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APPROXIMATION FOR MODIFIED BASKAKOV DURRMEYER TYPE OPERATORS

VIJAY GUPTA

ABSTRACT. In the present paper, we study a certain integral modification of the well-known Baskakov operators with the weight function of Beta basis function. We establish some local and global direct results in ordinary and simultaneous approximation for these new operators.

1. Introduction. For $x \in [0, \infty)$ and $\alpha > 0$, we consider a certain type of Baskakov-Durrmeyer operators as

(1.1)
$$\begin{aligned} B_{n,\alpha}(f(t), x) &= \sum_{k=1}^{\infty} p_{n,k,\alpha}(x) \int_0^{\infty} b_{n,k,\alpha}(t) f(t)\, dt + (1+\alpha x)^{-n/\alpha} f(0) \\ &= \int_0^{\infty} W_{n,\alpha}(x,t) f(t)\, dt \end{aligned}$$

where

$$p_{n,k,\alpha}(x) = \frac{\Gamma(n/\alpha + k)}{\Gamma(k+1)\Gamma(n/\alpha)} \cdot \frac{(\alpha x)^k}{(1+\alpha x)^{(n/\alpha)+k}}$$
$$b_{n,k,\alpha}(t) = \frac{\alpha\Gamma(n/\alpha + k + 1)}{\Gamma(k)\Gamma(n/\alpha + 1)} \cdot \frac{(\alpha t)^{k-1}}{(1+\alpha t)^{(n/\alpha)+k+1}}$$

and

$$W_{n,\alpha}(x,t) = \sum_{k=1}^{\infty} p_{n,k,\alpha}(x) b_{n,k,\alpha}(t) + (1+\alpha x)^{-n/\alpha}\delta(t),$$

$\delta(t)$ being the Dirac delta function.

The operators defined by (1.1) are the integral modification of the well-known Baskakov operators having weight functions of some beta

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basis functions. As a special case, i.e., $\alpha = 1$, the operators (1.1) reduce to the operators recently studied in [**3**]. These operators have different approximation properties than the operators recently studied by Srivastava and Gupta [**8**]. The present paper deals with the study of some local and global direct results in ordinary and simultaneous approximation for the operators (1.1).

2. Basic results. In this section we mention certain lemmas which will be used in the sequel.

Lemma 2.1 [**7**]. *For $m \in N \cup \{0\}$, if the mth order moment is defined as*

$$U_{n,m,\alpha}(x) = \sum_{k=0}^{\infty} p_{n,k,\alpha}(x)\left(\frac{k}{n} - x\right)^m,$$

then $U_{n,0,\alpha}(x) = 1, U_{n,1,\alpha}(x) = 0$ and

$$nU_{n,m+1,\alpha}(x) = x(1+\alpha x)(U^{(1)}_{n,m,\alpha}(x) + mU_{n,m-1,\alpha}(x)).$$

Consequently, we have $U_{n,m,\alpha}(x) = O(n^{-[(m+1)/2]})$.

Lemma 2.2. *Let the function $T_{n,m,\alpha}(x)$, $m \in N \cup \{0\}$, be defined as*

$$\begin{aligned} T_{n,m,\alpha}(x) &= B_{n,\alpha}\big((t-x)^m, x\big) \\ &= \sum_{k=1}^{\infty} p_{n,k,\alpha}(x) \int_0^{\infty} b_{n,k,\alpha}(t)(t-x)^m \, dt \\ &\quad + (1+\alpha x)^{-n/\alpha}(-x)^m. \end{aligned}$$

Then $T_{n,0,\alpha}(x) = 1, T_{n,1,\alpha} = 0$,

$$T_{n,2,\alpha}(x) = \frac{2x(1+\alpha x)}{n-\alpha},$$

and also the following recurrence relation holds:

$$\begin{aligned} (n-\alpha m)T_{n,m+1,\alpha}(x) &= x(1+\alpha x)\lfloor T^{(1)}_{n,m,\alpha}(x) + 2mT_{n,m-1,\alpha}(x)\rfloor \\ &\quad + m(1+2\alpha x)T_{n,m,\alpha}(x). \end{aligned}$$

Proof. By definition, we have

$$\begin{aligned}
T^{(1)}_{n,m,\alpha}(x) = &\sum_{k=1}^{\infty} p^{(1)}_{n,k,\alpha}(x) \int_0^{\infty} b_{n,k,\alpha}(t)(t-x)^m \, dt \\
&- m \sum_{k=1}^{\infty} p_{n,k,\alpha}(x) \int_0^{\infty} b_{n,k,\alpha}(t)(t-x)^{m-1} \, dt \\
&- n(1+\alpha x)^{-n/\alpha - 1}(-x)^m \\
&- m(1+\alpha x)^{-n/\alpha}(-x)^{m-1}.
\end{aligned}$$

Using the identities $x(1+\alpha x)p^{(1)}_{n,k,\alpha}(x) = (k-nx)p_{n,k,\alpha}(x)$ and $t(1+\alpha t)b^{(1)}_{n,k,\alpha}(t) = [(k-1)-(n+2\alpha)t]b_{n,k,\alpha}(t)$, we have

$$\begin{aligned}
&x(1+\alpha x)[T^{(1)}_{n,m,\alpha}(x) + mT_{n,m-1,\alpha}(x)] \\
&\quad = \sum_{k=1}^{\infty} p_{n,k,\alpha}(x) \int_0^{\infty} (k-nx) b_{n,k,\alpha}(t)(t-x)^m \, dt \\
&\qquad + n(1+\alpha x)^{-n/\alpha}(-x)^{m+1} = \sum_{k=1}^{\infty} p_{n,k,\alpha}(x) \\
&\qquad \times \int_0^{\infty} [(k-1)-(n+2\alpha)t + (n+2\alpha)(t-x) + (1+2\alpha x)] \\
&\qquad \cdot b_{n,k,\alpha}(t)(t-x)^m \, dt + n(1+\alpha x)^{-n/\alpha}(-x)^{m+1} \\
&\quad = \sum_{k=1}^{\infty} p_{n,k,\alpha}(x) \int_0^{\infty} t(1+\alpha t) b^{(1)}_{n,k,\alpha}(t)(t-x)^m \, dt \\
&\qquad + (n+2\alpha)[T_{n,m+1,\alpha}(x) - (1+\alpha x)^{-n/\alpha}(-x)^{m+1}] \\
&\qquad + (1+2\alpha x)[T_{n,m,\alpha}(x) - (1+\alpha x)^{-n/\alpha}(-x)^m] \\
&\qquad + n(1+\alpha x)^{-n/\alpha}(-x)^{m+1} \\
&\quad = -(m+1)(1+2\alpha x)[T_{n,m,\alpha}(x) - (1+\alpha x)^{-n/\alpha}(-x)^m] \\
&\qquad - \alpha(m+2)[T_{n,m+1,\alpha} - (1+\alpha x)^{-n/\alpha}(-x)^{m+1}] \\
&\qquad - mx(1+\alpha x)[T_{n,m-1,\alpha}(x) - (1+\alpha x)^{-n/\alpha}(-x)^{m-1}] \\
&\qquad + (n+2\alpha)[T_{n,m+1,\alpha} - (1+\alpha x)^{-n/\alpha}(-x)^{m+1}] \\
&\qquad + (1+2\alpha x)[T_{n,m,\alpha}(x) - (1+\alpha x)^{-n/\alpha}(-x)^m] \\
&\qquad + n(1+\alpha x)^{-n/\alpha}(-x)^{m+1}.
\end{aligned}$$

Thus, we get

$$(n-\alpha m)T_{n,m+1,\alpha}(x)$$
$$= x(1+\alpha x)[T^{(1)}_{n,m,\alpha}(x) + 2mT_{n,m-1,\alpha}(x)] + m(1+2\alpha x)T_{n,m,\alpha}(x).$$

This completes the proof of the recurrence relation. From the above recurrence relation, it is easily verified, for all $x \in [0,\infty)$, that

$$T_{n,m,\alpha}(x) = O\big(n^{-[(m+1)/2]}\big). \qquad \square$$

Remark 1. It is easily verified from Lemma 2.1 that, for each $x \in (0,\infty)$,

$$B_{n,\alpha}(t^i,x) = \frac{\Gamma(n/\alpha+i)\Gamma(n/\alpha-i+1)}{\Gamma(n/\alpha+1)\Gamma(n/\alpha)}x^i + i(i-1)\frac{\Gamma(n/\alpha+i-1)\Gamma(n/\alpha-i+1)}{\Gamma(n/\alpha+1)\Gamma(n/\alpha)}x^{i-1} + O(n^{-2}).$$

The following corollary is a direct consequence of Lemma 2.2 and Remark 1.

Corollary 2.3. *Let δ be a positive number. Then, for every $\gamma > 0$, $x \in (0,\infty)$, there exists a constant $M(s,x)$ independent of n and depending on s and x such that*

$$\int_{|t-x|>\delta} W_{n,\alpha}(x,t)t^\gamma \, dt \le M(s,x)n^{-s}, \quad s = 1,2,3,\dots .$$

Lemma 2.4 [**7**]. *The polynomials $Q_{i,j,r,\alpha}(x)$ exist independent of n and k such that*

$$\{x(1+\alpha x)\}^r D^r\left[p_{n,k,\alpha}(x)\right] = \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i(k-nx)^j Q_{i,j,r,\alpha}(x)p_{n,k,\alpha}(x), \quad \text{where } D \equiv \frac{d}{dx}.$$

Lemma 2.5. *If f is r times differentiable on $[0,\infty)$, such that $f^{(r-1)} = O(t^\gamma), \gamma > 0$ as $t \to \infty$, then for $r = 1, 2, 3, \dots$ and $n > \gamma + r$ we have*

$$B_{n,\alpha}^{(r)}(f,x) = \frac{\Gamma(n/\alpha + r)(n/\alpha - r + 1)}{\Gamma(n/\alpha + 1)\Gamma(n/\alpha)} \sum_{k=0}^{\infty} p_{n+\alpha r,k}(x) \times \int_0^\infty b_{n-\alpha r,k+r}(t) f^{(r)}(t)\, dt.$$

The proof of the above lemma follows along the lines of [**6**, Lemma 2.7]; thus, we omit the details. $\square$

3. Local and global approximation. By $C_B[0,\infty)$ we denote the space of real-valued continuous bounded functions f on the interval $[0,\infty)$; the norm-$||\cdot||$ on the space $C_B[0,\infty)$ is given by

$$||f|| = \sup_{0 \le x < \infty} |f(x)|.$$

The Peetre's K-functional is defined by

$$K_2(f,\delta) = \inf[\{||f-g|| + \delta||g''|| : g \in W_\infty^2\},$$

where $W_\infty^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By [**1**, page 177], there exists a positive constant $M > 0$ such that $K_2(f,\delta) \le M\omega_2(f,\delta^{1/2}), \delta > 0$ and the second order modulus of smoothness is given by

$$\omega_2(f,\delta^{1/2}) = \sup_{0<h\le\delta^{1/2}} \sup_{0\le x<\infty} |f(x+2h) - 2f(x+h) + f(x)|.$$

Also for $f \in C_B[0,\infty)$ the usual modulus of continuity is given by

$$\omega(f,\delta) = \sup_{0<h\le\delta} \sup_{0\le x<\infty} |f(x+h) - f(x)|.$$

Our first theorem in this section is the following local result:

Theorem 3.1. *Let $f \in C_B[0,\infty)$. Then for all $x \in [0,\infty)$ and $n \in N, n > \alpha$, there exists an absolute positive constant $M > 0$ such that*

$$|B_{n,\alpha}(f,x) - f(x)| \leq M\omega_2\left(f, \sqrt{\frac{2x(1+\alpha x)}{n-\alpha}}\right).$$

Proof. Let $g \in W_\infty^2$ and $x, t \in [0,\infty)$. By Taylor's expansion, we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\,du.$$

Applying Lemma 2.2, we obtain

$$B_{n,\alpha}(g,x) - g(x) = B_{n,\alpha}\left(\int_x^t (t-u)g''(u)\,du, x\right).$$

Also it is easily seen that $|\int_x^t (t-u)g''(u)\,du| \leq (t-x)^2||g''||$. Therefore,

$$|B_{n,\alpha}(g,x) - g(x)| \leq B_{n,\alpha}((t-x)^2, x)||g''|| = \frac{2x(1+\alpha x)}{n-\alpha}||g''||.$$

Using Lemma 2.2, we have

$$\begin{aligned}|B_{n,\alpha}(f,x)| \leq \sum_{k=1}^\infty p_{n,k,\alpha}(x) &\int_0^\infty b_{n,k,\alpha}(t)|f(t)|\,dt \\ &+ (1+\alpha x)^{-n/\alpha}|f(0)| \leq ||f||.\end{aligned}$$

Thus,

$$\begin{aligned}|B_{n,\alpha}(f,x) - f(x)| &\leq |B_{n,\alpha}(f-g,x) - (f-g)(x)| \\ &\quad + |B_{n,\alpha}(g,x) - g(x)| \\ &\leq 2||f-g|| + \frac{2x(1+\alpha x)}{n-\alpha}||g''||.\end{aligned}$$

Finally, taking the infimum over all $g \in W_\infty^2$ and using the inequality $K_2(f,\delta) \leq M\omega_2(f,\delta^{1/2})$, $\delta > 0$, we get the required result. This completes the proof of Theorem 3.1. $\square$

Our next result in this section is the global direct theorem.

Theorem 3.2. *Let $f \in C_B[0,\infty)$ and $n \geq \alpha + 1$. Then*

$$||B_{n,\alpha}f - f|| \leq C\omega_{\varphi}^2(f, n^{-1/2}),$$

where $C > 0$ is an absolute constant and $\varphi(x) = \sqrt{2x(1+\alpha x)}, x \in [0,\infty)$ and

$$\omega_{\varphi}^2(f,\delta) = \sup_{0<h\leq\delta} \sup_{x\pm h\varphi(x)\in[0,\infty)} |f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x))|$$

is the Ditzian Totik modulus of smoothness of second order.

Proof. Let

$$W_{\infty}^2(\varphi) = \{g \in C_B[0,\infty) : g', g'', \varphi^2 g'' \in C_B[0,\infty)\}.$$

For $g \in W_{\infty}^2(\varphi)$, using Taylor's expansion, we have

$$g(t) = g\left(\frac{k}{n}\right) + g'\left(\frac{k}{n}\right)\left(t - \frac{k}{n}\right) + \int_{k/n}^{t} (t-u)g''(u)\,du.$$

Also by Lemma 2.2, we have

$$\begin{aligned} B_{n,\alpha}(g,x) &- \widehat{B}_{n,\alpha}(g,x) \\ &= \sum_{k=1}^{\infty} p_{n,k,\alpha}(x) \int_0^{\infty} \left[g(t) - g\left(\frac{k}{n}\right)\right] b_{n,k,\alpha}(t)\,dt \\ &= \sum_{k=1}^{\infty} p_{n,k,\alpha}(x) \int_0^{\infty} b_{n,k,\alpha}(t) \left[\int_{k/n}^{t} (t-u)g''(u)\,du\right] dt, \end{aligned}$$

where $\widehat{B}_{n,\alpha}(f,x)$ denotes the modified Baskakov operators given by

$$\sum_{k=0}^{\infty} \frac{\Gamma(n/\alpha + k)}{\Gamma(k+1)\Gamma(n/\alpha)} \frac{(\alpha x)^k}{(1+\alpha x)^{n/\alpha+k}} f(k/n).$$

Therefore, by [**2**, page 140], we have

$$
\begin{aligned}
(3.1)\quad & |B_{n,\alpha}(g,x)-\widehat{B}_{n,\alpha}(g,x)| \\
&\le \sum_{k=1}^{\infty} p_{n,k,\alpha}(x)\int_0^{\infty} b_{n,k,\alpha}(t)\left|\int_{k/n}^{t}\frac{|t-u|}{2u(1+\alpha u)}\,du\right| dt\cdot\|\varphi^2 g''\| \\
&\le \frac{1}{2}\sum_{k=1}^{\infty} p_{n,k,\alpha}(x)\int_0^{\infty} b_{n,k,\alpha}(t)\frac{(t-k/n)^2}{k/n} \\
&\cdot\left(\frac{1}{1+\alpha k/n}+\frac{1}{1+\alpha t}\right) dt\|\varphi^2 g''\| \\
&= \frac{1}{2}\sum_{k=1}^{\infty} p_{n,k,\alpha}(x)\frac{n}{k}\frac{n}{n+\alpha k}\int_0^{\infty} b_{n,k,\alpha}(t)\left(t-\frac{k}{n}\right)^2 dt\|\varphi^2 g''\| \\
&\quad + \frac{1}{2}\sum_{k=1}^{\infty} p_{n,k,\alpha}(x)\frac{n}{k}\int_0^{\infty} b_{n,k,\alpha}(t)\frac{(t-k/n)^2}{1+\alpha t}\,dt\|\varphi^2 g''\|.
\end{aligned}
$$

Also by easy computation, we have

$$
\int_0^{\infty} b_{n,k,\alpha}(t)(t-k/n)^2\,dt = \frac{k}{n}\cdot\frac{n+\alpha k}{n}\cdot\frac{1}{n-\alpha} \tag{3.2}
$$

and

$$
\int_0^{\infty} b_{n,k,\alpha}(t)\frac{(t-k/n)^2}{1+\alpha t}\,dt = \frac{k}{n}\cdot\frac{1}{n}\cdot\frac{n+\alpha k}{n+\alpha(k+1)}. \tag{3.3}
$$

Therefore combining (3.1)–(3.3), we get

$$
\begin{aligned}
(3.4)\quad & |B_{n,\alpha}(g,x)-\widehat{B}_{n,\alpha}(g,x)| \\
&\le \frac{1}{2}\left\{\sum_{k=1}^{\infty} p_{n,k,\alpha}(x)\frac{1}{n-\alpha}+\sum_{k=1}^{\infty} p_{n,k,\alpha}(x)\frac{1}{n}\cdot\frac{n+\alpha k}{n+\alpha(k+1)}\right\}\|\varphi^2 g''\| \\
&\le \frac{1}{2}\left\{\frac{1}{n-\alpha}+\sum_{k=1}^{\infty} p_{n,k,\alpha}(x)\frac{1}{n}\right\}\|\varphi^2 g''\| \le \frac{1}{n-\alpha}\cdot\|\varphi^2 g''\|.
\end{aligned}
$$

Applying Lemma 2.2, for all $f\in C_B[0,\infty)$, we have

$$
\|B_{n,\alpha} f\| \le \|f\|. \tag{3.5}
$$

Next, applying Taylor's expansion of g, the Szhwarz inequality and using the fact that $\widehat{B}_{n,\alpha}(1,x)=1$, $\widehat{B}_{n,\alpha}(t-x,x)=0$ and $\widehat{B}_{n,\alpha}((t-x)^2,x)=(x(1+\alpha x))/n$, we have

(3.6)
$$\begin{aligned}
|\widehat{B}_{n,\alpha}(g,x)-g(x)| &\leq \left|\widehat{B}_{n,\alpha}\left(\int_{k/n}^{t}(t-u)g''(u)\,du,x\right)\right| \\
&\leq \widehat{B}_{n,\alpha}\left(\left|\int_{k/n}^{t}|t-u|\cdot|g''(u)|\,du\right|,x\right) \\
&\leq \widehat{B}_{n,\alpha}((t-x)^2,x)\cdot\|g''\| \leq \frac{x(1+\alpha x)}{n}\|g''\| \\
&= \frac{1}{n}\|\hat{\varphi}^2 g''\|,
\end{aligned}$$

where $\hat{\varphi}(x)=\sqrt{x(1+\alpha x)}$, $x\in[0,\infty)$. Combining (3.4), (3.5) and (3.6), we obtain

$$\begin{aligned}
(3.7)\quad & |B_{n,\alpha}(f,x)-f(x)| \\
&\leq |B_{n,\alpha}(f-g,x)-(f-g)(x)| \\
&\quad + |B_{n,\alpha}(g,x)-\widehat{B}_{n,\alpha}(g,x)| + |\hat{B}_{n,\alpha}(g,x)-g(x)| \\
&\leq 2\|f-g\| + \frac{1}{n-\alpha}\|\varphi^2 g''\| + \frac{1}{n}\|\hat{\varphi}^2 g''\| \\
&\leq 2\left\{\|f-g\| + \frac{1}{n-\alpha}\|\varphi^2 g''\|\right\} + \left\{\|f-g\| + \frac{1}{n}\|\hat{\varphi}^2 g''\|\right\}.
\end{aligned}$$

Proceeding along the lines of [**2**, page 10], we consider the K-functionals:

$$K_{\varphi}^2(f,\delta)=\inf\{\|f-g\|+\delta\|\varphi^2 g''\| : g\in W_\infty^2(\varphi)\},\quad \delta>0,$$

and

$$K_{\hat{\varphi}}^2(f,\delta)=\inf\{\|f-g\|+\delta\|\hat{\varphi}^2 g''\| : g\in W_\infty^2(\varphi)\},\quad \delta>0.$$

By (3.7), for all $f\in C_B[0,\infty)$ and $n\geq\alpha+1$, we obtain

$$\|B_{n,\alpha}f-f\|\leq 2K_{\varphi}^2(f,(n-\alpha)^{-1})+K_{\hat{\varphi}}^2(f,n^{-1}).$$

Using [**2**, page 11], the inequality $\hat{\varphi}(x) \leq \varphi(x)$ and [**2**, page 37], we get the desired result. This completes the proof of Theorem 3.2. $\square$

4. Simultaneous approximation. By $C_\gamma[0,\infty)$ we denote the class $\{f \in C[0,\infty) : |f(t)| \leq Mt^\gamma \text{ for some } M > 0, \gamma > 0\}$, and the norm $||\cdot||_\gamma$ on the class $C_\gamma[0,\infty)$ is defined as $||f||_\gamma = \sup_{0\leq t<\infty} |f(t)|t^{-\gamma}$. In this section we present the following results.

Theorem 4.1. *Let $f \in C_\gamma[0,\infty)$ and $f^{(r)}$ exist at a point $x \in (0,\infty)$. Then we have*

$$\lim_{n\to\infty} B_{n,\alpha}^{(r)}(f,x) = f^{(r)}(x).$$

Proof. By Taylor expansion of f, we have

$$f(t) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^r,$$

where $\varepsilon(t,x) \to 0$ as $t \to x$. Hence,

$$\begin{aligned} B_{n,\alpha}^{(r)}(f,x) &= \int_0^\infty W_{n,\alpha}^{(r)}(t,x)f(t)\,dt \\ &= \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \int_0^\infty W_{n,\alpha}^{(r)}(t,x)(t-x)^i\,dt \\ &\quad + \int_0^\infty W_{n,\alpha}^{(r)}(t,x)\varepsilon(t,x)(t-x)^r\,dt \\ &=: R_1 + R_2. \end{aligned}$$

First to estimate R_1, using binomial expansion of $(t-x)^i$ and Remark 1, we have

$$\begin{aligned} R_1 &= \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \sum_{v=0}^{i} \left[\begin{matrix} i \\ v \end{matrix}\right] (-x)^{i-v} \frac{\partial^r}{\partial x^r} \int_0^\infty W_{n,\alpha}(t,x)t^v\,dt \\ &= \frac{f^{(r)}(x)}{r!} \frac{d^r}{dx^r}\bigg[\frac{\Gamma(n/\alpha+r)\Gamma(n/\alpha-r+1)!}{\Gamma(n/\alpha+1)\Gamma(n/\alpha)}x^r \\ &\qquad + \text{ terms containing lower powers of } x\bigg] \\ &= f^{(r)}(x)\left[\frac{\Gamma(n/\alpha+r)\Gamma(n/\alpha-r+1)!}{\Gamma(n/\alpha+1)\Gamma(n/\alpha)}\right] \longrightarrow f^{(r)}(x) \end{aligned}$$

as $n \to \infty$. Next applying Lemma 2.4, we obtain

$$\begin{aligned}
R_2 &= \int_0^\infty W_{n,\alpha}^{(r)}(t,x)\varepsilon(t,x)(t-x)^r\,dt,\\
|R_2| &\le \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \frac{|Q_{i,j,r,\alpha}(x)|}{\{x(1+\alpha x)\}^r} \sum_{k=1}^\infty |k-nx|^j p_{n,k,\alpha}(x)\\
&\quad \int_0^\infty b_{n,k,\alpha}(t)|\varepsilon(t,x)||t-x|^r\,dt\\
&\quad + \frac{\Gamma(n/\alpha+r+2)}{\Gamma(n/\alpha)}(1+\alpha x)^{-n/\alpha-r}|\varepsilon(0,x)|x^r.
\end{aligned}$$

The second term in the above expression tends to zero as $n \to \infty$. Since $\varepsilon(t,x) \to 0$ as $t \to x$ for a given $\varepsilon > 0$, there exists a δ such that $|\varepsilon(t,x)| < \varepsilon$ whenever $0 < |t-x| < \delta$. If $\mu \ge \max\{\gamma, r\}$, where μ is any integer, then we can find a constant $M_3 > 0$, $|\varepsilon(t,x)(t-x)^r| \le M_3|t-x|^\mu$, for $|t-x| \ge \delta$. Therefore,

$$\begin{aligned}
|R_2| &\le M_3 \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \sum_{k=0}^\infty p_{n,k,\alpha}(x)|k-nx|^j\\
&\quad \cdot \left\{\varepsilon \int_{|t-x|<\delta} b_{n,k,\alpha}(t)|t-x|^r\,dt\right.\\
&\quad\quad \left. + \int_{|t-x|\ge\delta} b_{n,k,\alpha}(t)|t-x|^\mu\,dt\right\}\\
&=: R_3 + R_4.
\end{aligned}$$

Applying the Cauchy-Schwarz inequality for integration and summation, respectively, we obtain

$$\begin{aligned}
R_3 &\le \varepsilon M_3 \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \left\{\sum_{k=1}^\infty p_{n,k,\alpha}(x)(k-nx)^{2j}\right\}^{1/2}\\
&\quad \times \left\{\sum_{k=1}^\infty p_{n,k,\alpha}(x)\int_0^\infty b_{n,k,\alpha}(t)(t-x)^{2r}\,dt\right\}^{1/2}.
\end{aligned}$$

Using Lemmas 2.1 and 2.2, we get

$$R_3 = \varepsilon \cdot O(n^{r/2})O(n^{-r/2}) = \varepsilon \cdot O(1).$$

Again using the Cauchy-Schwarz inequality, Lemma 2.1 and Corollary 2.3, we get

$$\begin{aligned} R_4 &\le M_4 \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \sum_{k=1}^{\infty} p_{n,k,\alpha}(x)|k-nx|^j \int_{|t-x|\ge\delta} b_{n,k,\alpha}(t)|t-x|^{\mu}\,dt \\ &\le M_4 \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \sum_{k=1}^{\infty} p_{n,k,\alpha}(x)|k-nx|^j \left\{\int_{|t-x|\ge\delta} b_{n,k,\alpha}(t)\,dt\right\}^{1/2} \\ &\quad \times \left\{\int_{|t-x|\ge\delta} b_{n,k,\alpha}(t)(t-x)^{2\mu}\,dt\right\}^{1/2} \\ &\le M_4 \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \left\{\sum_{k=1}^{\infty} p_{n,k,\alpha}(x)(k-nx)^{2j}\right\}^{1/2} \\ &\quad \times \left\{\sum_{k=1}^{\infty} p_{n,k,\alpha}(x)\int_0^{\infty} b_{n,k,\alpha}(t)(t-x)^{2\mu} dt\right\}^{1/2} \\ &= \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i O(n^{j/2}) O(n^{-\mu/2}) = O(n^{(r-\mu)/2}) = o(1). \end{aligned}$$

Collecting the estimates of $R_1 - R_4$, we get the required result.

Theorem 4.2. *Let $f \in C_\gamma[0,\infty)$. If $f^{(r+2)}$ exists at a point $x \in (0,\infty)$, then*

$$\begin{aligned} &\lim_{n\to\infty} n[B_{n,\alpha}^{(r)}(f,x) - f^{(r)}(x)] \\ &\quad = \alpha r(r-1) f^{(r)}(x) + r(1+2\alpha x) f^{(r+1)}(x) + x(1+\alpha x) f^{(r+2)}(x). \end{aligned}$$

Proof. Using Taylor's expansion of f, we have

$$f(t) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^{r+2},$$

where $\varepsilon(t,x) \to 0$ as $t \to x$ and $\varepsilon(t,x) = O((t-x)^{\gamma}), t \to \infty$ for $\gamma > 0$. Applying Lemma 2.2, we have

$$\begin{aligned} n[B_{n,\alpha}^{(r)}(f(t),x) - f^{(r)}(x)] &\\ = n\Big[\sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} \int_0^{\infty} W_{n,\alpha}^{(r)}(t,x)(t-x)^i dt - f^{(r)}(x)\Big] &\\ + n\int_0^{\infty} W_{n,\alpha}^{(r)}(t,x)\varepsilon(t,x)(t-x)^{r+2}\,dt =: E_1 + E_2. \end{aligned}$$

First, we have

$$\begin{aligned} E_1 &= n\sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \begin{bmatrix} i \\ j \end{bmatrix} (-x)^{i-j} \int_0^{\infty} W_{n,\alpha}^{(r)}(t,x)t^j\,dt - nf^{(r)}(x) \\ &= \frac{f^{(r)}(x)}{r!} n\left[B_{n,\alpha}^{(r)}(t^r,x) - (r!)\right] \\ &\quad + \frac{f^{(r+1)}(x)}{(r+1)!} n[(r+1)(-x)B_{n,\alpha}^{(r)}(t^r,x) + B_{n,\alpha}^{(r)}(t^{r+1},x)] \\ &\quad \frac{f^{(r+2)}(x)}{(r+2)!} n\Big[\frac{(r+2)(r+1)}{2} x^2 B_{n,\alpha}^{(r)}(t^r,x) \\ &\quad + (r+2)(-x)B_{n,\alpha}^{(r)}(t^{r+1},x) + B_{n,\alpha}^{(r)}(t^{r+2},x)\Big]. \end{aligned}$$

Therefore, by Remark 1, we have

$$\begin{aligned} E_1 &= nf^{(r)}(x)\Big[\frac{\Gamma(n/\alpha+r)\Gamma(n/\alpha-r+1)}{\Gamma(n/\alpha+1)\Gamma(n/\alpha)} - 1\Big] \\ &\quad + \frac{nf^{(r+1)}(x)}{(r+1)!}\Big[(x-1)(-x)\Big\{\frac{\Gamma(n/\alpha+r)\Gamma(n/\alpha-r+1)}{\Gamma(n/\alpha+1)\Gamma(n/\alpha)}\Big\} \\ &\quad + \Big\{\frac{\Gamma(n/\alpha+r+1)\Gamma(n/\alpha-r)}{\Gamma(n/\alpha+1)\Gamma(n/\alpha)!}(r+1)!x \\ &\quad + (r+1)r\frac{\Gamma(n/\alpha+r)\Gamma(n/\alpha-r)}{\Gamma(n/\alpha+1)\Gamma(n/\alpha)} r!\Big\}\Big] \\ &\quad + \frac{nf^{(r+2)}(x)}{(r+2)!}\Big[\frac{(r+2)(r+1)}{2} x^2 \frac{\Gamma(n/\alpha+r)\Gamma(n/\alpha-r+1)}{\Gamma(n/\alpha+1)\Gamma(n/\alpha)!} r!\Big] \end{aligned}$$

$$+ (r+2)(-x)\left\{\frac{\Gamma(n/\alpha+r+1)\Gamma(n/\alpha-r)}{2}x(r+1)!\right.$$
$$\left.+ (r+1)r\frac{\Gamma(n/\alpha-r)\Gamma(n/\alpha-r)}{\Gamma(n/\alpha+1)\Gamma(n/\alpha)}r!\right\}$$
$$+\left\{\frac{\Gamma(n/\alpha+r+2)\Gamma(n/\alpha-r-1)}{\Gamma(n/\alpha+1)\Gamma(n/\alpha)}\frac{(r+2)!}{2}x^2\right.$$
$$\left.\left.+ (r+2)(r+1)\frac{\Gamma(n/\alpha+r+1)\Gamma(n/\alpha-r-1)}{\Gamma(n/\alpha+1)\Gamma(n/\alpha)}(r+1)!x\right\}\right]$$
$$+ O(n^{-2}).$$

In order to complete the proof of the theorem it is sufficient to show that $E_2 \to 0$ as $n \to \infty$ which easily follows proceeding along the lines of the proof of Theorem 4.1 and by using Lemmas 2.1, 2.2 and 2.4.

Theorem 4.3 *Let $f \in C_\gamma[0,\infty)$ and suppose $0 < a < a_1 < b_1 < b < \infty$. Then, for all n sufficiently large, we have*

$$\|B_{n,\alpha}^{(r)}(f,\bullet) - f^{(r)}\|_{C[a_1,b_1]} \le \left\{M_1\omega_2(f^{(r)}, n^{-1/2}, a, b) + M_2 n^{-1}\|f\|_\gamma\right\}$$

where $M_1 = M_1(r), M_2 = M_2(r,f)$ and $\omega_2(f^{(r)}, n^{-1/2}, a, b) = \sup\{|\Delta_h^2 f^{(r)}(x)| : |h| \le n^{-1/2}; x, x+2h \in [a,b]\}$.

Proof. For sufficiently small $\delta > 0$, we define a function $f_{2,\delta}(t)$ corresponding to $f \in C_\gamma[0,\infty)$ by

$$f_{2,\delta}(t) = \delta^{-2}\int_{-\delta/2}^{\delta/2}\int_{-\delta/2}^{\delta/2}\left(f(t) - \Delta_\eta^2 f(t)\right)dt_1\,dt_2,$$

where $\eta = (t_1+t_2)/2$, $t \in [a,b]$, and $\Delta_\eta^2 f(t)$ is the second forward difference of f with step length η. Following [**4**] it is easily checked that:

(i) $f_{2,\delta}$ has continuous derivatives up to order 2 on $[a,b]$,

(ii) $\|f_{2,\delta}^{(r)}\|_{C[a_1,b_1]} \le \widehat{M}_1\delta^{-r}\omega_2(f,\delta,a,b)$,

(iii) $\|f - f_{2,\delta}\|_{C[a_1,b_1]} \le \widehat{M}_2\omega_2(f,\delta,a,b)$,

(iv) $\|f_{2,\delta}\|_{C[a,b]} \le \widehat{M}_3\|f\|_\gamma$, where $\widehat{M}_i$, $i = 1, 2, 3$, are certain constants that depend on $[a, b]$ but are independent of f and n.

We can write

$$\begin{aligned}
&||B_{n,\alpha}^{(r)}(f,\bullet) - f^{(r)}||_{C[a_1,b_1]} \\
&\qquad \le ||B_{n,\alpha}^{(r)}(f - f_{2,\delta},\bullet)||_{C[a_1,b_1]} + \|B_{n,\alpha}^{(r)}(f_{2,\delta},\bullet) - f_{2,\delta}^{(r)}\|_{C[a_1,b_1]} \\
&\qquad\quad + ||f^{(r)} - f_{2,\delta}^{(r)}||_{C[a_1,b_1]} =: J_1 + J_2 + J_3.
\end{aligned}$$

Since $f_{2,\delta}^{(r)} = (f^{(r)})_{2,\delta}(t)$, by property (iii) of the function $f_{2,\delta}$, we get

$$J_3 \le \widehat{M}_4 \omega_2(f^{(r)}, \delta, a, b).$$

Next, on an application of Theorem 4.2, it follows that

$$J_2 \le \widehat{M}_5 n^{-1} \sum_{j=r}^{r+2} \|f_{2,\delta}^{(j)}\|_{C[a,b]}.$$

Using the interpolation property due to Goldberg and Meir [**5**], for each $j = r,\ r+1,\ r+2$, it follows that

$$||f_{2,\delta}^{(j)}||_{C[a,b]} \le \widehat{M}_6 \big\{||f_{2,\delta}||_{C[a,b]} + ||f_{2,\delta}^{(r+2)}||_{C[a,b]}\big\}.$$

Therefore, by applying properties (iii) and (iv) of the function $f_{2,\delta}$, we obtain

$$J_2 \le \widehat{M}_7 . n^{-1} \big\{||f||_\gamma + \delta^{-2} \omega_2(f^{(r)}, \delta, a, b)\big\}.$$

Finally we shall estimate J_1, choosing a^*, b^* satisfying the conditions $0 < a < a^* < a_1 < b_1 < b^* < b < \infty$. Suppose $\psi(t)$ denotes the characteristic function of the interval $[a^*, b^*]$. Then

$$\begin{aligned}
J_1 &\le ||B_{n,\alpha}^{(r)}(\psi(t)(f(t) - f_{2,\delta}(t)),\bullet)\|_{C[a_1,b_1]} \\
&\quad + \|B_{n,\alpha}^{(r)}((1-\psi(t))(f(t) - f_{2,\delta}(t)),\bullet)\|_{C[a_1,b_1]} \\
&=: J_4 + J_5.
\end{aligned}$$

Using Lemma 2.5, it is clear that

$$\begin{aligned}
&B_{n,\alpha}^{(r)}\big(\psi(t)(f(t) - f_{2,\delta}(t)), x\big) \\
&\qquad = \frac{\Gamma(n/\alpha + r)\Gamma(n/\alpha - r + 1)}{\Gamma(n/\alpha + 1)\Gamma(n/\alpha)} \sum_{k=0}^{\infty} p_{n+r,k,\alpha}(x) \\
&\qquad\qquad \times \int_0^\infty b_{n-r,k+r,\alpha}(t)\psi(t)(f^{(r)}(t) - f_{2,\delta}^{(r)}(t))\, dt.
\end{aligned}$$

Hence,

$$||B_{n,\alpha}^{(r)}(\psi(t)(f(t)-f_{2,\delta}(t)),\bullet)||_{C[a_1,b_1]} \le \widehat{M}_8||f^{(r)}-f_{2,\delta}^{(r)}||_{C[a^*,b^*]}.$$

Next, for $x \in [a_1,b_1]$ and $t \in [0,\infty) \setminus [a^*,b^*]$, we choose a $\delta_1 > 0$ satisfying $|t-x| \ge \delta_1$.

Therefore, by Lemma 2.4 and the Cauchy-Schwarz inequality, we have

$$I \equiv B_{n,\alpha}^{(r)}((1-\psi(t))(f(t)-f_{2,\delta}(t),x)$$

and

$$\begin{aligned}|I| \le &\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \frac{|Q_{i,j,r,\alpha}(x)|}{\{x(1+\alpha x)\}^r}\sum_{k=1}^{\infty} p_{n,k,\alpha}(x)|k-nx|^j\\ &\times \int_0^\infty b_{n,k,\alpha}(t)(1-\psi(t))|f(t)-f_{2,\delta}(t)|\,dt\\ &+\frac{\Gamma(n/\alpha+r)}{\Gamma(n/\alpha)}(1+\alpha x)^{-n/\alpha-r}(1-\psi(0))|f(0)-f_{2,\delta}(0)|.\end{aligned}$$

For sufficiently large n, the second term tends to zero. Thus,

$$\begin{aligned}|I| &\le \widehat{M}_9||f||_\gamma \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \sum_{k=1}^{\infty} p_{n,k,\alpha}(x)|k-nx|^j \int_{|t-x|\ge\delta_1} b_{n,k,\alpha}(t)\,dt\\ &\le \widehat{M}_9||f||_\gamma \delta_1^{-2m} \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \sum_{k=1}^{\infty} p_{n,k,\alpha}(x)|k-nx|^j\\ &\quad\times\left(\int_0^\infty b_{n,k,\alpha}(t)\,dt\right)^{1/2}\\ &\quad\times\left(\int_0^\infty b_{n,k,\alpha}(t)(t-x)^{4m}\,dt\right)^{1/2}\\ &\le \widehat{M}_9||f||_\gamma \delta_1^{-2m} \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \left\{\sum_{k=1}^{\infty} p_{n,k,\alpha}(x)(k-nx)^{2j}\right\}^{1/2}\\ &\quad\times\left\{\sum_{k=1}^{\infty} p_{n,k,\alpha}(x)\int_0^\infty b_{n,k,\alpha}(t)(t-x)^{4m}\,dt\right\}^{1/2}.\end{aligned}$$

Hence, by using Lemmas 2.1 and 2.2, we have

$$I \leq \widehat{M}_{10}||f||_\gamma \delta_1^{-2m} O(n^{(i+(j/2)-m)}) \leq \widehat{M}_{11} n^{-q}||f||_\gamma,$$

where $q = m - (r/2)$. Now choosing $m > 0$ satisfying $q \geq 1$, we obtain $I \leq \widehat{M}_{11} n^{-1}\|f\|_\gamma$. Therefore, by property (iii) of the function $f_{2,\delta}(t)$, we get

$$\begin{aligned} J_1 &\leq \widehat{M}_8||f^{(r)} - f_{2,\delta}^{(r)}||_{C[a^*,b^*]} + \widehat{M}_{11} n^{-1}||f||_\gamma \\ &\leq \widehat{M}_{12}\omega_2(f^{(r)}, \delta, a, b) + \widehat{M}_{11} n^{-1}||f|||_\gamma. \end{aligned}$$

Choosing $\delta = n^{-1/2}$, the theorem follows. □

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School of Applied Sciences, Netaji Subhas Institute of Technology
Sector 3 Dwarka, New Delhi 110078, India
Email address: vijaygupta2001@hotmail.com