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A VORONOVSKAYA-TYPE THEOREM FOR A GENERAL CLASS OF DISCRETE OPERATORS

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Dedicated to Professor Paul Leo Butzer, on the occasion of his 80th birthday, with warm friendship and high esteem.

ABSTRACT. Here we introduce a general class of discrete operators, not necessarily positive and we give a Voronovskaya-type formula for this class. Applications to generalized sampling-type operators and to a further generalization of the classical Szász-Mirak'jan operator are given. Finally a survey on Voronovskaya's formula for classical discrete operators is treated.

1. Introduction. In this paper we deal with the pointwise approximation properties of a general class of discrete, not necessarily positive, operators acting on functions defined on an interval of the real line, having the form

$$(S_n f)(t) = \sum_{k=0}^{+\infty} K_n(t, \nu_{n,k}) f(\nu_{n,k}), \quad n \in \mathbf{N}, \quad t \in I,$$

where I is a fixed interval (bounded or not) in $\mathbf{R}$ and, for every fixed $n \in \mathbf{N}$, $(\nu_{n,k})_{k \in \mathbf{N}_0} \subset I$ is a sequence satisfying suitable assumptions. These kinds of operators may be considered as particular cases of a class of abstract (also nonlinear) operators which was introduced and widely studied in [**7, 8**], in connection with uniform and modular convergence in modular function spaces (see [**27**]), under specific assumptions on the "kernel functions" K_n.

Probably the most classical discrete operator is the celebrated Bernstein operator (polynomial), which gives an elegant approach to the famous Weierstrass approximation theorem for continuous functions, see e.g., [**15, 25**]. Later, Voronovskaya [**36**] gave the first asymptotic

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formula for the pointwise approximation of continuous functions which have a second derivative at a point $t \in I = [0,1]$. Since then, a lot of generalized versions of Voronovskaya's formula have been studied, for various discrete operators, which represent generalized versions of the classical Bernstein operator. We quote here the operators of Szász-Mirak'jan, Chlodovsky, Baskakov, Meyer-König and Zeller, Bleimann-Butzer-Hahn and their various extensions (see also the book [**19**]). The contributions in this direction are so numerous that we can certainly speak of a "Vornovskaya-type approximation theory," which involves essentially the pointwise convergence. This theory was developed also for classical integral operators of convolution type, see [**33**], and recently for Mellin-type convolution operators in [**9**]. Another important fact is that this theory reveals deep links with the theory of semigroups of operators, see [**4**].

Our aim is to give a unitary approach to the study of pointwise asymptotic formulas of Voronovskaya type, using a general class of discrete operators, which includes as special cases all the above-mentioned operators. In Section 2 we determine some basic assumptions which allow us to state a general Voronovskaya's formula (Theorem 1). It turns out that, in order to apply a general method to various kinds of discrete operators, it is convenient to consider, as a first step, only bounded functions f. As main applications, some kind of generalized sampling operators generated by a compact support function (see [**10, 16, 17, 30**]) and a further generalized version of the Szász-Mirak'jan operator is considered in Sections 3 and 4, respectively. Section 5 represents a survey on various asymptotic formulas for the classical operators, which can be deduced as special cases of our general approach.

Finally, in Section 6, we give an extension of the asymptotic formula proved in Section 2 to a nonlinear operator of the form

$$(S_n f)(t) = \sum_{k=0}^{+\infty} K_n(t, \nu_{n,k}, f(\nu_{n,k})), \quad n \in \mathbf{N}, \quad t \in I.$$

In this case the theory is quite different because we can obtain only some estimates of the error of approximation in terms of limsup which involves only the first derivative of the function f. In particular, it seems to us that it is not possible to obtain an exact order of pointwise approximation.

2. A general Voronovskaya-type theorem. In the following we will denote by I a fixed interval (bounded or not) in $\mathbf{R}$ and, for every fixed $n \in \mathbf{N}$, by $\Gamma_n = (\nu_{n,k})_{k \in \mathbf{N}_0} \subset I$ a sequence such that

$$0 < \nu_{n,k+1} - \nu_{n,k} \leq \lambda_n,$$

where λ_n are positive real numbers and $\lim_{n \to +\infty} \lambda_n = 0$.

Let us consider a sequence $S = (S_n)$ of operators of the form

$$(S_n f)(t) = \sum_{k=0}^{+\infty} K_n(t, \nu_{n,k}) f(\nu_{n,k}), \quad n \in \mathbf{N}, \quad t \in I. \tag{1}$$

We will put $\operatorname{Dom} S := \cap_{n \in \mathbf{N}} \operatorname{Dom} S_n$ where $\operatorname{Dom} S_n$ is the set of all functions $f : I \to \mathbf{R}$ for which (1) is well defined.

The family of functions $(K_n)_{n \in \mathbf{N}}$, $K_n : I \times \Gamma_n \to \mathbf{R}$ satisfies the following assumptions

1) $\sum_{k=0}^{+\infty} K_n(t, \nu_{n,k}) = 1$, for every $n \in \mathbf{N}$ and $t \in I$.

2) Putting for $j \in \mathbf{N}$

$$m_j(K_n, t) := \sum_{k=0}^{+\infty} K_n(t, \nu_{n,k})(\nu_{n,k} - t)^j$$

we have, for every $t \in I$, $n \in \mathbf{N}$ and $j = 1, 2$,

$$-\infty < m_j(K_n, t) < +\infty,$$

and there are $\alpha > 0$ and real numbers $\ell_j(t)$ such that

$$\lim_{n \to +\infty} n^\alpha m_j(K_n, t) = \ell_j(t), \quad j = 1, 2. \tag{2}$$

3) For the above $\alpha > 0$, putting

$$M_2(K_n, t) := \sum_{k=0}^{+\infty} |K_n(t, \nu_{n,k})| (\nu_{n,k} - t)^2,$$

for every $t \in I$, there is a positive constant $H(t)$ and $\overline{n} \in \mathbf{N}$ such that

$$n^\alpha M_2(K_n, t) \leq H(t)$$

for every $n \geq \overline{n}$ and, for every $\delta > 0$,

$$\sum_{|\nu_{n,k}-t|\geq\delta} |K_n(t,\nu_{n,k})|(\nu_{n,k}-t)^2 = o(n^{-\alpha}), \quad n \to +\infty,$$

for $t \in I$.

Theorem 1. *Let $f \in \mathrm{Dom}\, S \cap L^\infty(I)$ be a function such that $f''(t)$ exists at a point $t \in I$. Under the above assumptions there holds*

$$\lim_{n\to+\infty} n^\alpha[(S_nf)(t) - f(t)] = f'(t)\ell_1(t) + \frac{f''(t)}{2}\ell_2(t). \tag{3}$$

Proof. Using a local Taylor's formula for the function f, there exists a bounded function h such that $\lim_{y\to 0} h(y) = 0$ and

$$f(\nu_{n,k}) = f(t) + f'(t)(\nu_{n,k}-t) + \frac{f''(t)}{2}(\nu_{n,k}-t)^2 + h(\nu_{n,k}-t)(\nu_{n,k}-t)^2.$$

Thus, we have

$$\begin{aligned} n^\alpha[(S_nf)(t) - f(t)] &= n^\alpha f'(t) \sum_{k=0}^{+\infty} K_n(t,\nu_{n,k})(\nu_{n,k}-t) \\ &\quad + n^\alpha \frac{f''(t)}{2} \sum_{k=0}^{+\infty} K_n(t,\nu_{n,k})(\nu_{n,k}-t)^2 \\ &\quad + n^\alpha \sum_{k=0}^{+\infty} K_n(t,\nu_{n,k}) h(\nu_{n,k}-t)(\nu_{n,k}-t)^2 \\ &= I_1 + I_2 + I_3. \end{aligned}$$

We immediately have

$$I_1 = n^\alpha f'(t) m_1(K_n,t), \qquad I_2 = n^\alpha \frac{f''(t)}{2} m_2(K_n,t).$$

Now we estimate I_3. Let $\varepsilon > 0$ be fixed. There exists a $\delta > 0$ such that $|h(y)| \leq \varepsilon$ for every $|y| \leq \delta$. Hence,

$$\begin{aligned} |I_3| &\leq n^\alpha \sum_{|\nu_{n,k}-t|<\delta} |K_n(t,\nu_{n,k}) h(\nu_{n,k}-t)|(\nu_{n,k}-t)^2 \\ &\quad + n^\alpha \sum_{|\nu_{n,k}-t|\geq\delta} |K_n(t,\nu_{n,k}) h(\nu_{n,k}-t)|(\nu_{n,k}-t)^2 \\ &= I_3' + I_3''. \end{aligned}$$

We have, by assumption 3),

$$|I_3'| \leq \varepsilon H(t)$$

for a sufficiently large n. Moreover, choosing a constant $M > 0$ such that $|h(y)| \leq M$, we have

$$|I_3''| \leq M n^\alpha \sum_{|\nu_{n,k}-t|\geq\delta} |K_n(t, \nu_{n,k})|(\nu_{n,k} - t)^2 = o(1)$$

for $n \to +\infty$. Thus, using (2),

$$\limsup_{n\to+\infty} n^\alpha[(S_n f)(t) - f(t)] \leq f'(t)\ell_1(t) + \frac{f''(t)}{2}\ell_2(t) + \varepsilon H(t),$$

and analogously,

$$\liminf_{n\to+\infty} n^\alpha[(S_n f)(t) - f(t)] \geq f'(t)\ell_1(t) + \frac{f''(t)}{2}\ell_2(t) - \varepsilon H(t).$$

So (3) follows. □

Remark 1. If I is bounded, condition 3) above is implied by the following simpler one

3$'$) For every $t \in I$, there is a positive constant $H(t)$ such that

$$n^\alpha M_2(K_n, t) \leq H(t)$$

for every $n \geq \overline{n}$ and, for every $\delta > 0$,

$$\sum_{|\nu_{n,k}-t|\geq\delta} |K_n(t, \nu_{n,k})| = o(n^{-\alpha}), \quad n \to +\infty.$$

for $t \in I$.

Indeed, we easily have

$$\sum_{|\nu_{n,k}-t|\geq\delta} |K_n(t, \nu_{n,k})|(\nu_{n,k} - t)^2$$
$$\leq |I|^2 \sum_{|\nu_{n,k}-t|\geq\delta} |K_n(t, \nu_{n,k})| = o(n^{-\alpha}),$$

$|I|$ denoting the length of I. In this instance it is easy to show that $L^\infty(I) \subset \operatorname{Dom} S$.

Remark 2. Note that we can obtain a generalization of Theorem 1 without the assumptions (2). We only assume that

$$\limsup_{n\to+\infty} n^\alpha m_j(K_n,t) = \ell'_j(t) \in \mathbf{R}, \quad j=1,2,$$
$$\liminf_{n\to+\infty} n^\alpha m_j(K_n,t) = \ell''_j(t) \in \mathbf{R}, \quad j=1,2.$$

In this instance we obtain an estimate of the type

$$\begin{aligned} f'(t)\ell''_1(t) + \frac{f''(t)}{2}\ell''_2(t) &\le \liminf_{n\to+\infty} n^\alpha[(S_nf)(t)-f(t)] \\ &\le \limsup_{n\to+\infty} n^\alpha[(S_nf)(t)-f(t)] \le f'(t)\ell'_1(t) + \frac{f''(t)}{2}\ell'_2(t). \end{aligned}$$

Remark 3. If I is an unbounded interval, we can relax the boundedness assumption on f in the following way. Assume that there are two positive constants a and b such that

$$|f(x)| \le a + bx^2, \quad \text{for every } x \in I.$$

Then, putting

$$P_2(x) = f(t) + f'(t)(x-t) + \frac{f''(t)}{2}(x-t)^2,$$

Taylor's polynomial of second order centered at point t, by Taylor's formula we can write

$$\frac{f(x)-P_2(x)}{(x-t)^2} = h(x-t),$$

where h is a function such that $\lim_{y\to 0} h(y) = 0$. Then h is bounded in a neighborhood of t, say $[t-\delta, t+\delta]$, while for $|x-t| > \delta$, we have

$$|h(x-t)| \le \frac{a+bx^2}{(x-t)^2} + \frac{|P_2(x)|}{(x-t)^2},$$

and the second righthand side of the above inequality is bounded for $|x-t| > \delta$, Thus $h(\cdot - t)$ is bounded on I, and we can proceed as in the proof of Theorem 1, obtaining the same Voronovskaya's formula.

Remark 4. If we assume that there is a positive constant a_n such that

$$a_n \leq \nu_{n,k+1} - \nu_{n,k}, \quad n \in \mathbf{N},$$

then $L^\infty(I) \subset \operatorname{Dom} S$ for any interval I. Indeed, we can write

$$\sum_{k=0}^{\infty} |K_n(t, \nu_{n,k})| = \Bigg(\sum_{|\nu_{n,k}-t|<1} + \sum_{|\nu_{n,k}-t|\geq 1} \Bigg) |K_n(t, \nu_{n,k})| := I_1 + I_2.$$

Then I_1 has only a finite number of terms while for I_2, using assumption 2), we get

$$I_2 \leq \sum_{|\nu_{n,k}-t|\geq 1} |K_n(t, \nu_{n,k})|(\nu_{n,k} - t)^2 \leq M_2(K_n, t) < +\infty.$$

Thus, for $f \in L^\infty(I)$, we have

$$\sum_{k=0}^{\infty} |K_n(t, \nu_{n,k})||f(\nu_{n,k})| \leq \|f\|_\infty \sum_{k=0}^{\infty} |K_n(t, \nu_{n,k})| < +\infty.$$

If simply (K_n) is nonnegative, then $L^\infty(I) \subset \operatorname{Dom} S$ is a direct consequence of assumption 1).

3. Generalized sampling series. Let us consider now the whole real line as base interval I and $\nu_{n,k} = k/n$, $k \in \mathbf{Z}$, $n \in \mathbf{N}$. Let $\varphi : \mathbf{R} \to \mathbf{R}$ be a continuous function with compact support $J \subset \mathbf{R}$ satisfying the following assumptions:

i) we have

$$\sum_{k=-\infty}^{+\infty} \varphi(u-k) = 1, \quad u \in \mathbf{R}$$

and

$$M_0(\varphi) := \sup_{u \in \mathbf{R}} \sum_{k=-\infty}^{+\infty} |\varphi(u-k)| < +\infty.$$

ii) Assume that, for every $u \in \mathbf{R}$,

$$m_1(\varphi) := \sum_{k=-\infty}^{+\infty} \varphi(u-k)(k-u) = 0$$
$$m_2(\varphi) := \sum_{k=-\infty}^{+\infty} \varphi(u-k)(k-u)^2 = C$$

for a given constant $C \in \mathbf{R}$.

For $n \in \mathbf{N}$, the generalized sampling operator is defined as (see, e.g., [**10, 16, 17, 30**])

$$(G_n f)(t) = \sum_{k=-\infty}^{+\infty} \varphi\left(n\left(t-\frac{k}{n}\right)\right) f\left(\frac{k}{n}\right).$$

Note that, in this case, since φ has a compact support, for every fixed n and $t \in \mathbf{R}$, only a finite number of sample values k/n occurs in the series defining $(G_n f)(t)$. Therefore, $\operatorname{Dom} G = \cap_{n\in\mathbf{N}} \operatorname{Dom} G_n$ contains every function $f : \mathbf{R} \to \mathbf{R}$. In particular, $L^\infty(\mathbf{R}) \subset \operatorname{Dom} G$.

Here

$$K_n\left(t, \frac{k}{n}\right) = \varphi\left(n\left(t-\frac{k}{n}\right)\right), \quad t \in \mathbf{R}, \quad k \in \mathbf{Z}.$$

By assumption i) we easily obtain

$$\sum_{k=-\infty}^{+\infty} K_n\left(t, \frac{k}{n}\right) = \sum_{k=-\infty}^{+\infty} \varphi(nt-k) = 1.$$

Moreover for $j = 1, 2$, we have

$$\begin{aligned} m_j(K_n, t) &= \sum_{k=-\infty}^{+\infty} \varphi\left(n\left(t-\frac{k}{n}\right)\right)\left(\frac{k}{n}-t\right)^j \\ &= \frac{1}{n^j}\sum_{k=-\infty}^{+\infty} \varphi(nt-k)(k-nt)^j \\ &= \frac{1}{n^j} m_j(\varphi). \end{aligned}$$

Thus, we have

$$\lim_{n\to+\infty} n^2 m_1(K_n,t) = 0$$

and

$$\lim_{n\to+\infty} n^2 m_2(K_n,t) = C.$$

Moreover, in order to prove assumption 3) note that, denoting by $[-R,R]$ the support of φ there holds, with $\alpha = 2$,

$$\begin{aligned} n^2 M_2(K_n,t) &= n^2 \sum_{k=-\infty}^{+\infty} |\varphi(nt-k)| \left(\frac{k}{n}-t\right)^2 \\ &= \sum_{k=-\infty}^{+\infty} |\varphi(nt-k)|(nt-k)^2 \le R^2 M_0(\varphi) \\ &:= H(t) \end{aligned}$$

Moreover, let $\delta > 0$ be fixed, and let $\overline{n}$ be such that $\delta n > R$ for every $n > \overline{n}$. Thus, for $n > \overline{n}$,

$$\sum_{|k-nt|\ge n\delta} |\varphi(nt-k)| \left(\frac{k}{n}-t\right)^2 = 0$$

so property 3) holds for any $\alpha > 0$.

So we obtain the following result as a corollary

Corollary 1. *Let $f \in L^\infty(\mathbf{R})$. Then we have*

$$\lim_{n\to+\infty} n^2[(G_n f)(t) - f(t)] = C\frac{f''(t)}{2}$$

at every point $t \in \mathbf{R}$ in which $f''(t)$ exists.

Now, using a method developed in [**17**], we give an explicit example of kernel φ satisfying all the previous assumptions. In order to do that, let us define the central B-splines of order $h \in \mathbf{N}$ as

$$B_h(t) := \frac{1}{(h-1)!} \sum_{j=0}^{h} (-1)^j \binom{h}{j} \left(\frac{h}{2}+t-j\right)_+^{h-1}$$

where $x_+^r := \max\{x^r, 0\}$. It is well known that the Fourier transform of the functions B_h is given by

$$\widehat{B}_h(v) = \left(\frac{\sin v/2}{v/2}\right)^h, \quad v \in \mathbf{R}, \quad h \in \mathbf{N},$$

(see [**17, 31**]). Given real numbers $\varepsilon_0, \varepsilon_1, \varepsilon_2$ with $\varepsilon_0 < \varepsilon_1 < \varepsilon_2$, we will construct a linear combination of translates of B_h, with $h \geq 3$, of type

$$\varphi(t) = a_0 B_h(t - \varepsilon_0) + a_1 B_h(t - \varepsilon_1) + a_2 B_h(t - \varepsilon_2)$$

in such a way that i) and ii) are satisfied. Using Poisson's summation formula,

$$(-i)^j \sum_{k=-\infty}^{+\infty} \varphi(u-k)(u-k)^j \sim \sum_{k=-\infty}^{+\infty} \widehat{\varphi}^{(j)}(2\pi k) e^{i2\pi k u},$$

we have to find constants a_0, a_1 and a_2 such that

$$\widehat{\varphi}(2\pi k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0. \end{cases}$$
$$\widehat{\varphi}'(2\pi k) = 0 \text{ for every } k \in \mathbf{Z}.$$
$$\widehat{\varphi}''(2\pi k) = \begin{cases} -C & k = 0 \\ 0 & k \neq 0. \end{cases}$$

The Fourier transform of φ is given by

$$\widehat{\varphi}(v) = \widehat{B}_h(v) \left(\sum_{\mu=0}^{2} a_\mu e^{-i\varepsilon_\mu v} \right).$$

By an elementary calculation, we have

$$\widehat{B}_h'(2k\pi) = 0 \text{ for every } k \in \mathbf{Z}$$

and

$$\widehat{B}_h''(2k\pi) = \begin{cases} -h/12 & k = 0 \\ 0 & k \neq 0; \end{cases}$$

thus, we obtain the system

$$\begin{aligned}\widehat{\varphi}(0) &= a_0 + a_1 + a_2 = 1\\ \widehat{\varphi}'(0) &= -i(\varepsilon_0 a_0 + \varepsilon_1 a_1 + \varepsilon_2 a_2) = 0\\ \widehat{\varphi}''(0) &= -(\varepsilon_0^2 a_0 + \varepsilon_1^2 a_1 + \varepsilon_2^2 a_2) - \frac{h}{12} = -C,\end{aligned}$$

while for $k \neq 0$ we obtain identities $0 = 0$. Solving the above linear system, we obtain the unique solution

$$\begin{aligned}a_0 &= \frac{C - (h/12) + \varepsilon_1\varepsilon_2}{(\varepsilon_1 - \varepsilon_0)(\varepsilon_2 - \varepsilon_0)},\\ a_1 &= -\frac{C - (h/12) + \varepsilon_0\varepsilon_2}{(\varepsilon_2 - \varepsilon_1)(\varepsilon_1 - \varepsilon_0)},\\ a_2 &= \frac{C - (h/12) + \varepsilon_1\varepsilon_0}{(\varepsilon_2 - \varepsilon_1)(\varepsilon_2 - \varepsilon_0)}.\end{aligned}$$

Moreover, it is easy to see that the support of function φ is contained in the interval $[\varepsilon_0 - h/2, \varepsilon_2 + h/2]$.

4. A Szász-Mirak'jan type operator. In this section we will study a Szász-Mirak'jan type operator generated by functions of the type $\varphi(t) = p(t)e^t$, where $p(t)$ is a polynomial and $t \in I = [0, +\infty[$, using a set of sample values of the type $k/(n+\beta)$, where β is a nonnegative constant. We will apply our general theorem in order to obtain an asymptotic formula for this operator.

Let us consider $I = [0, +\infty[$ and $\nu_{n,k} = k/(n+\beta)$, $k \in \mathbf{N}_0$, $n \in \mathbf{N}$. We define the operator

$$(T_n f)(t) = \frac{1}{\varphi(nt)} \sum_{k=0}^{+\infty} a_k (nt)^k f\left(\frac{k}{n+\beta}\right),$$

where the coefficients a_k are given (uniquely) by the Taylor expansion

$$\varphi(t) = \sum_{k=0}^{+\infty} a_k t^k.$$

The polynomial $p(t)$ has the form $p(t) = t^r + b_1 t^{r-1} + \cdots + b_r$, and we will assume that $p(t) > 0$ for every $t \geq 0$. Note that in this case the domain of operator T_n contains very large classes of functions; for example, we can also consider functions with exponential growth (for the Szász-Mirak'jan operator, see e.g., [**4, 20, 34**]). In what follows, we will simply assume that $f \in \operatorname{Dom} T = \cap_{n \in \mathbf{N}} \operatorname{Dom} T_n$. In particular, $L^\infty(I) \subset \operatorname{Dom} T$.

Here

$$K_n\left(t, \frac{k}{n+\beta}\right) = \frac{1}{\varphi(nt)} a_k (nt)^k.$$

It is easy to show that assumption 1) holds. The following lemma gives an expression for the moments $m_\nu(K_n, t)$ for $\nu \in \mathbf{N}$.

Lemma 1. *Putting, for $j \in \mathbf{N}$,*

$$S_j = \sum_{k=0}^{+\infty} a_k k^j (nt)^k,$$

there holds, for $\nu \in \mathbf{N}$,

$$m_\nu(K_n, t) = \sum_{j=0}^{\nu} \frac{(-1)^{\nu-j}}{\varphi(nt)} \binom{\nu}{j} \frac{t^{\nu-j}}{(n+\beta)^j} S_j.$$

Proof. We have

$$\begin{aligned} m_\nu(K_n, t) &= \frac{1}{\varphi(nt)} \sum_{k=0}^{+\infty} a_k (nt)^k \left(\frac{k}{n+\beta} - t\right)^\nu \\ &= \frac{1}{\varphi(nt)} \sum_{j=0}^{\nu} (-1)^{\nu-j} \binom{\nu}{j} \frac{t^{\nu-j}}{(n+\beta)^j} \sum_{k=0}^{+\infty} a_k k^j (nt)^k \\ &= \sum_{j=0}^{\nu} \frac{(-1)^{\nu-j}}{\varphi(nt)} \binom{\nu}{j} \frac{t^{\nu-j}}{(n+\beta)^j} S_j. \qquad \square \end{aligned}$$

If we now calculate S_j for $j = 1, 2, 3, 4$, by elementary calculations we have

$$S_1 = nt\varphi'(nt)$$
$$S_2 = nt\varphi'(nt) + (nt)^2\varphi''(nt)$$
$$S_3 = nt\varphi'(nt) + 3(nt)^2\varphi''(nt) + (nt)^3\varphi'''(nt)$$
$$S_4 = nt\varphi'(nt) + 7(nt)^2\varphi''(nt) + 6(nt)^3\varphi'''(nt) + (nt)^4\varphi^{iv}(nt).$$

Thus, we obtain the following expressions for the first four moments:

$$m_1(K_n, t) = \frac{nt}{n+\beta}\frac{\varphi'(nt)}{\varphi(nt)} - t$$
$$m_2(K_n, t) = \frac{nt}{n+\beta}\left(\frac{\varphi'(nt)}{\varphi(nt)}\left(\frac{1}{n+\beta} - 2t\right) + \frac{\varphi''(nt)}{\varphi(nt)}\frac{nt}{n+\beta}\right) + t^2$$
$$\begin{aligned} m_3(K_n, t) = {} & \frac{1}{\varphi(nt)}\frac{nt}{(n+\beta)^3}(\varphi'(nt) + 3nt\varphi''(nt) + n^2t^2\varphi'''(nt)) \\ & - \frac{3t}{\varphi(nt)}\frac{nt}{(n+\beta)^2}(\varphi'(nt) + nt\varphi''(nt)) \\ & + \frac{3t^2}{\varphi(nt)}\frac{nt}{(n+\beta)}\varphi'(nt) - t^3 \end{aligned}$$
$$\begin{aligned} m_4(K_n, t) = {} & \frac{1}{\varphi(nt)}\frac{nt}{(n+\beta)^4}(\varphi'(nt) + 7nt\varphi''(nt) \\ & + 6n^2t^2\varphi'''(nt) + n^3t^3\varphi^{iv}(nt)) \\ & - \frac{4t}{\varphi(nt)}\frac{nt}{(n+\beta)^3}(\varphi'(nt) + 3nt\varphi''(nt) + n^2t^2\varphi'''(nt)) \\ & + \frac{6t^2}{\varphi(nt)}\frac{nt}{(n+\beta)^2}(\varphi'(nt) + nt\varphi''(nt)) \\ & - \frac{4t^3}{\varphi(nt)}\frac{nt}{(n+\beta)}\varphi'(nt) + t^4. \end{aligned}$$

We now state the following

Lemma 2. *There hold*

$$\lim_{n\to+\infty} nm_1(K_n, t) = r - \beta t,$$
$$\lim_{n\to+\infty} nm_2(K_n, t) = t,$$
$$\lim_{n\to+\infty} nm_4(K_n, t) = 0.$$

Proof. First note that, for every $j = 1, 2, 3, 4$, there holds

$$\varphi^{(j)}(t) = e^t \sum_{\nu=0}^{j} \binom{j}{\nu} p^{(j-\nu)}(t).$$

Thus, it is easy to see that

$$\lim_{t\to+\infty} \frac{\varphi^{(j)}(t)}{\varphi(t)} = 1, \quad j = 1, 2, 3, 4. \tag{4}$$

Taking into account the degree of polynomial p, by elementary calculations we obtain

$$\lim_{n\to+\infty} nm_1(K_n, t) = \lim_{n\to+\infty} n\frac{ntp'(nt) - \beta tp(nt)}{(n+\beta)p(nt)} = r - \beta t,$$

$$\begin{aligned}\lim_{n\to+\infty} & nm_2(K_n, t) \\ &= \lim_{n\to+\infty} \frac{nt}{n+\beta} \\ &\times \frac{np(nt) + \beta^2 tp(nt) + np'(nt) - 2\beta ntp'(nt) + n^2tp''(nt)}{(n+\beta)p(nt)} = t.\end{aligned}$$

Finally, we calculate the limit

$$\lim_{n\to+\infty} nm_4(K_n, t).$$

Taking into account (4), we can reduce the calculation of the limit in the following way

$$\begin{aligned}\lim_{n\to+\infty} nm_4(K_n, t) = \lim_{n\to+\infty} \Bigg[& \frac{n^5t^4}{(n+\beta)^4} \frac{\varphi^{iv}(nt)}{\varphi(nt)} - \frac{4n^4t^4}{(n+\beta)^3} \frac{\varphi'''(nt)}{\varphi(nt)} \\ & + \frac{6n^3t^4}{(n+\beta)^4} \frac{\varphi''(nt)}{\varphi(nt)} - \frac{4n^2t^4}{n+\beta} \frac{\varphi'(nt)}{\varphi(nt)} + nt^4 \Bigg].\end{aligned}$$

Reducing now the last term in a unique fraction, we determine a ratio P/Q of two polynomials in n, P and Q, having degree $r+3$ and $r+4$, respectively. Thus,

$$\lim_{n\to+\infty} nm_4(K_n,t)=0,$$

and the proof is now complete. □

So we obtain the following result as a corollary

Corollary 2. *Let $f\in L^\infty(I)$. We have*

$$\lim_{n\to+\infty} n[(T_nf)(t)-f(t)]=(r-\beta t)f'(t)+t\frac{f''(t)}{2},$$

at every point t in which $f''(t)$ exists.

Proof. We have only to check assumption 3), with $\alpha=1$. The first part is satisfied for sufficiently large n by taking $H(t)=t+1$ for every $t\in[0,+\infty[$. For the second part, we can write, for $\delta>0$,

$$\sum_{|(k/n+\beta)-t|\ge\delta}\left|K_n\left(t,\frac{k}{n+\beta}\right)\right|\left(\frac{k}{n+\beta}-t\right)^2\le\frac{1}{\delta^2}m_4(K_n,t)=o(n^{-1})$$

for $n\to+\infty$. □

In the cases $r=0$ and $\beta=0$, T_n is the classical Szász-Mirak'jan operator:

$$(S_nf)(t)=\sum_{k=0}^{\infty}e^{-nt}\frac{(nt)^k}{k!}f\left(\frac{k}{n}\right),\quad t\ge 0.$$

The asymptotic formula reduces to the classical one:

$$\lim_{n\to+\infty} n[(S_nf)(t)-f(t)]=t\frac{f''(t)}{2},$$

when $f''(t)$ exists.

The cases $r=0$ and $\beta>0$ have been discussed in [**29**], in which the authors introduced the following slight modification of the Szász-Mirak'jan operator:

$$(\widetilde{S}_n^\beta f)(t)=\sum_{k=0}^{+\infty}e^{-nt}\frac{(nt)^k}{k!}f\left(\frac{k}{n+\beta}\right),\quad t\ge 0.$$

In this case the asymptotic formula reduces to ([**29**])

$$\lim_{n\to+\infty} n[(\widetilde{S}_n^{\beta} f)(t) - f(t)] = -\beta t f'(t) + t\frac{f''(t)}{2}.$$

Another interesting generalization of the Szász-Mirak'jan operator is studied in [**29**], in which the following operator is considered:

$$(H_n f)(t) = \frac{1}{\cosh nt}\sum_{k=0}^{+\infty}\frac{(nt)^{2k}}{(2k)!} f\left(\frac{2k}{n}\right).$$

Here $I = [0,+\infty[$ and $\nu_{n,k} = 2k/n$ for $k \in \mathbf{N}$. The kernel is given now by

$$K_n\left(t, \frac{2k}{n}\right) = \frac{1}{\cosh nt}\frac{(nt)^{2k}}{(2k)!}.$$

We will consider functions $f : [0,+\infty[\to \mathbf{R}$ such that $f \in \mathrm{Dom}\, H = \cap_{n\in\mathbf{N}}\mathrm{Dom}\, H_n$.

It is easy to show that assumption 1) holds. As to the moments $m_j(K_n, t)$ for $j = 1, 2$, we have (see [**28**])

$$m_1(K_n, t) = \frac{1}{\cosh nt}\sum_{k=0}^{+\infty}\frac{(nt)^{2k}}{(2k)!}\left(\frac{2k}{n} - t\right) = -t(1 - \tanh nt)$$

and

$$m_2(K_n, t) = \left(2t^2 - \frac{t}{n}\right)(1 - \tanh nt) + \frac{t}{n},$$

and so

$$\lim_{n\to+\infty} n m_1(K_n, t) = 0, \quad \lim_{n\to+\infty} n m_2(K_n, t) = t.$$

Finally, let us remark that for sufficiently large n, (depending on t) we have

$$n M_2(K_n, t) \le t + 1,$$

and so putting $\alpha = 1$ and $H(t) = t + 1$ the first part of 3) is satisfied.

For the second part of 3) we apply [**28**, Lemma 3] which states the following estimate for the fourth order moment:

$$m_4(K_n, t) := \frac{1}{\cosh nt}\sum_{k=0}^{+\infty}\frac{(nt)^{2k}}{(2k)!}\left(\frac{2k}{n} - t\right)^4 \le C(t) n^{-2},$$

for every $n \in \mathbf{N}$, since C is a positive constant depending only on t. Thus, we have

$$\sum_{|(2k/n)-t|\geq\delta}\left|K_n\left(t,\frac{k}{n}\right)\right|\left(\frac{2k}{n}-t\right)^2\leq\frac{1}{\delta^2}m_4(K_n,t)=o(n^{-1}),$$
$$n\to+\infty.$$

So we obtain the following result as a corollary (see [**28**, Theorem 1]).

Corollary 3. *Let $f\in L^\infty(\mathbf{R})$. For the hyperbolic Szász-Mirak'jan operator H_nf, we have*

$$\lim_{n\to+\infty}n[(H_nf)(t)-f(t)]=t\frac{f''(t)}{2},$$

at every point $t\geq 0$ in which $f''(t)$ exists.

We can consider also a variant of the above operator which involves the function $\sinh t$. This operator was also studied in [**28**], and it is defined as

$$(\widetilde{H}_nf)(t)=\frac{f(0)}{1+\sinh nt}+\frac{1}{1+\sinh nt}\sum_{k=0}^{+\infty}\frac{(nt)^{2k+1}}{(2k+1)!}f\left(\frac{2k+1}{n}\right).$$

For this operator we obtain the same asymptotic formula

$$\lim_{n\to+\infty}n[(\widetilde{H}_nf)(t)-f(t)]=t\frac{f''(t)}{2}.$$

5. Classical examples. In this section we will give a survey about Voronovskaya's formula for various classical discrete operators by showing that all of them are particular cases of our general theory. Here we will always assume that $f\in L^\infty(I)$.

5.1. Bernstein polynomials. Let us consider $I=[0,1]$ and $\nu_{n,k}=k/n$ for $k=0,\dots,n$. The Bernstein polynomials of a function $f:[0,1]\to\mathbf{R}$ are defined as (see, e.g., [**6, 25**])

$$(B_nf)(t)=\sum_{k=0}^{n}\binom{n}{k}f\left(\frac{k}{n}\right)t^k(1-t)^{n-k}.$$

Here,

$$K_n\left(t, \frac{k}{n}\right) = \binom{n}{k} t^k (1-t)^{n-k}.$$

It is easy to show that assumption 1) holds. It is very well known that (see [**6, 25**])

$$m_1(K_n, t) = 0, \;\; m_2(K_n, t) = \frac{t(1-t)}{n}.$$

Thus, we have

$$nM_2(K_n, t) = t(1-t),$$

and so putting $\alpha = 1$ and $H(t) = t(1-t)$, the first part of 3) is satisfied. For the second part note that (see [**6**])

$$M_4(K_n, t) = m_4(K_n, t) = \frac{1}{n^4}(3n^2t^2(1-t)^2 + (1-6t(1-t))nt(1-t)),$$

and so

$$\sum_{|(k/n)-t|\geq\delta} \left|K_n\left(t, \frac{k}{n}\right)\right| \left(\frac{k}{n} - t\right)^2 \leq \frac{1}{\delta^2} M_4(K_n, t),$$

and so 3) is completely proved. The asymptotic formula for the Bernstein polynomials of f now reads (see [**36**]):

$$\lim_{n\to+\infty} n[(B_n f)(t) - f(t)] = t(1-t)\frac{f''(t)}{2},$$

at every point $t \in [0,1]$ in which $f''(t)$ exists.

Let us consider now the modification of the Bernstein operator introduced by Chlodovsky (see [**3, 18, 21**]) and defined, for $f : [0, +\infty[\to \mathbf{R}$, by

$$(C_n f)(t) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n} b_n\right) \left(\frac{t}{b_n}\right)^k \left(1 - \frac{t}{b_n}\right)^{n-k}$$

where $0 \leq t \leq b_n$ and (b_n) is a sequence of positive numbers such that $\lim_{n\to+\infty} b_n = +\infty$ and $\lim_{n\to+\infty} n^{\alpha-1} b_n = c$ for nonnegative constants $\alpha < 1$ and c. Here $I = [0, +\infty[$, $\nu_{n,k} = b_n k/n$, for $k \in \mathbf{N}$, and

$$K_n\left(t, b_n \frac{k}{n}\right) = \begin{cases} \binom{n}{k} (t/b_n)^k (1-(t/b_n))^{n-k} & 0 \le t \le b_n, \\ 0 & t \ge b_n. \end{cases}$$

It is easy to show that assumption 1) holds. In [**21**] it is proved that, for $0 \le t \le b_n$,

$$m_1(K_n, t) = 0, \qquad m_2(K_n, t) = \frac{t(b_n - t)}{n}.$$

So we have, for the above constants α and c,

$$\lim_{n \to +\infty} n^\alpha m_2(K_n, t) = ct.$$

Thus, we also have

$$n^\alpha M_2(K_n, t) = \frac{t}{n^{1-\alpha}}(b_n - t),$$

and so for the constant $\alpha < 1$, the first part of 3) is satisfied with $H(t) = ct + 1$. For the second part note that, putting

$$X = \frac{t}{b_n}\left(1 - \frac{t}{b_n}\right)$$

and using the expression of the fourth order moment of the Bernstein operator (see, e.g., [**6**]), we have

$$M_4(K_n, t) = m_4(K_n, t) = \frac{b_n^4}{n^4}[3n^2 X^2 + nX(1 - 6X)],$$

and so

$$\lim_{n \to +\infty} n^\alpha M_4(K_n, t) = 0.$$

Since

$$\sum_{|(b_n k/n) - t| \ge \delta} \left| K_n\left(t, \frac{kb_n}{n}\right)\right| \left(\frac{kb_n}{n} - t\right)^2 \le \frac{1}{\delta^2} M_4(K_n, t),$$

assumption 3) is completely proved. The asymptotic formula for the Bernstein-Chlodowsky polynomials reads

$$\lim_{n\to+\infty} n^{\alpha}[(C_n f)(t) - f(t)] = ct\frac{f''(t)}{2},$$

at every point $t \in [0, +\infty[$ in which $f''(t)$ exists.

5.2. Baskakov operator. Let us consider $I = [0, +\infty]$ and $\nu_{n,k} = k/n$, $k \in \mathbf{N}_0$, $n \in \mathbf{N}$. The Baskakov operator is defined as

$$(\widetilde{B}_n f)(t) = \sum_{k=0}^{+\infty} \binom{n+k-1}{k} \frac{t^k}{(1+t)^{n+k}} f\left(\frac{k}{n}\right),$$

for $f \in \mathrm{Dom}\, \widetilde{B} = \cap_{n\in\mathbf{N}} \mathrm{Dom}\, \widetilde{B}_n$. Here

$$K_n\left(t, \frac{k}{n}\right) = \binom{n+k-1}{k} \frac{t^k}{(1+t)^{n+k}}.$$

It is easy to show that assumption 1) holds. For the moments $m_j(K_n, t)$, for $j = 1, 2$, we have, using elementary calculations (see also [**11**])

$$\begin{aligned} m_1(K_n, t) &= \sum_{k=0}^{+\infty} \binom{n+k-1}{k} \frac{t^k}{(1+t)^{n+k}} \left(\frac{k}{n} - t\right) \\ &= \frac{1}{n(1+t)^n} \sum_{k=1}^{+\infty} \binom{n+k-1}{k} \left(\frac{t}{1+t}\right)^k k - t = 0, \end{aligned}$$

and

$$m_2(K_n, t) = \frac{t(1+t)}{n}.$$

Thus, we have

$$nM_2(K_n, t) = t(1+t),$$

and so putting $\alpha = 1$ and $H(t) = t(1+t)$, the first part of 3) is satisfied.

For the second part of 3) we refer to [**11**, Lemma 4] in which it is proved that

$$m_4(K_n, t) = \frac{c_n}{n^3} H(t) + \frac{d_n}{n^2} H^2(t),$$

where c_n and d_n are bounded sequences of positive numbers. So we can write

$$\sum_{|(k/n)-t|\geq\delta} K_n\left(t, \frac{k}{n}\right)\left(\frac{k}{n} - t\right)^2 \leq \frac{1}{\delta^2} m_4(K_n, t)$$

and so the second part of 3) is satisfied with $\alpha = 1$. So we get the following asymptotic formula for the Baskakov operator (see [**19**]):

$$\lim_{n\to+\infty} n[(\widetilde{B}_n f)(t) - f(t)] = t(1+t)\frac{f''(t)}{2},$$

at every point $t \in [0, +\infty[$ in which $f''(t)$ exists.

5.3. The Meyer-König and Zeller operators. Let us consider $I = [0, 1]$ and $\nu_{n,k} = k/(k+n)$, $k \in \mathbf{N}_0$, $n \in \mathbf{N}$. The Meyer-König and Zeller operator is defined as (see [**11, 26, 35**])

$$(M_n f)(t) = (1-t)^{n+1} \sum_{k=0}^{+\infty} \binom{n+k}{k} t^k f\left(\frac{k}{k+n}\right),$$

for $f \in \mathrm{Dom}\, M = \cap_{n\in\mathbf{N}} \mathrm{Dom}\, M_n$. Here

$$K_n\left(t, \frac{k}{k+n}\right) = (1-t)^{n+1}\binom{n+k}{k} t^k.$$

It is easy to show that assumption 1) holds. Now we calculate $m_j(K_n, t)$ for $j = 1, 2$. We have, using an elementary calculation,

$$\begin{aligned} m_1(K_n, t) &= (1-t)^{n+1} \sum_{k=0}^{+\infty} \binom{n+k}{k} t^k \left(\frac{k}{k+n} - t\right) \\ &= (1-t)^{n+1} \sum_{k=1}^{+\infty} \frac{(k+n-1)!}{(k-1)!n!} t^k - t = 0. \end{aligned}$$

Moreover, for the moment $m_2(K_n, t)$, we have (see [**32**])

$$m_2(K_n, t) = \frac{t(1-t)^2}{n} + \frac{t(1-t)^2(2t-1)}{n^2} + \mathcal{O}(n^{-3}).$$

Thus, we have

$$nM_2(K_n,t) = t(1-t)^2 + \frac{t(1-t)^2(2t-1)}{n} + \mathcal{O}(n^{-2}),$$

and so putting $\alpha = 1$ and $H(t) = t(1-t)^2 + 1$, the first part of 3) is satisfied.

For the second part of 3), we apply the representation of the fourth moment proved in [**32**]

$$m_4(K_n,t) = \frac{3t^2(1-t)^4}{n^2} + \mathcal{O}(n^{-3}).$$

So we can write

$$\sum_{|k/(k+n)-t|\geq\delta} K_n\left(t, \frac{k}{k+n}\right)\left(\frac{k}{k+n} - t\right)^2 \leq \frac{1}{\delta^2} m_4(K_n,t),$$

and so the second part of 3) is satisfied with $\alpha = 1$. So we get the following Voronovskaya's formula for the Meyer-König and Zeller operators

$$\lim_{n\to+\infty} n[(M_n f)(t) - f(t)] = t(1-t)^2 \frac{f''(t)}{2},$$

at every point $t \in [0,1]$ in which $f''(t)$ exists.

For a generalized version of the Meyer-König and Zeller operators, see [**2**].

5.4. The Bleimann-Butzer-Hahn operator. Let us consider $I = [0,+\infty[$ and $\nu_{n,k} = k/(n+1-k)$, $k \in \mathbf{N}_0$, $n \in \mathbf{N}$. The Bleimann-Butzer Hahn operator is defined as (see, e.g., [**1, 12, 23**])

$$(L_n f)(t) = \frac{1}{(t+1)^n} \sum_{k=0}^{n} \binom{n}{k} t^k f\left(\frac{k}{n+1-k}\right),$$
$$t \in [0,+\infty[, \quad n \in \mathbf{N},$$

for $f \in \mathrm{Dom}\, L = \cap_{n\in\mathbf{N}} \mathrm{Dom}\, L_n$. Here

$$K_n\left(t, \frac{k}{n+1-k}\right) = \frac{1}{(t+1)^n} \binom{n}{k} t^k.$$

As before, it is easy to show that assumption 1) holds. Moreover, for the first order moment of the kernel K_n, using elementary calculations, we have

$$m_1(K_n, t) = -t\left(\frac{t}{t+1}\right)^n, \quad t \geq 0,$$

and so

$$\lim_{n\to+\infty} nm_1(K_n, t) = 0, \quad t \geq 0.$$

For the second order moment, we have the estimate (see [**1, 4, 12**])

$$m_2(K_n, t) \leq C\frac{t(1+t)^2}{n+2}, \quad t \geq 0, \quad n \in \mathbf{N}$$

for a suitable constant $C > 0$ and

$$\lim_{n\to+\infty} nm_2(K_n, t) = t(1+t)^2.$$

Thus, taking $\alpha = 1$ and $H(t) = Ct(1+t)^2$, the first part of assumption 3) is satisfied. Finally, the following estimate holds, for sufficiently small $\delta > 0$, (see [**22**])

$$\frac{1}{(t+1)^{n+1}} \sum_{|k/(n+1-k)-t|\geq\delta} \binom{n+1}{k} t^k \leq 2\exp\left(\frac{-(n+1)\delta^2}{16t(1+t)^2}\right);$$

hence, we also have

$$\begin{aligned}\frac{1}{(t+1)^n} &\sum_{|k/(n+1-k)-t|\geq\delta} \binom{n}{k} t^k \\ &= \frac{1}{(t+1)^n} \sum_{|k/(n+1-k)-t|\geq\delta} \binom{n+1}{k} \frac{n+1-k}{n+1} t^k \\ &\leq (1+t)\frac{1}{(t+1)^{n+1}} \sum_{|k/(n+1-k)-t|\geq\delta} \binom{n+1}{k} t^k \\ &\leq 2(1+t)\exp\left(\frac{-(n+1)\delta^2}{16t(1+t)^2}\right).\end{aligned}$$

Thus,

$$
\begin{aligned}
n \sum_{|k/(n+1-k)-t|\geq\delta} K_n\Big(t, \frac{k}{n+1-k}\Big)\Big(\frac{k}{n+1-k}-t\Big)^2 \\
&\leq n(n+|t|)^2 \sum_{|k/(n+1-k)-t|\geq\delta} K_n\Big(t, \frac{k}{n+1-k}\Big) \\
&\leq 2(1+t)n(n+|t|)^2 \exp\left(\frac{-(n+1)\delta^2}{16t(1+t)^2}\right).
\end{aligned}
$$

Hence, also the second part of assumption 3) is satisfied, with $\alpha = 1$.

The related asymptotic formula for the Bleimann-Butzer-Hahn operator takes the form (see [**23**])

$$
\lim_{n\to+\infty} n[(L_n f)(t) - f(t)] = t(1+t)^2 \frac{f''(t)}{2},
$$

at every point $t \geq 0$ in which $f''(t)$ exists.

6. An extension to the nonlinear case. Here we will consider a nonlinear version of operator (1), of the form

$$
(S_n f)(t) = \sum_{k=0}^{+\infty} K_n(t, \nu_{n,k}, f(\nu_{n,k})), \quad n \in \mathbf{N}, \quad t \in I. \tag{5}
$$

We remark that in this instance the theory is quite different because we can obtain now only some estimates of the error of approximation in terms of limsup and of absolute moments. In particular, it seems to us that it is not possible to obtain an exact order of pointwise approximation.

The results given here involve only the first derivative of the function f. This is in some sense natural, as we will show later.

In (5) $\Gamma_n = (\nu_{n,k})_{k\in\mathbf{N}_0} \subset I$ is a sequence satisfying the same assumption of Section 2 and $K_n : I \times \Gamma_n \times \mathbf{R} \to \mathbf{R}$ is a function satisfying a Hölder condition of type

$$
|K_n(t, \nu_{n,k}, u) - K_n(t, \nu_{n,k}, v)| \leq L_n(t, \nu_{n,k})|u-v|^\gamma,
$$

where $0 < \gamma \leq 1$ and $K_n(t, \nu_{n,k}, 0) = 0$ for every $t \in I$, $\nu_{n,k} \in \Gamma_n$ and $n \in \mathbf{N}$. Here the sequence $(L_n)_{n \in \mathbf{N}}$ satisfies the conditions

(L.1) there is a constant $D > 0$ such that

$$\sum_{k=0}^{+\infty} L_n(t, \nu_{n,k}) \leq D, \quad n \in \mathbf{N}, \quad t \in I.$$

(L.2) There holds

$$M_1(L_n, t, \gamma) := \sum_{k=0}^{+\infty} L_n(t, \nu_{n,k}) |\nu_{n,k} - t|^{\gamma} < +\infty$$

for $n \in \mathbf{N}$, $t \in I$.

By assumption $(L.1)$ it is easy to see that for every $n \in \mathbf{N}$ the domain of S_n contains $L^{\infty}(\mathbf{R})$. Indeed, we have

$$|(S_n f)(t)| \leq \|f\|_{\infty}^{\gamma} \sum_{k=0}^{+\infty} L_n(t, \nu_{n,k}) \leq \|f\|_{\infty}^{\gamma} D.$$

We will say that the sequence $(K_n)_{n \in \mathbf{N}}$ is α-singular if for a fixed $\alpha > 0$ the following assumptions are satisfied

(K.1) for every $t \in I$ and $\delta > 0$ there holds

$$\sum_{|\nu_{n,k} - t| \geq \delta} L_n(t, \nu_{n,k}) |\nu_{n,k} - t|^{\gamma} = o(n^{-\alpha}), \quad (n \to +\infty).$$

(K.2) For every $u \in \mathbf{R}$ and for every $t \in I$ we have

$$\lim_{n \to +\infty} n^{\alpha} \left[\sum_{k=0}^{+\infty} K_n(t, \nu_{n,k}, u) - u \right] = 0.$$

Theorem 2. *Let $f \in L^{\infty}(\mathbf{R})$ be a function such that $f'(t)$ exists at a point $t \in I$. Let us assume that the sequence $(K_n)_{n \in \mathbf{N}}$ is α-singular and*

$$\limsup_{n \to +\infty} n^{\alpha} M_1(L_n, t, \gamma) = \ell_1(t) \in \mathbf{R}. \tag{6}$$

Then

$$\limsup_{n\to+\infty} n^{\alpha}|(S_n f)(t) - f(t)| \le \ell_1(t)|f'(t)|^{\gamma}. \tag{7}$$

Proof. Since f is differentiable at point t, there exists a bounded function h such that $\lim_{y\to 0} h(y) = 0$ and

$$f(\nu_{n,k}) = f(t) + f'(t)(\nu_{n,k} - t) + h(\nu_{n,k} - t)(\nu_{n,k} - t).$$

Now we have

$$\begin{aligned} n^{\alpha}|(S_n f)(t) - f(t)| &\le n^{\alpha}\sum_{k=0}^{+\infty} L_n(t,\nu_{n,k})|f(\nu_{n,k}) - f(t)|^{\gamma} \\ &\quad + n^{\alpha}\left|\sum_{k=0}^{+\infty} K_n(t,\nu_{n,k}, f(t)) - f(t)\right| = I_1 + I_2. \end{aligned}$$

By assumption (K.2), term I_2 tends to zero. Now we evaluate the term I_1. Using concavity of the function $g(x) = x^{\gamma}$, $x \ge 0$, we have

$$\begin{aligned} I_1 &\le n^{\alpha}|f'(t)|^{\gamma} M_1(L_n, t, \gamma) \\ &\quad + n^{\alpha}\sum_{k=0}^{+\infty} L_n(t,\nu_{n,k})|h(\nu_{n,k} - t)|^{\gamma}|\nu_{n,k} - t|^{\gamma}. \end{aligned}$$

Denoting by J the last term on the righthand side of the previous inequality and using assumption (K.1), as in the proof of Theorem 1 we have

$$J \le \varepsilon^{\gamma} n^{\alpha} M_1(L_n, t, \gamma) + o(1), \quad (n \to +\infty).$$

Thus, by (6) we have

$$\limsup_{n\to+\infty} n^{\alpha}|(S_n f)(t) - f(t)| \le \ell_1(t)|f'(t)|^{\gamma}. \qquad \square$$

Remark. Note that in this case it is not meaningful to assume the existence of a second derivative at a point t along with

$$M_2(L_n, t, \gamma) := \sum_{k=0}^{+\infty} L_n(t,\nu_{n,k})|\nu_{n,k} - t|^{2\gamma} < +\infty$$

and

$$\sum_{|\nu_{n,k}-t|\geq\delta} L_n(t,\nu_{n,k})|\nu_{n,k}-t|^{2\gamma} = o(n^{-\alpha}),$$

for every δ positive. Indeed, in this case, we also have

$$\sum_{|\nu_{n,k}-t|\geq\delta} L_n(t,\nu_{n,k})|\nu_{n,k}-t|^{\gamma} \leq \delta^{-\gamma} \sum_{|\nu_{n,k}-t|\geq\delta} L_n(t,\nu_{n,k})|\nu_{n,k}-t|^{2\gamma} = o(n^{-\alpha}),$$

so that all the assumptions of Theorem 2 are satisfied and we obtain (7). This means that estimate (7) cannot be improved if the function f is more regular.

Examples. 1. A nonlinear Bernstein operator. Let us consider $I = [0,1]$ and $\nu_{n,k} = k/n$, for $k = 0,\dots,n$. A nonlinear version of the Bernstein operator may be defined as (see [**8**])

$$(B_n f)(t) = \sum_{k=0}^{n} \binom{n}{k} G_n\left(f\left(\frac{k}{n}\right)\right) t^k(1-t)^{n-k},$$

where $G_n : \mathbf{R} \to \mathbf{R}$ satisfies a Hölder condition of the form

$$|G_n(u) - G_n(v)| \leq R|u-v|^{\gamma}$$

for every $n \in \mathbf{N}$, $0 < \gamma \leq 1$ and a suitable constant $R > 0$ and

$$\lim_{n\to+\infty} n^{\alpha}(G_n(u) - u) = 0$$

for every $u \in \mathbf{R}$. Here

$$K_n\left(t, \frac{k}{n}, u\right) = \binom{n}{k} t^k(1-t)^{n-k} G_n(u).$$

It is easy to show that all the previous assumptions are satisfied with

$$L_n\left(t, \frac{k}{n}\right) = R\binom{n}{k} t^k(1-t)^{n-k} := R\widetilde{L}_n\left(t, \frac{k}{n}\right)$$

and $D = R$. Using the relations (see [**13, 37**])

$$R\sum_{k=0}^{n}\widetilde{L}_n\left(t,\frac{k}{n}\right)\left|\frac{k}{n}-t\right| = R\left(\frac{2t(1-t)}{\pi}\right)^{1/2}\frac{1}{\sqrt{n}} + \mathcal{O}(n^{-1}(t(1-t))^{-1/2})$$

for $t \in]0,1[$, by concavity of the function $g(x) = x^{\gamma}$, $x \geq 0$, we have

$$\begin{aligned} M_1(L_n,t,\gamma) &= R\sum_{k=0}^{n}\widetilde{L}_n\left(t,\frac{k}{n}\right)\left|\frac{k}{n}-t\right|^{\gamma} \\ &\leq R\left(\sum_{k=0}^{n}\widetilde{L}_n\left(t,\frac{k}{n}\right)\left|\frac{k}{n}-t\right|\right)^{\gamma} \\ &\leq R\left(\frac{2t(1-t)}{\pi}\right)^{\gamma/2}\left(\frac{1}{\sqrt{n}}\right)^{\gamma} \\ &\quad + \mathcal{O}(n^{-\gamma}(t(1-t))^{-\gamma/2}). \end{aligned}$$

Thus, for $\alpha = \gamma/2$, we have

$$\ell_1(t) \leq R\left(\frac{t(1-t)}{n}\right)^{\gamma/2}.$$

Moreover (see Section 5) using again the concavity of the function g, we get

$$\begin{aligned} M_2(L_n,t,\gamma) &= R\sum_{k=0}^{n}\widetilde{L}_n\left(t,\frac{k}{n}\right)\left|\frac{k}{n}-t\right|^{2\gamma} \\ &\leq R\left(\sum_{k=0}^{n}\widetilde{L}_n\left(t,\frac{k}{n}\right)\left|\frac{k}{n}-t\right|^{2}\right)^{\gamma} \\ &\leq R\left(\frac{t(1-t)}{n}\right)^{\gamma}. \end{aligned}$$

Then we have

$$\sum_{|(k/n)-t|\geq\delta} L_n\left(t,\frac{k}{n}\right)\left|\frac{k}{n}-t\right|^{\gamma} \leq \frac{1}{\delta^{\gamma}}M_2(L_n,t,\gamma) = o(n^{-\gamma/2}).$$

Therefore, for $\alpha = \gamma/2$, we obtain the following asymptotic formula

$$\limsup_{n\to+\infty} n^{\gamma/2}|(B_n f)(t) - f(t)| \le R\left(\frac{2t(1-t)}{\pi}\right)^{\gamma/2} |f'(t)|^{\gamma}$$

at every point $t \in]0,1[$ in which $f'(t)$ exists.

2. A nonlinear Szász-Mirak'jan operator. Let us consider $I = [0,+\infty[$ and $\nu_{n,k} = k/n$ for $k \in \mathbf{N}_0$. A nonlinear version of the Szász-Mirak'jan operator may be defined as (see [**8**])

$$(S_n f)(t) = \sum_{k=0}^{+\infty} e^{-nt}\frac{(nt)^k}{k!} G_n\left(f\left(\frac{k}{n}\right)\right),$$

where $G_n : \mathbf{R} \to \mathbf{R}$ satisfies the previous conditions. Here

$$K_n\left(t, \frac{k}{n}, u\right) = e^{-nt}\frac{(nt)^k}{k!} G_n(u)$$

and

$$L_n\left(t, \frac{k}{n}\right) = Re^{-nt}\frac{(nt)^k}{k!} := R\widetilde{L}_n\left(t, \frac{k}{n}\right).$$

As in the previous example, it is easy to show that all the previous assumptions are satisfied with $D = R$. Using the relations (see [**14, 37**])

$$R\sum_{k=0}^{+\infty} \widetilde{L}_n\left(t, \frac{k}{n}\right)\left|\frac{k}{n} - t\right| = R\left(\frac{2t}{\pi}\right)^{1/2}\frac{1}{\sqrt{n}} + \mathcal{O}(n^{-1})$$

for $t \in [0,+\infty[$, we have

$$\begin{aligned} M_1(L_n, t, \gamma) &= R\sum_{k=0}^{n} \widetilde{L}_n\left(t, \frac{k}{n}\right)\left|\frac{k}{n} - t\right|^{\gamma} \\ &\le R\left(\frac{2t}{\pi}\right)^{\gamma/2}\left(\frac{1}{\sqrt{n}}\right)^{\gamma} + \mathcal{O}(n^{-\gamma}). \end{aligned}$$

Moreover, as in Section 4, there holds

$$M_2(L_n, t, \gamma) \le R\left(\frac{t}{n}\right)^{\gamma}.$$

Therefore, as in the previous example, for $\alpha = \gamma/2$, we obtain the following asymptotic formula

$$\limsup_{n\to+\infty} n^{\gamma/2}|(S_n f)(t) - f(t)| \leq R\left(\frac{2t}{\pi}\right)^{\gamma/2}|f'(t)|^{\gamma}$$

at every point $t \in [0, +\infty[$ in which $f'(t)$ exists.

Further examples may be deduced, using the same methods, from other classical linear operators such as Bleimann-Butzer-Hahn, Baskakov, Meyer-König, the Zeller operator, and so on.

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