holomorphic map, one easily obtains

$$\begin{aligned}\nabla^N_{df X} df Y + \nabla^N_{df \varphi X} df \varphi Y &= -J[J df X, df Y] - [df X, df Y]\\ &= -df(\varphi[\varphi X, Y] + [X, Y]),\end{aligned}$$

for any $X, Y \in \chi(M)$. We have

$$\begin{aligned}(\nabla df)(X, Y) + (\nabla df)(\varphi X, \varphi Y) &= \nabla^N_{df X} df Y + \nabla^N_{df \varphi X} df \varphi Y\\ &\quad - df(\nabla^M_X Y + \nabla^M_{\varphi X}\varphi Y) = 0,\end{aligned}$$

for any $X, Y \in \Gamma(\mathcal{H})$.

For $X \in \chi(M)$, we get

$$\begin{aligned}(\nabla df)(X, \xi) + (\nabla df)(\varphi X, \varphi\xi) &= -df(\nabla^M_X \xi) = -df \varphi X\\ &= \mp J df X \neq 0,\end{aligned}$$

since f is $\pm(\varphi, J)$-holomorphic. So f is generalized $\mathcal{H}$-pluriharmonic but it is not generalized φ-pluriharmonic. □

Proposition 5.5. *Let $(M, \varphi, \xi, \eta, g)$ be a normal almost contact metric manifold and $(N, \psi, \zeta, \theta, h)$ a Sasakian or a cosymplectic manifold. If $f : M \to N$ is a $\pm(\varphi, \psi)$-holomorphic map, then f is generalized $\mathcal{H}$-pluriharmonic. Moreover, if $(d\eta)_x \neq 0$, for any $x \in M$, N is a Sasakian manifold and f is $\pm(\varphi, \psi)$-holomorphic with $df\xi = \pm\zeta$, then f is generalized φ-pluriharmonic. If N is a cosymplectic manifold, then f cannot be generalized φ-pluriharmonic.*

Proof. The fact that f is generalized $\mathcal{H}$-pluriharmonic can be obtained exactly as in the proof of Proposition 5.4, since for a cosymplectic or a Sasakian manifold, N, one obtains easily that, (see [**1, 10**]),

$$\nabla^N_X Y + \nabla^N_{\varphi X}\varphi Y = \psi[\psi X, Y] + [X, Y],$$

for any $X, Y \in \Gamma(\mathcal{H}')$, where ∇^N is the Levi-Civita connection on N.

On the other hand, if $(d\eta)_x \neq 0$, for any $x \in M$, then $df\xi = a\zeta$, $a \in \mathbf{R}$, by the meaning of Proposition 3.7. Assume that N is a Sasakian manifold. Then, for any $X \in \chi(M)$, we have

$$\begin{aligned}(\nabla df)(X,\xi) &= \widetilde{\nabla}^N_X df\xi - df\nabla^M_X\xi = \nabla^N_{dfX} df\xi + df\varphi X\\ &= -a\psi dfX + df\varphi X = (\mp a + 1)df\varphi X.\end{aligned}$$

Thus, f is generalized φ-pluriharmonic if and only if $a = \pm 1$. If N is a cosymplectic manifold, $\nabla^N_X\zeta = 0$, for any $X \in \chi(N)$, and we obtain $(\nabla df)(X,\xi) = df\varphi X$, for any $X \in \chi(N)$, then f cannot be generalized φ-pluriharmonic. □

6. Examples. We end by giving some examples of φ-harmonic maps.

1. Let (M, J, g) be a Hermitian manifold such that $(\nabla^M_X J)X = 0$, for any vector field $X \in \chi(M)$. On $M \times \mathbf{R}$, set $\eta = dt$ and $\xi = \partial/\partial t$, where t is the coordinate on $\mathbf{R}$, and define the tensor field φ of type (1,1) by $\varphi\xi = 0$ and $\varphi(X, 0) = JX$ for $X \in \chi(M)$. Then (φ, ξ, η) is an almost contact metric structure on $M \times \mathbf{R}$ with $d\eta = 0$.

Since (M, J, g) is Hermitian, we have $N_J = 0$, where N_J is the Nijenhuis tensor of J. Then, for the Nijenhuis tensor of φ, N_φ, one obtains $N_\varphi((X,0),(Y,0)) = N_J(X,Y)$ and $N_\varphi((X,0),(0,(\partial/\partial t))) = 0$ for any $X, Y \in \chi(M)$. Thus $N_\varphi = 0$, and since $d\eta = 0$, one obtains that $(M \times \mathbf{R}, \varphi, \xi, \eta)$ is a normal almost contact manifold.

If $\pi : M \times \mathbf{R} \to M$ is the canonical projection, then π is a (φ, J)-holomorphic map and, by Theorem 3.6, a φ-harmonic map. If M is a Kähler manifold, then π is a generalized $\mathcal{H}$-pluriharmonic map.

2. Now let us recall some results obtained in [**13, 14**]. First, let $\pi : M^{2n+1} \to N^{2n+2}$ be a principal circle bundle over a complex manifold N, and assume that there exists a connection form η such that $d\eta = \pi^*\Psi$, where Ψ is a form of bidegree $(1,1)$ on N. Define $\varphi X = \widetilde{\pi} J d\pi X$, where J is the almost contact structure on N and $\widetilde{\pi}$ is the horizontal lift with respect to η, (see [**1**] for details). If ξ is a vertical vector field with $\eta(\xi) = 1$, then (φ, ξ, η) is a normal almost contact manifold, [**13**].

Conversely, if $(M^{2n+1}, \varphi, \xi, \eta)$ is a compact normal almost contact with ξ regular, that is, for any point $p \in N$ there exists a cubic neighborhood of p such that any integral curve of ξ passes through the neighborhood at most once, then M is the bundle space of a principal circle bundle $\pi : M^{2n+1} \to N^{2n+2}$ over a complex manifold N. Moreover, η is a connection form and the 2-form Ψ on N such that $d\eta = \pi^*\Psi$ is of bidegree $(1,1)$, [**14**].

Note that this fibration is similar to the well known Boothby-Wang fibration of compact regular manifolds.

Next, assume that M^{2n+1} and N^{2n+2} are two manifolds as above and in addition N is Hermitian with $(\nabla^N_X J)X = 0$, for any vector field $X \in \chi(N)$. Since, from the definition of horizontal lifts, it follows easily that π is (φ, J)-holomorphic, and then π is φ-harmonic. If N is Kähler, then π is generalized $\mathcal{H}$-pluriharmonic.

3. Let $(M^{2m+1}, \varphi, \xi, \eta)$ and $(N^{2n+1}, \psi, \zeta, \theta, h)$ be two normal almost contact metric manifolds, and consider Hermitian manifolds $(M \times \mathbf{R}, J_1, G)$, $(N \times \mathbf{R}, J_2, H)$ where structure tensors J_1 and J_2 are given by (2.3) and G, H are the product metrics $G = g + dt^2$, $H = h + dt^2$, which are Hermitian, see [**1**]. Assume that $(d\eta)_x \neq 0$, for any $x \in M$, and let $F : M \times \mathbf{R} \to N \times \mathbf{R}$ be a smooth map such that $F(x,t) = (f(x), t)$, for any $(x,t) \in M \times \mathbf{R}$, where $f : M \to N$. Then it is easy to obtain that F is $\pm$holomorphic if and only if f is a (φ, ψ)-holomorphic map and, moreover, $df\xi = \zeta$. Thus, if F is $\pm$holomorphic, then f is a φ-harmonic map. If $N \times \mathbf{R}$ is Kähler and f is $\pm(\varphi, \psi)$-holomorphic with $df\xi = \pm\zeta$, then F is Hermitian harmonic. Finally, assume that H is conformally equivalent to a Kähler metric H' such that $H = e^{-2t}H'$. In this case N is a Sasakian manifold, see [**1**]. Hence, F is $\pm$holomorphic implies that f is generalized φ-pluriharmonic, by Proposition 5.5.

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