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AN EXTENSION OF THE FIRST-ORDER STARK CONJECTURE

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ABSTRACT. We extend the first-order abelian Stark conjecture to include situations where every imprimitive L_S-function vanishes at $s = 0$ but no prime in S splits completely. We reduce the problem to finding certain exponential factors in the original Stark units and prove that the extended conjecture follows from the original conjecture under certain circumstances.

1. Introduction. Broadly speaking, the Stark conjectures form a link between the leading term of the Taylor series of imprimitive Artin L_S-functions at $s = 0$ for Galois extensions K/k and certain algebraic objects in K. When the Galois group is abelian and every L_S-function of K/k has at least a first-order zero at $s = 0$, the refined first-order abelian Stark conjecture states there is an S-unit in K which evaluates $L'_S(0, \chi)$ for all characters χ of the Galois group and whose nth roots generate abelian extensions over the base field.

Previously, the first-order abelian Stark conjecture assumed that some prime in S splits completely in K/k. This condition ensures that all L_S-functions have at least a first-order zero at $s = 0$. In [**2**], Dummit and Hayes consider totally real cubic base fields with a totally positive system of fundamental units. For these base fields, the L-functions for the narrow Hilbert class field all vanish, although none of the infinite primes splits completely. Dummit has computationally verified an unpublished conjecture which includes these situations. "This 'robust Stark Conjecture' suggests the possibility of a version of the Conjecture that removes the requirement of a distinguished prime in S that splits completely in the extension" [**1**, page 51]. However, the notion of a single L_S-function evaluator is lost in Dummit's version. In conversations between Dummit and Stark, Stark suggested that a

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single unit for the top field could be formed by multiplying the units from the various intermediate fields where the original conjectures hold.

The purpose of this paper is to formulate an extended version of the first-order abelian Stark conjecture to include first-order vanishing situations where no prime in S splits completely. After demonstrating some basic functorality properties, we show that the extension is equivalent to the original first-order abelian Stark conjecture under certain conditions. In particular, we show that the original conjecture is a special case of the extended conjecture in Lemma 4.2. We then reduce the extended conjecture to assuming the original conjecture holds for all intermediate fields and finding "extra powers" in the Stark units in Theorem 5.2. Finally, we prove that the extension follows from the original conjecture in several situations in Section 6.

2. Preliminaries. Throughout this paper, K/k will denote a finite abelian extension of number fields with Galois group G and group of characters $\widehat{G}$. Let S be a finite set of primes of k including all infinite primes of k and all finite primes that ramify in the extension K/k. For a given character $\chi \in \widehat{G}$, the *imprimitive L-function of K/k and S* is defined for $\mathrm{Re}\,(s) > 1$ as

$$L_S(s,\chi) = \prod_{v \notin S} \left(1 - \frac{\chi(\sigma_v)}{(\mathbf{N}v)^s}\right)^{-1},$$

where $\mathbf{N}v$ is the norm of v and σ_v is the Frobenius automorphism of v (which is well-defined since v is unramified). The product converges absolutely and uniformly on compact subsets of $\{s \in \mathbf{C} \mid \mathrm{Re}\,(s) > 1\}$ and can be analytically continued to the entire complex plane, except for a simple pole at $s = 1$ when χ is the trivial character.

The order of vanishing $r_S(\chi)$ of $L_S(s,\chi)$ at $s = 0$ is well understood and given by the following lemma. Let G_v denote the decomposition group of the prime w in K dividing the prime v in k (which only depends on v since K/k is abelian).

Lemma 2.1.

$$r_S(\chi) = \begin{cases} |S| - 1 & \text{if } \chi = \mathbf{1}_G \\ |\{v \in S \mid \chi|_{G_v} = 1\}| & \text{otherwise.} \end{cases}$$

Proof. See [**14**, pages 24–25]. □

We wish to fix a choice of primes above primes in k for all finite abelian extensions of k enjoying compatibility with respect to field inclusion. To this end, let $\mathcal{P}_k$ denote the set of all primes in k. Let k^{ab} be the maximal abelian extension of k. For each prime $v \in \mathcal{P}_k$, choose a prime v^{ab} in k^{ab} lying above v, and let $\mathcal{P}_k^{\mathrm{ab}}$ be the set of these primes in k^{ab}. For any finite abelian extension K/k, let $\mathcal{P}_k^K$ denote the set of all restrictions of primes in $\mathcal{P}_k^{\mathrm{ab}}$ to K. There will be precisely one prime $w \in \mathcal{P}_k^K$ lying above each $v \in \mathcal{P}_k$. Furthermore, if K' is an intermediate field of K/k, then there is precisely one prime $w' \in \mathcal{P}_k^{K'}$ such that w' divides v and w divides w'.

The S-units predicted by the first-order abelian Stark conjecture depend upon this choice of primes in $\mathcal{P}_k^{\mathrm{ab}}$, though the truth of the conjectures is independent of this choice.

For each prime $v \in \mathcal{P}_k$, define the (normalized) v-adic absolute value:

$$|\alpha|_v = \begin{cases} |\alpha| & \text{if } v \text{ is real infinite,} \\ |\alpha|^2 & \text{if } v \text{ is complex infinite,} \\ \big(\mathbf{N}v\big)^{-\mathrm{ord}_v(\alpha)} & \text{if } v \text{ is finite.} \end{cases}$$

The next lemma, which is established by checking cases, provides the proper normalization for lifting absolute values to abelian extensions.

Lemma 2.2. *Let K/k be an abelian extension of number fields with $w \in \mathcal{P}_k^K$ lying above $v \in \mathcal{P}_k$. Then, for any $\alpha \in k$, $|\alpha|_v = |\alpha|_w^{1/|G_v|}$.*

For a given finite abelian extension K/k, define the following "adelic" absolute value:

$$|\alpha|_{K/k} = \prod_{w \in \mathcal{P}_k^K} |\alpha|_w^{1/|G_v|}.$$

Note that the product converges for any $\alpha \in K$, since all but finitely many terms are equal to 1. It is also not identically trivial on K, since only one prime w is chosen above each $v \in \mathcal{P}_k$. Furthermore, if $k \subseteq K' \subseteq K$ and $\alpha \in K'$, then $|\alpha|_{K'/k} = |\alpha|_{K/k}$. The main technical advantage will be to remove the reliance of the extended conjecture on the unstable $S_{\min}$, which will appear in Definition 2.4.

We now state the first-order abelian Stark conjecture. Denote W_K as the number of roots of unity in K.

Conjecture 2.3 (St $(K/k, S)$). *Let K/k be an abelian extension of number fields. Let G be the Galois group of K/k, and let $\widehat{G}$ be the group of characters on G. Let S be a set of primes in k containing all infinite and ramified primes. Suppose that $|S| \geq 2$ and S contains a $v_0 \in \mathcal{P}_k$ which splits completely in K. Let $w_0 \in \mathcal{P}_k^K$ be the prime in K above v_0. Then there exists an $\varepsilon \in K^\times$, unique up to root of unity, with the following properties*:

i. *$|\varepsilon|_w = 1$ for all primes w not lying above a prime in S, i.e., ε is an S-unit. If $S = \{v_0, v'\}$, then for a fixed w' lying above v', $|\varepsilon^\sigma|_{w'} = |\varepsilon|_{w'}$ for all $\sigma \in G$. If $|S| \geq 3$, then $|\varepsilon|_w = 1$ for all w not lying above v_0.*

ii. *For all $\chi \in \widehat{G}$,*

$$L'_S(0, \chi) = -\frac{1}{W_K} \sum_{\sigma \in G} \chi(\sigma) \log |\varepsilon^\sigma|_{w_0}.$$

iii. *$K(\varepsilon^{1/W_K})$ is an abelian extension of k.*

Conjecture 2.3 is known to hold when $k = \mathbf{Q}$, when k is an imaginary quadratic field [**13**] and when k is a function field [**9**]. It is also known for quadratic extensions [**14**] and for most multi-quadratic fields [**3**]. It has been computationally verified for many other base fields [**2, 4, 5**].

The reason for the assumption of $|S| \geq 2$ and S containing a prime which splits completely is to ensure that $r_S(\chi) \geq 1$ for all $\chi \in \widehat{G}$, see Lemma 2.1. To loosen this assumption, we introduce the notion of a 1-*cover*. Let X be any subset of $\widehat{G}$.

Definition 2.4. i. A 1-*cover of* X is a finite set S of primes in k with the property that, for each $\chi \in X$, there is a $v \in S$ (depending on χ) such that $\chi|_{G_v} = 1$. If S contains all infinite primes of k and all ramified primes of K/k, then S being a 1-cover of X is equivalent to $L_S(0, \chi) = 0$ for all $\chi \in X$.

ii. A 1-*subcover of* X is a subset S' of a 1-cover S of X with the property that, for each $\chi \in X$, there is a $v \in S'$ such that $\chi|_{G_v} = 1$.

iii. $\widehat{G}_{1,S}$ is the subset of characters χ for which $L_S(s,\chi)$ has precisely first-order vanishing at $s=0$. Equivalently for nontrivial characters, $\chi \in \widehat{G}_{1,S}$ if and only if $\chi|_{G_v} = 1$ for precisely one prime $v \in S$.

iv. The *minimal* 1*-subcover of* $\widehat{G}_{1,S}$, denoted $S_{\min}$, is the set of primes $v \in S$ such that there is a $\chi \in \widehat{G}_{1,S}$ with $\chi|_{G_v} = 1$. Equivalently,

$$S_{\min} = \bigcap S'$$

where S' runs through all 1-subcovers of $\widehat{G}_{1,S}$.

Lemma 2.5.

$$S_{\min} = \bigcap S'$$

where S' runs through all 1*-subcovers of* $\widehat{G}$. *Thus, we may refer to* $S_{\min}$ *as the minimal* 1*-subcover of* $\widehat{G}$.

Proof. $S_{\min} \subseteq \cap S'$ is clear, since every 1-subcover of $\widehat{G}$ is also a 1-subcover of $\widehat{G}_{1,S}$. If $v \notin S_{\min}$, then for any $S' \subseteq S$ which is a 1-subcover of $\widehat{G}$, $S' \setminus \{v\}$ is a 1-subcover of $\widehat{G}_{1,S}$. But then $S' \setminus \{v\}$ is also a 1-cover of $\widehat{G}$, since removing any prime reduces the order of vanishing by at most 1. Hence, $v \notin \cap S'$. $\square$

We note that $S_{\min}$ is inherently unstable. It can change when the top field K changes and when the set S changes. Some examples of 1-covers are given in [**8**, Chapter 5].

3. Regulator theory. The goal of this section is to establish the connection between the general Stark conjecture "over $\mathbf{Q}$" and the extended conjectures when $r(\chi) = 1$ and K/k is abelian. We follow the discussion in [**14**].

We use the following notation for this section. If χ is a character on a finite group G, let $\mathbf{Q}(\chi)$ be the cyclotomic field found by adjoining the image of χ to $\mathbf{Q}$. If $\mathbf{F}$ is a subfield of $\mathbf{C}$ and A is a $\mathbf{Z}$-module, then $\mathbf{F}A = \mathbf{F} \otimes_{\mathbf{Z}} A$. If V and W are $\mathbf{F}[G]$-modules, then $\mathrm{Hom}_G(V,W)$ represents all $\mathbf{F}[G]$-module homomorphisms from V to W. Given an $\mathbf{F}[G]$-module V, $V^* = \mathrm{Hom}\,(V,\mathbf{F})$ is the dual space, which is an $\mathbf{F}[G]$-module by $(g\varphi)(v) = \varphi(g^{-1}v)$.

As before, let K/k be a finite abelian extension of number fields with Galois group G. Let S be a nonempty finite set of primes in k and S_K the set of all primes in K lying above some prime in S. For simplicity, we generally suppress the S and S_K notation.

Let $Y = Y_{S_K}$ be the free abelian group generated by S_K, and let $X = X_{S_K}$ be the kernel of the augmentation map $\sum n_w w \mapsto \sum n_w$. Let $U = U_{K,S_K} = \{u \in K^\times \mid |u|_w = 1 \text{ for all } w \notin S_K\}$. Let $\lambda : U \to \mathbf{R}X$ be defined as

$$\lambda(u) = \sum_{w \in S_K} \log |u|_w w.$$

Note by the product formula that the image of λ is contained in $\mathbf{R}X$, not just $\mathbf{R}Y$.

Proposition 3.1 (Unit theorem). *The kernel of λ is the set of all roots of unity in K, and the image of λ is a lattice of full rank $|S| - 1$ in $\mathbf{R}X$. By extension of scalars, $1 \otimes \lambda : \mathbf{C}U \to \mathbf{C}X$ is an isomorphism of $\mathbf{C}[G]$-modules.*

Proof. See [**10**, V.I].

Since X and U are both $\mathbf{Q}[G]$-modules of the same rank, there exists a (noncanonical) $\mathbf{Q}[G]$-module isomorphism $f : \mathbf{Q}X \to \mathbf{Q}U$. This map can be complexified to obtain a $\mathbf{C}[G]$-module isomorphism $f : \mathbf{C}X \to \mathbf{C}U$. Such a map is said to be "defined over $\mathbf{Q}$."

If V is any finite-dimensional $\mathbf{C}[G]$-module with character χ, then the map

$$\begin{aligned}(\lambda \circ f)_V : \mathrm{Hom}_G(V^*, \mathbf{C}X) &\longrightarrow \mathrm{Hom}_G(V^*, \mathbf{C}X)\\ \phi &\longmapsto \lambda \circ f \circ \phi\end{aligned}$$

is a $\mathbf{C}$-linear automorphism. The *Stark regulator* is defined as

$$R(\chi, f) = \det\left((\lambda \circ f)_V\right).$$

Conjecture 3.2 (Stark, "over $\mathbf{Q}$"). *Let $A(\chi, f) = (R(\chi, f))/c(\chi) \in \mathbf{C}$ be the ratio of the Stark regulator and the leading term $c(\chi)$ of Taylor series of $L_S(s, \chi)$ at $s = 0$. Then $A(\chi, f) \in \mathbf{Q}(\chi)$ and for all $\alpha \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})$, $A(\chi, f)^\alpha = A(\chi^\alpha, f)$.*

In the case when $r(\chi) = 1$, then $c(\chi) = L'(0, \chi)$. The Stark conjecture "over $\mathbf{Q}$" for χ simply states that $R(\chi, f) = A(\chi, f) \cdot L'(0, \chi)$ for some $A(\chi, f) \in \mathbf{Q}(\chi)$ which respects conjugation of χ. From now on, assume $\chi \in \widehat{G}_{1,S}$ and v is the unique prime in S such that $\chi|_{G_v} = 1$.

Let

$$e_\chi = \frac{1}{|G|} \sum_{\sigma \in G} \overline{\chi}(\sigma)\sigma$$

be the χ-component orthogonal idempotent of $\mathbf{C}[G]$. Since $\chi \in \widehat{G}_{1,S}$ implies that $\chi^\alpha \in \widehat{G}_{1,S}$ for all $\alpha \in \mathrm{Gal}\,(\mathbf{Q}(\chi)/\mathbf{Q})$, it follows from [**14**, Proposition III.2.1] that Conjecture 3.2 for an irreducible character χ with $r(\chi) = 1$ implies that

$$\sum_{\chi \in \widehat{G}_{1,S}} L_S'(0, \chi) e_{\overline{\chi}} X \subseteq \lambda \mathbf{Q} U.$$

Note that $\mathbf{1}_G \notin \widehat{G}_{1,S}$ when $|S| \geq 3$, so we may replace X with Y in the previous statement since the trivial representation is annihilated by each e_χ. Also note that $L_S'(0, \chi) = 0$ if $r(\chi) > 1$ and $e_\chi \mathbf{Q} X = 0$ if $r(\chi) = 0$ (since χ cannot vanish on any G_v, the sum $\sum_{\tau \in G_v} \chi(\tau) = 0$). Hence, we may increase the sum to all $\chi \in \widehat{G}$, so that for $w \in \mathcal{P}_k^K$ lying above v,

$$m \sum_{\chi \in G} L_S'(0, \chi) e_{\overline{\chi}} w = \lambda(\varepsilon_v) \tag{1}$$

for some nonzero integer m and some $\varepsilon_v \in U$ (called the *Stark unit*). In analogy with the Dirichlet class number formula, Stark's first-order refinement predicts that the denominator m will be W_K, the number of roots of unity in K. Stark further conjectures that $K(\varepsilon_v^{1/W_K})/k$ is an abelian extension.

Note that $e_{\overline{\chi}} w$ is supported at primes in S_K lying above the fixed prime $v \in S$ such that $\chi|_{G_v} = 1$. In particular, ε_v has trivial valuation at any prime of K not dividing v. Simplifying the two sides of equation (1),

$$\begin{aligned}
W_K \sum_{\chi \in G} L_S'(0,\chi) e_{\overline{\chi}}\, w &= \frac{W_K}{|G|} \sum_{\chi \in \widehat{G}} L_S'(0,\chi) \Big(\sum_{\sigma \in G} \chi(\sigma)\sigma \Big) w \\
&= \frac{W_K |G_v|}{|G|} \sum_{\sigma \in G/G_v} \Big(\sum_{\chi \in \widehat{G}} L_S'(0,\chi)\chi(\sigma) \Big) w^{\sigma} \\
\lambda(\varepsilon_v) = \sum_{\sigma \in G/G_v} & \log |\varepsilon_v^{\sigma^{-1}}|_w w^{\sigma}.
\end{aligned}$$

Equating the coefficients of w^{σ}, we have for each $\sigma \in G/G_v$,

$$\log |\varepsilon_v^{\sigma^{-1}}|_w = \frac{W_K |G_v|}{|G|} \sum_{\chi \in \widehat{G}} L_S'(0,\chi)\chi(\sigma) \tag{2}$$

or equivalently (by Mobius inversion and replacing ε_v with its inverse), for each $\chi \in \widehat{G}$ such that $\chi|_{G_v} = 1$,

$$L_S'(0,\chi) = -\frac{1}{W_K} \sum_{\sigma \in G} \chi(\sigma) \log |\varepsilon_v^{\sigma}|_w^{1/|G_v|}. \tag{3}$$

If S is a 1-cover and $S_{\min}$ is the minimal 1-subcover, equation (3) implies that there is a v-unit ε_v for each $v \in S_{\min}$ which evaluates $L_S'(0,\chi)$ when $\chi|_{G_v} = 1$. Furthermore, ε_v will be in the fixed field K^{G_v} by [**14**, page 76].

The philosophy of Stark is that since each ε_v evaluates $L'(0,\chi)$ for those $\chi \in \widehat{G}_{1,S}$ such that $\chi|_{G_v} = 1$, $\varepsilon = \prod \varepsilon_v$ evaluates $L'(0,\chi)$ for all $\chi \in \widehat{G}_{1,S}$, in the following sense. Recalling that each sum vanishes except when $\chi \in \widehat{G}_{1,S}$ and $\chi|_{G_v} = 1$,

$$\begin{aligned}
L_S'(0,\chi) &= \sum_{v \in S_{\min}} -\frac{1}{W_K} \sum_{\sigma \in G} \chi(\sigma) \log |\varepsilon_v^{\sigma}|^{1/|G_v|}|_w \\
&= -\frac{1}{W_K} \sum_{\sigma \in G} \chi(\sigma) \log \Big(\prod_{v \in S_{\min}} |\varepsilon_v^{\sigma}|_w^{1/|G_v|} \Big) \qquad (4) \\
&= -\frac{1}{W_K} \sum_{\sigma \in G} \chi(\sigma) \log |\varepsilon^{\sigma}|_{K/k}.
\end{aligned}$$

We have shown that an integral refinement similar to Conjecture 2.3 follows from the Stark conjecture "over $\mathbf{Q}$" which holds for any 1-cover S, not just when S contains a prime which splits completely. In the next section, we explicitly state the extended conjecture.

4. The extended first-order abelian Stark conjecture. With equation (4) and the first-order abelian Stark conjecture as our motivation, we now state the extended first-order abelian Stark conjecture.

Conjecture 4.1 ($\widetilde{\mathrm{St}}(K/k, S)$). *Let K/k be a finite abelian extension of number fields with Galois group G and character group $\widehat{G}$. Let S be a 1-cover of $\widehat{G}$, and let $S_{\min}$ be the minimal 1-subcover. Assume $|S| \geq 3$ and $S \neq S_{\min}$. Then there exists an $\varepsilon \in K^{\times}$, unique up to root of unity, with the following properties*:

i. *ε is an $S_{\min}$-unit, that is, $|\varepsilon|_w = 1$ for all w not dividing some $v \in S_{\min}$.*

ii. *For any $\chi \in \widehat{G}$,*

$$L'_S(0,\chi) = -\frac{1}{W_K} \sum_{\sigma \in G} \chi(\sigma) \log |\varepsilon^{\sigma}|_{K/k}. \tag{5}$$

iii. *$K(\varepsilon^{1/W_K})/k$ is an abelian extension.*

It is possible to formulate the extended conjecture when $|S| = 2$. However, the original and extended conjectures are equivalent in these cases, and the original conjecture is known to hold in this situation. See [**8**] for more details.

On the other hand, the condition that $S \neq S_{\min}$ cannot be relaxed. There are examples (Dummit's original totally real cubic examples in [**2**]) where the abelian condition of the extended conjecture is not satisfied. This condition is not fully understood at this point. Emmons has proposed that a more natural condition may be that the decomposition groups of primes in S generate the Galois group G [**6**, subsection 5.4].

Here are some immediate functorality results arising from the statement of the general question.

Lemma 4.2. *An affirmative answer to the extended first-order abelian Stark conjecture implies that the first-order abelian Stark conjecture is true.*

Proof. When S contains a prime v_0 which splits completely, then $S_{\min} = \{v_0\}$ and G_{v_0} is trivial. Hence, the original conjecture is a special case of the extended conjecture. $\square$

Lemma 4.3. *The following statements hold for the extended first-order abelian Stark conjecture*:

i. $\widetilde{\mathrm{St}}\,(K/k, S)$ *is true if all the* L*-functions for* K/k *and* S *have order of vanishing at least two at* $s = 0$.

ii. $\widetilde{\mathrm{St}}\,(k/k, S)$ *is true.*

iii. *If* $S \subseteq S'$, *then* $\widetilde{\mathrm{St}}\,(K/k, S)$ *implies* $\widetilde{\mathrm{St}}\,(K/k, S')$.

iv. *If* $k \subseteq K' \subseteq K$, *then* $\widetilde{\mathrm{St}}\,(K/k, S)$ *implies* $\widetilde{\mathrm{St}}\,(K'/k, S)$.

Proof. The proofs are similar to [**14**, Propositions IV.3.1, 3.2, 3.4, and 3.5]. In the first two cases, we may take $\varepsilon = 1$ if $|S| \geq 3$. For the third statement, let ε be the Stark unit for $\widetilde{\mathrm{St}}\,(K/k, S)$. Then the Stark unit for $\widetilde{\mathrm{St}}\,(K/k, S \cup \{v\})$ is $\varepsilon^{1-\sigma_v^{-1}}$ for $v \in S' \setminus S$ by the following calculation:

$$
\begin{aligned}
L_{S\cup\{v\}}(s,\chi) &= \left(1 - \frac{\chi(\sigma_v)}{\mathbf{N}v^s}\right) \cdot L_S(s,\chi) \\
L'_{S\cup\{v\}}(0,\chi) &= (1 - \chi(\sigma_v)) \cdot L'_S(0,\chi) \\
&= -\frac{1}{W_K} \sum_{\sigma\in G} (\chi(\sigma) - \chi(\sigma\sigma_v)) \log |\varepsilon^{\sigma}|_{K/k} \\
&= -\frac{1}{W_K} \sum_{\sigma\in G} \chi(\sigma) \log \left| \left(\varepsilon^{1-\sigma_v^{-1}}\right)^{\sigma} \right|_{K/k}.
\end{aligned}
\tag{6}
$$

For the fourth assertion, let w' be the prime in K' lying between v and w. Let $G' = \mathrm{Gal}\,(K/K') \subseteq G$. Then $\mathrm{Gal}\,(K'/k)$ is isomorphic to G/G'. Denote $G_{w|v}$, $G_{w|w'}$, and $G_{w'|v}$ as the relative decomposition groups of the three primes in their respective Galois groups. Note that

$|G_{w|v}| = |G_{w|w'}| \cdot |G_{w'|v}|$. By Lemma 2.2, $|\alpha|_w^{1/|G_{w|v}|} = |\alpha|_{w'}^{1/|G_{w'|v}|}$ for any $\alpha \in K'$.

Let ε be the Stark unit for $\widetilde{\mathrm{St}}\,(K/k, S)$. Choose any $\chi \in \widehat{G}$ such that $\chi|_{G'} = 1$, that is, $\chi \in \widehat{(G/G')}$. Denote $\mathbf{N} = \mathbf{N}_{K/K'}$. Then

$$
\begin{aligned}
L'_S(0,\chi) &= -\frac{1}{W_K}\sum_{\sigma\in G}\chi(\sigma)\log\Bigg[\prod_{v\in S_{\min}}|\varepsilon^{\sigma}|_w^{1/|G_{w|v}|}\Bigg] \\
&= -\frac{1}{W_K}\sum_{\tau\in G/G'}\chi(\tau)\log\Bigg[\prod_{v\in S_{\min}}\Bigg(\prod_{\sigma\in G'}|\varepsilon^{\sigma\tau}|_w^{1/|G_{w|v}|}\Bigg)\Bigg] \qquad (7)\\
&= -\frac{1}{W_K}\sum_{\tau\in G/G'}\chi(\tau)\log\Bigg[\prod_{v\in S_{\min}}|\big(\mathbf{N}\varepsilon\big)^{\tau}|_w^{1/|G_{w|v}|}\Bigg] \\
&= -\frac{1}{W_{K'}}\sum_{\tau\in G/G'}\chi(\tau)\log\Bigg[\prod_{v\in S_{\min}}|\big(\mathbf{N}\varepsilon^{W_{K'}/W_K}\big)^{\tau}|_{w'}^{1/|G_{w'|v}|}\Bigg].
\end{aligned}
$$

Note that the minimal 1-subcover of $\widehat{G}$ and $\widehat{(G/G')}$ may be different (although the latter is always a subset of the former). Denote $S'_{\min}$ as the minimal 1-subcover of $\widehat{(G/G')}$. We may think of $\widehat{(G/G')}$ as the subgroup of characters in $\widehat{G}$ which contains G' in the kernel. If v is in $S_{\min}$ but not $S'_{\min}$, then the characters $\chi \in \widehat{G}_{\mathbf{1},S}$ such that $\chi|_{G_v} = 1$ are not in $\widehat{(G/G')}$. Thus, the character sum for $\chi \in \widehat{(G/G')}$

$$\sum_{\tau\in G/G'}\chi(\tau)\log|(\mathbf{N}\varepsilon)^{\tau}|_{w'}$$

must equal zero, since the original sum at the prime w does not contribute to the value of $L'_S(0,\chi)$. Hence, we may restrict the product from $S_{\min}$ to $S'_{\min}$.

Tate shows that there exists an $\varepsilon' \in K'$ such that $(\varepsilon')^{W_K/W_{K'}} = \zeta\cdot\mathbf{N}\varepsilon$ for some root of unity $\zeta \in K'$, see [**14**, IV.3.5]. Equation (7) becomes

$$L'_S(0,\chi) = -\frac{1}{W_{K'}}\sum_{\tau\in G/G'}\chi(\tau)\log|(\varepsilon')^{\tau}|_{K'/k}.$$

Thus, the Stark unit for $\widetilde{\mathrm{St}}\,(K'/k, S)$ is ε'. The abelian condition for $\widetilde{\mathrm{St}}\,(K'/k, S)$ follows from the abelian condition for $\widetilde{\mathrm{St}}\,(K/k, S)$ and [**14**, Proposition IV.1.2]. $\square$

Lemma 4.4. *When G is cyclic, $\widetilde{\mathrm{St}}\,(K/k,S)$ and $\mathrm{St}\,(K/k,S)$ are equivalent.*

Proof. Recall that $\widehat{G}$ is isomorphic to G. Let χ_0 be a generator for $\widehat{G}$. Since S is a 1-cover, there exists a $v_0 \in S$ such that $\chi_0|_{G_{v_0}} = 1$. Then, for any $\chi \in \widehat{G}$, $\chi = \chi_0^k$ for some integer k, and so $\chi|_{G_{v_0}} = 1$. By the nondegeneracy of the character group, the only subgroup of G on which every character vanishes is the trivial subgroup. Hence, G_{v_0} is trivial and v_0 splits completely. □

The previous lemma shows that the only instances different from the original conjectures occur when $|S| \geq 3$ and when G is a noncyclic abelian group.

5. Reduction. Our basic approach has been to assume the first-order abelian Stark conjecture is true for certain intermediate fields of K/k. The Stark unit for $\widetilde{\mathrm{St}}\,(K/k,S)$ then arises from the Stark units of the intermediate fields.

Fix some $v \in S_{\min}$, and let $w \in \mathcal{P}_k^K$ lie above v. Suppose $\chi \in \widehat{G}$ is a character such that $\chi|_{G_v} = 1$. Let K^v denote the fixed field of G_v. Then K^v is the maximal intermediate field of K/k in which v splits completely, and $\mathrm{Gal}\,(K^v/k) \cong G/G_v$. Let $w' \in \mathcal{P}_k^{K^v}$ be the prime which lies between v and w. Since G_v is in the kernel of χ, we may think of χ as a character on G/G_v. Assuming $\mathrm{St}\,(K^v/k,S)$, there exists an $\varepsilon_v \in (K^v)^\times$ such that ε_v is a v-unit, and for all $\chi \in \widehat{(G/G')}$,

$$L_S'(0,\chi) = -\frac{1}{W_{K^v}} \sum_{\sigma \in G/G_v} \chi(\sigma) \log |\varepsilon_v^\sigma|_{w'} \tag{8}$$

and $K(\varepsilon_v^{1/W_{K^v}})$ is an abelian extension over k.

Since ε_v is in the fixed field of G_v, $\varepsilon_v^\sigma = \varepsilon_v$ for all $\sigma \in G_v$. Hence, lifting the sum from G/G_v to G has the effect of multiplying by a factor of $|G_v|$:

$$L_S'(0,\chi) = -\frac{1}{W_{K^v}} \sum_{\sigma \in G} \chi(\sigma) \log |\varepsilon_v^\sigma|_{w'}^{1/|G_v|}.$$

By Lemma 2.2, lifting the absolute value from K^v to K introduces another factor of $|G_v|$ to the denominator:

$$L_S'(0,\chi) = -\frac{1}{W_{K^v}} \sum_{\sigma\in G} \chi(\sigma) \log |\varepsilon_v^\sigma|_w^{1/|G_v|^2}.$$

Finally, we adjust the denominator outside the sum to be W_K, which is possible since W_{K^v} divides W_K:

$$L_S'(0,\chi) = -\frac{1}{W_K} \sum_{\sigma\in G} \chi(\sigma) \log \left|\left(\varepsilon_v^{W_K/W_{K^v}}\right)^\sigma\right|_w^{1/|G_v|^2}. \tag{9}$$

We need to show this sum vanishes except when $\chi \in \widehat{G}_{\mathbf{1},S}$ and $\chi|_{G_v} = 1$. Let

$$\mathcal{S}_v = \sum_{\sigma\in G} \chi(\sigma) \log |\varepsilon_v^\sigma|_w.$$

We prove the following consistency conditions:

Lemma 5.1. *Fix some $v \in S_{\min}$ and $\chi \in \widehat{G}$.*

i. *If $\chi|_{G_v} \neq 1$, then $\mathcal{S}_v = 0$.*

ii. *If $\chi|_{G_v} = 1$ and there exists some other $v' \in S_{\min}$ such that $\chi|_{G_{v'}} = 1$, then $\mathcal{S}_v = 0$.*

Proof. i. In this case, let R be a set of coset representatives of G/G_v. Note that $\varepsilon_v \in K^v$ implies that $\varepsilon_v^\tau = \varepsilon_v$ for all $\tau \in G_v$. Hence,

$$\begin{aligned}\mathcal{S}_v &= \sum_{\sigma\in R}\sum_{\tau\in G_v} \chi(\sigma\tau) \log |\varepsilon_v^\sigma|_w \\ &= \sum_{\sigma\in R} \chi(\sigma) \log |\varepsilon_v^\sigma|_w \left(\sum_{\tau\in G_v} \chi(\tau)\right).\end{aligned}$$

Since χ is nontrivial on G_v, $\sum_{\tau\in G_v} \chi(\tau) = 0$.

ii. Here, χ may be considered as a character of G/G_v. Since there are at least two primes in S for which χ is trivial on their decomposition groups in K (and therefore in K^v), the value of $L_S'(0,\chi) = 0$ in equation

(8). Since $\mathcal{S}_v$ is simply the lift of the character sum in equation (8), we conclude that $\mathcal{S}_v = 0$. $\square$

We now have the following criterion for the extended first-order abelian Stark conjecture.

Theorem 5.2. *Let K/k be an abelian extension with Galois group G. Let S be a 1-cover of $\widehat{G}$, and let $S_{\min}$ be its minimal 1-subcover. Suppose that $\mathrm{St}\,(K^v/k, S)$ holds for each $v \in S_{\min}$, with Stark unit ε_v. Suppose further that for each $v \in S_{\min}$, there exists an $\eta_v \in K^\times$ such that $\varepsilon_v = \eta_v^{|G_v|}$ and such that $K(\eta_v^{1/W_{K^v}})$ is abelian over k. Then $\widetilde{\mathrm{St}}\,(K/k, S)$ is true.*

Proof. Suppose the above conditions are met. Let the Stark unit ε for K/k and S be defined as follows:

$$\varepsilon = \prod_{v \in S_{\min}} \eta_v^{W_K/W_{K^v}}.$$

By construction, ε is an $S_{\min}$-unit, since it only has nontrivial valuation above primes in $S_{\min}$. By Lemma 5.1, the character sums $\mathcal{S}_v$ vanish unless $\chi \in \widehat{G}_{\mathbf{1},S}$ and $\chi|_{G_v} = 1$. Finally, η_v is a v-unit implies that $|\varepsilon|_w = |\eta_v^{W_K/W_{K^v}}|_w$ for the fixed w lying above v. Using equation (9), for any $\chi \in \widehat{G}$,

$$\begin{aligned}
L'_S(0,\chi) &= \sum_{v \in S_{\min}} \Bigg(-\frac{1}{W_K} \sum_{\sigma \in G} \chi(\sigma) \log \Big| \Big(\varepsilon_v^{W_K/W_{K^v}}\Big)^{\sigma} \Big|_w^{1/|G_v|^2} \Bigg) \\
&= \sum_{v \in S_{\min}} \Bigg(-\frac{1}{W_K} \sum_{\sigma \in G} \chi(\sigma) \log \Big| \Big(\eta_v^{W_K/W_{K^v}}\Big)^{\sigma} \Big|_w^{1/|G_v|} \Bigg) \\
&= -\frac{1}{W_K} \sum_{\sigma \in G} \chi(\sigma) \log \Bigg(\prod_{v \in S_{\min}} \Big| \Big(\eta_v^{W_K/W_{K^v}}\Big)^{\sigma} \Big|_w^{1/|G_v|} \Bigg) \\
&= -\frac{1}{W_K} \sum_{\sigma \in G} \chi(\sigma) \log |\varepsilon^{\sigma}|_{K/k}.
\end{aligned}$$

Since $K(\eta_v^{1/W_{K^v}})$ is abelian over k for each $v \in S_{\min}$, $K(\prod \eta_v^{1/W_{K^v}}) = K(\varepsilon^{1/W_K})$ is also abelian over k. This follows from the fact that the composition of abelian fields over a common base field is abelian. Therefore, the abelian condition of $\widetilde{\mathrm{St}}\,(K/k, S)$ is satisfied. □

6. Results.

6.1. Unramified case. With Theorem 5.2 from the previous section in hand, we show that the extended first-order abelian Stark conjecture follows from the first-order abelian Stark conjecture when S contains a 1-subcover consisting solely of unramified primes.

Theorem 6.1. *Let K/k be an abelian extension of number fields, let G and $\widehat{G}$ be the Galois group and the group of characters, respectively, and let S be a 1-cover of $\widehat{G}$. Suppose S contains a 1-subcover $S' = \{v_1, \dots, v_t\}$ consisting of only unramified finite primes in K/k. Denote $S_0 = S \setminus S'$ and $S_i = S_0 \cup \{v_i\}$. Suppose* $\mathrm{St}\,(K_i/k, S_i)$ *holds true for all $k \subseteq K_i \subseteq K$, where K_i is the decomposition field for v_i. Then* $\widetilde{\mathrm{St}}\,(K/k, S)$ *has an affirmative answer.*

Proof. Fix some $v_i \in S_{\min}$ and some prime w_i lying above v_i. Since v_i is unramified, the decomposition group G_i of v_i is generated by the Frobenius automorphism σ_i for v_i. Let $\mathbf{f}_i = |G_i| = \mathrm{ord}\,(\sigma_i)$ be the residual degree for w_i over v_i. Then $\mathrm{Gal}\,(K_i/k) \cong G/G_i$ and v_i splits completely in K_i. Note that S_0 contains all ramified primes of K_i (since v ramifies in K'/k implies v ramifies in K/k whenever $k \subseteq K' \subseteq K$). Hence, there is a Stark unit $\varepsilon_i \in K_i^\times$ for the field extension K_i/k and for the set $S_i = S_0 \cup \{v_i\}$.

By equation (6), adding v_j to the set S_i has the effect of applying the group ring element $1 - \sigma_j^{-1} \in \mathbf{Z}[G/G_i]$ to the Stark unit ε_i. As we run through all $j \neq i$, the Stark unit for K'/k and $S = S_0 \cup \{v_1, \dots, v_t\}$ is $\varepsilon_i^{\rho_i}$, where

$$\rho_i = \prod_{j \neq i} \bigl(1 - \sigma_j^{-1}\bigr) \in \mathbf{Z}[G/G_i].$$

We show that ρ_i is divisible by $\mathbf{f}_i$ in $\mathbf{Z}[G/G_i]$. Recall that the Artin map defines a map from the finite unramified primes of k onto elements of G, which sends a prime v to its Frobenius element σ_v. Hence, it suffices to prove some more general facts about subsets $\mathfrak{S}$ of G which are "1-covers" of $\widehat{G}$.

Lemma 6.2. *Let $\mathfrak{S}$ be a subset of a finite abelian group G such that, for any $\chi \in \widehat{G}$, $\chi(\sigma) = 1$ for some $\sigma \in \mathfrak{S}$. Then in $\mathbf{Z}[G]$,*

$$\rho := \prod_{\sigma \in \mathfrak{S}} \bigl(1 - \sigma\bigr) = 0.$$

Proof. Write $\rho = \sum_{\sigma \in G} a_\sigma \sigma$ with $a_\sigma \in \mathbf{Z}$. Applying χ to both sides,

$$\chi(\rho) = \sum_{\sigma \in G} a_\sigma \chi(\sigma) = \prod_{\sigma \in \mathfrak{S}} \bigl(1 - \chi(\sigma)\bigr).$$

Since $\mathfrak{S}$ is a 1-cover of $\widehat{G}$, the product vanishes for all $\chi \in \widehat{G}$. Thus, we have a system of linear equations in the coefficients a_σ, all equal to zero. Since the characters of a group are mutually orthogonal to one another, the linear equations are linearly independent. Therefore, each a_σ must be zero. $\square$

Lemma 6.3. *Let G and $\mathfrak{S}$ be defined as in the previous lemma. Fix some $\sigma_0 \in \mathfrak{S}$. Then*

$$\rho_0 := \prod_{\substack{\sigma \in \mathfrak{S} \\ \sigma \neq \sigma_0}} \bigl(1 - \sigma\bigr)$$

is divisible by $\mathbf{N}\sigma_0 = 1 + \sigma_0 + \cdots + \sigma_0^{\operatorname{ord}(\sigma_0) - 1}$ in $\mathbf{Z}[G]$.

Proof. Write $\rho_0 = \sum_{\sigma \in G} b_\sigma \sigma$ with $b_\sigma \in \mathbf{Z}$. Let ρ be defined as in the previous lemma. Then in $\mathbf{Z}[G]$,

$$\begin{aligned} 0 = \rho = \bigl(1 - \sigma_0\bigr)\rho_0 &= \bigl(1 - \sigma_0\bigr) \sum_{\sigma \in G} b_\sigma \sigma \\ &= \sum_{\sigma \in G} \bigl(b_\sigma \sigma - b_\sigma \sigma\sigma_0\bigr) = \sum_{\sigma \in G} \bigl(b_\sigma - b_{\sigma\sigma_0^{-1}}\bigr)\sigma. \end{aligned}$$

Therefore, $b_\sigma = b_{\sigma\sigma_0^{-1}}$, and so b_σ is constant on cosets of $\langle\sigma_0\rangle$. Let R_0 be a set of coset representatives of $G/\langle\sigma_0\rangle$. Partitioning G into cosets of $\langle\sigma_0\rangle$, we have

$$\rho_0 = \sum_{\sigma \in R_0} b_\sigma \sigma \sum_{j=0}^{\operatorname{ord}(\sigma_0) - 1} \sigma_0^j = \Biggl(\sum_{\sigma \in R_0} b_\sigma \sigma\Biggr) \mathbf{N}\sigma_0. \qquad \square$$

Corollary 6.4. *In* $\mathbf{Z}[G/\langle\sigma_0\rangle]$, ρ_0 *is divisible by* $\mathrm{ord}\,(\sigma_0)$.

Proof. $\mathbf{N}\sigma_0$ reduces to $\mathrm{ord}\,(\sigma_0)$ in $\mathbf{Z}[G/\langle\sigma_0\rangle]$. □

These lemmas apply to the situation in Theorem 6.1 by setting $\rho_0 = \rho_i$, $\sigma_0 = \sigma_i$ and $\mathrm{ord}\,(\sigma_0) = \mathbf{f}_i = |G_i|$. In particular, ρ_i is divisible by $\mathbf{f}_i$ in $\mathbf{Z}[G/G_i]$. Hence, the Stark unit $\varepsilon_i^{\rho_i}$ for K_i/k and S is an $\mathbf{f}_i$th power in K_i. Let $\eta_i = \varepsilon_i^{\rho_i/\mathbf{f}_i}$. The abelian condition for η_i follows from the abelian condition for ε_i with K_i/k and S_i. Therefore, the conditions of Theorem 5.2 are satisfied. This ends the proof of Theorem 6.1. □

Theorem 6.5. *Suppose* S *contains a* 1*-subcover* $S' = \{v_0, v_1, \dots, v_t\}$ *consisting of one real infinite prime* v_0 *and unramified finite primes* v_i *for* $1 \le i \le t$. *Let* $G_i = G_{v_i}$, $K_i = K^{G_i}$, $S_0 = (S \setminus S_{\min}) \cup \{v_0\}$ *and* $S_i = S_0 \cup \{v_i\}$ *for* $1 \le i \le t$. *If* $\mathrm{St}\,(K_i/k, S_i)$ *holds for each* $v_i \in S_{\min}$, *then* $\widetilde{\mathrm{St}}\,(K/k, S)$ *is true.*

Proof. Let σ_i be the Frobenius automorphism of v_i and $\mathbf{f}_i = |G_i| = \mathrm{ord}\,(\sigma_i)$ as usual. Let $\tau = \sigma_0$ denote complex conjugation associated to v_0. By $\mathrm{St}\,(K_0/k, S_0)$, there exists an ε_0 such that for any character χ such that $\chi(\tau) = 1$,

$$L'_{S_0}(0,\chi) = -\frac{1}{2}\sum_{\sigma\in G/\langle\tau\rangle}\chi(\sigma)\log|\varepsilon_0^{\sigma}|_{w_0}.$$

In this case, the Stark unit for K_0/k and S is $\varepsilon_0^{\rho_0}$, where $\rho_0 = \prod_{i=1}^{t}(1-\sigma_i)$. The same argument in Theorem 6.1 shows that ρ_0 is divisible by $|G_0| = 2$. Hence, let $\eta_0 = \varepsilon_0^{\rho_0/2} \in K_0^{\times}$.

Now fix $v_i \in S_{\min}$ for some $1 \le i \le t$. Let ε_i be the Stark unit for $\mathrm{St}\,(K_i/k, S_i)$. Following the proof of Theorem 6.1, the Stark unit for $\mathrm{St}\,(K_i/k, S)$ is $\varepsilon_i^{\rho_i}$, where $\rho_i = \prod_{j\ne i}(1-\sigma_j)$. From Lemmas 6.2 and 6.3, $(1-\tau)\rho_i$ is divisible by $\mathbf{f}_i$ in $\mathbf{Z}[G/G_i]$.

Let $H = G/G_i = G/\langle\sigma_i\rangle$. Note that τ cannot be a power of σ_i. Otherwise, $\chi(\sigma_i) = 1$ implies $\chi(\tau) = 1$, which means v_i could not be an element of $S_{\min}$ ($L_S(s,\chi)$ would have a double zero at $s = 0$ for all $\chi \in \widehat{G}$ with $\chi(\sigma_i) = 1$).

Write $\rho_i = \sum_{\sigma \in H} a_\sigma \sigma$. Choose R to be some set of representatives for $H/\langle\tau\rangle$. Then

$$\begin{aligned} \rho_i &= \sum_{\sigma\in R} a_\sigma\sigma + \sum_{\sigma\in R} a_{\tau\sigma}\tau\sigma \\ &= \sum_{\sigma\in R} a_{\tau\sigma}(\sigma+\tau\sigma) + \sum_{\sigma\in R}(a_\sigma - a_{\tau\sigma})\sigma \\ &= (1+\tau)\sum_{\sigma\in R} a_{\tau\sigma}\sigma + \sum_{\sigma\in R}(a_\sigma - a_{\tau\sigma})\sigma. \end{aligned} \tag{10}$$

The first sum of equation (10) has a factor of $(1+\tau)$, and $\varepsilon_i^{1+\tau} = 1$ since ε_i is a unit at v_0. Therefore,

$$\varepsilon_i^{\rho_i} = \varepsilon_i^{\sum_{\sigma\in R}(a_\sigma - a_{\tau\sigma})\sigma}.$$

Applying $(1-\tau)$ to both sides of equation (10),

$$\begin{aligned} (1-\tau)\rho_i &= (1-\tau^2)\sum_{\sigma\in R} a_{\tau\sigma}\sigma + (1-\tau)\sum_{\sigma\in R}(a_\sigma - a_{\tau\sigma})\sigma \\ &= 0 + \sum_{\sigma\in R}(a_\sigma - a_{\tau\sigma})\sigma + \sum_{\sigma\in R}(a_{\tau\sigma} - a_\sigma)\sigma\tau \\ &= \sum_{\sigma\in H}(a_\sigma - a_{\tau\sigma})\sigma. \end{aligned}$$

Hence, $a_\sigma - a_{\tau\sigma}$ is divisible by $\mathbf{f}_i$ for all σ, and so $\varepsilon_i^{\rho_i}$ is an $\mathbf{f}_i$th power in K_i as desired. The abelian condition of Theorem 5.2 is satisfied by the same argument at the end of Theorem 6.1. $\square$

As an application, we apply Theorems 6.1 and 6.5 to cyclotomic extensions $K/k = \mathbf{Q}(\zeta_m)/\mathbf{Q}$ for certain values of m.

Theorem 6.6. *Let m be a positive integer which is either odd or divisible by 4. Let S be a 1-cover of $\widehat{G}$ for the cyclotomic extension $\mathbf{Q}(\zeta_m)/\mathbf{Q}$. For each prime p dividing m, write $m = p^a n$ with $(p,n) = 1$, and suppose that there exists a prime factor l of $\phi(p^a)$ which does not divide $\phi(n)$. Then S contains a 1-subcover S' consisting entirely of unramified primes and possibly the one real infinite prime of $\mathbf{Q}$. In particular, $\widetilde{\mathrm{St}}\,(\mathbf{Q}(\zeta_m)/\mathbf{Q}, S)$ is true for m satisfying the above condition.*

Proof. The primes which ramify in $\mathbf{Q}(\zeta_m)/\mathbf{Q}$ are precisely the primes p which divide m. It suffices to show that $p \notin S_{\min}$ for all p dividing m. Let l be the prime factor of $\phi(p^a)$. Separate $\widehat{G}$ into two disjoint subsets, X_1 consisting of all characters defined modulo n and X_2 consisting of all characters for which p divides the conductor. It suffices to show that if $S \setminus \{p\}$ is a 1-subcover of X_2, then it is a 1-subcover of X_1.

Let σ_p, σ_q and τ denote the Frobenius automorphisms for p, q and v_∞, respectively. Suppose $S \setminus \{p\}$ is not a 1-subcover of X_1. Let $\chi_1 \in X_1$ be a character such that $\chi_1(\sigma_p) = 1$, $\chi_1(\sigma_q) \neq 1$ for all $q \in S \setminus \{p\}$ and $\chi_1(\tau) = -1$ so that χ_1 is not covered by v_∞. Note that χ_1 takes values in the $\phi(n)$th roots of unity. Let $\chi_2 \in X_2$ be a character of conductor p^a and order l, where l is the prime factor of $\phi(p^a)$ which does not divide $\phi(n)$. Then χ_2 takes values in the lth roots of unity, and $\chi_2(\tau) = 1$ since l must be odd.

Consider the character $\chi = \chi_1 \cdot \chi_2 \in X_2$. For all $q \in S \setminus \{p\}$, $\chi_1(\sigma_q)$ is a nontrivial $\phi(n)$th root of unity and $\chi_2(\sigma_q)$ is an lth root of unity. Since $\phi(n)$ and l are relatively prime, $\chi(\sigma_q) \neq 1$ for all $q \in S \setminus \{p\}$. Furthermore, $\chi(\tau) = -1$ since χ_1 is odd and χ_2 is even, so χ is not covered by v_∞. Since $\chi(\sigma_p) = 0$, χ would not be covered by S. This contradicts S being a 1-cover, and thus $S \setminus \{p\}$ is a subcover of S. We conclude that S contains a 1-subcover S' with no ramified primes. Since the first-order abelian Stark conjecture is known to be true for all $K \subseteq \mathbf{Q}(\zeta_m)$ when $k = \mathbf{Q}$ [**14**, IV.3.9, IV.6.7], $\widetilde{\mathrm{St}}\,(\mathbf{Q}(\zeta_m)/\mathbf{Q}, S)$ is true by Theorem 6.1 if the subcover S' consists only of unramified primes and by Theorem 6.5 if S' contains the one real infinite prime of $\mathbf{Q}$. □

There exist 1-covers of $\widehat{G}$ with ramified primes in $S_{\min}$ when the condition on m in Theorem 6.6 is not satisfied. The smallest such example is $K/k = \mathbf{Q}(\zeta_{20})/\mathbf{Q}$, $S = \{v_\infty, 2, 3, 5, 11\}$, and $S_{\min} = \{v_\infty, 3, 5, 11\}$. Although Conjecture 4.1 can be verified for individual cases such as these, the techniques presented here do not apply to these cases. It is still open to show that the extended first-order abelian Stark conjecture follows from the first-order abelian Stark conjecture when $S_{\min}$ contains ramified primes or more than one infinite prime.

6.2. Multiquadratic case. We now turn our attention to extensions K/k for which the Galois group G is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^m$ for some $m \geq 2$. We call K/k a *multi-quadratic extension of rank* m. The original conjecture was proven to hold for these extensions under certain conditions in [**11, 12**] and extended to most multi-quadratic extensions in [**3**].

As before, the character group $\widehat{G}$ is isomorphic to G, so every nontrivial character has order 2. Denote χ_0 as the trivial character and χ_i as the nontrivial characters for $1 \leq i \leq 2^m - 1$. The kernels $G_i = \operatorname{Ker}\chi_i$ have index 2 in G which correspond to the $2^m - 1$ quadratic extensions K_i/k by Galois theory. Therefore, the Stark units for the multi-quadratic extension K/k should arise from the Stark units for the various quadratic extensions K_i/k.

Proposition 6.7 [**14**, IV.5.4]. *Suppose* $[K_i : k] = 2$. *If* S *contains two primes which split completely in* K_i/k, *let* $\eta_i = 1$. *If* S *contains a single split prime* v_i *in* K_i/k, *let* η_i *be a generator of the free part of the* v_i*-units of* K_i/k. *Let* $M_i = |\operatorname{Coker}(\operatorname{Cl}_k(S) \to \operatorname{Cl}_{K_i}(S))|$. *Then* $\operatorname{St}(K_i/k, S)$ *holds with Stark unit*

$$\varepsilon_i = \eta_i^{M_i \cdot 2^{|S|-3}}.$$

In [**3**], the factors M_i are shown to be divisible by 2^{rk}, where $rk = rk_k(S)$ is the 2-rank of the S_{fin}-class group of k. In fact, Theorem 1 from [**3**] shows that $\operatorname{St}(K/k, S)$ holds for multi-quadratic extensions K/k of rank m if $|S| > m + 1 - rk$. We use this fact to prove a similar result for $\widetilde{St}\,(K/k, S)$.

Theorem 6.8. *Let* K/k *be a multi-quadratic extension of rank* m. *Let* S *be a* 1*-cover of* $\widehat{G}$ *and* $S_{\min}$ *its minimal* 1*-subcover. Assume that* $|S| > m + 1 - rk$. *Then* $\widetilde{\operatorname{St}}\,(K/k, S)$ *is true.*

Proof. Fix some $v \in S_{\min}$. Let K' be the fixed field of G_v and n_v the rank of K'/k, that is, $[K' : k] = 2^{n_v}$. Note that $|G_v| = [K : K'] = 2^{m-n_v}$. As usual, K' is the maximal intermediate field in which v splits completely.

By assumption, $|S| > n_v + 1 - rk$. Applying Theorem 1 from [**3**], $\mathrm{St}\,(K'/k, S)$ holds with Stark unit

$$\varepsilon_v = \prod \eta_i^{M_i \cdot 2^{|S|-n_v-2}(W_{K'}/W_{K_i})}$$

where the product is taken over all quadratic subfields K_i contained in K'. In particular, $\varepsilon_v = \eta_v^N$ for some $\eta_v \in K'$, where $N = 2^{|S|-n_v-2+rk}$. Again by assumption, $|S| - n_v - 2 + rk \geq m - n_v$ and so $|G_v| = 2^{m-n_v}$ divides N. Furthermore, $\eta_i^{1/W_{K_i}}$ lies in an abelian extension of k by Proposition 6.7. Therefore, the conditions of Theorem 5.2 are satisfied. □

Theorem 6.9. $\widetilde{\mathrm{St}}\,(K/k, S)$ *is true for biquadratic extensions, that is, when* $m = 2$. *In fact, the assumption* $|S| \geq |S_{\min}| + 1$ *may be relaxed to include* $S = S_{\min}$.

Proof. If S contains a prime which splits completely in K/k, then the question was answered in [**3, 11, 12**]. Otherwise, each prime splits in at most one of the three intermediate quadratic fields K_i. If $|S| \geq 4$, then the question follows from Theorem 6.8. If $|S| = 3$, then each prime in S splits in a different quadratic subfield of K, which implies every prime of S is in $S_{\min}$. This violates the assumption in $\widetilde{\mathrm{St}}\,(K/k, S)$ that $|S| \geq |S_{\min}| + 1$. In fact, it is shown in [**8**] that there are no biquadratic extensions where $|S| = |S_{\min}| = 3$ by considering cases. □

Note that for almost all multi-quadratic extensions $K/\mathbf{Q}$ of rank m, there are at least m finite ramified primes and one infinite prime in S (the one type of exception being if $\mathbf{Q}(i, \sqrt{2})$ is contained in K). Theorem 6.8 proves that the extended first-order abelian Stark conjecture holds for multi-quadratic extensions of $\mathbf{Q}$, unless S consists of this minimal set of $m + 1$ primes. There do exist multi-quadratic extensions and 1-covers where the condition on the number of primes does not meet the condition in Theorem 6.8. An example is $K/k = \mathbf{Q}(\sqrt{5}, \sqrt{-7}, \sqrt{-11})/\mathbf{Q}$ and $S = S_{\min} = \{5, 7, 11, \infty\}$. Although the extended first-order abelian Stark conjecture is shown to hold for this example in [**8**], a general proof is still unclear in these situations.

7. Conclusion. There are several advantages that the extended first-order abelian Stark conjecture has over the first-order abelian Stark conjecture. It removes the splitting prime condition and thus applies to a wide variety of first order vanishing situations previously not considered by the original conjectures. It combines the Stark and Brumer-Stark conjectures, making no distinction between finite and infinite primes. In the case of multi-quadratic extensions, the extended question uses the "extra powers of two" found in [**3, 11, 12**]. It may be possible to explain the occurrence of these extra powers, which consistently appear in the computational evidence for the conjectures [**1**].

The techniques presented in the previous sections approach the extended question from two fundamentally different perspectives. In the case of unramified 1-subcovers, the necessary factors arose from the group ring elements obtained by adding unramified primes to the set S. In the multi-quadratic case, the necessary factors are often already present in the exponents of the Stark units. For a general proof that the original conjectures for intermediate fields implies the extended question, it would be necessary to combine these techniques.

It is also interesting to consider first order vanishing situations when $S = S_{\min}$. Two such examples are considered in [**8**]. The first example is multi-quadratic of rank 3 with $k = \mathbf{Q}$. The second example is Dummit's original motivating example over totally real cubic fields. Initially, there are not enough factors of two in the exponents and no reason to expect a unit in the top field to exist. In the first case, the units which are not fourth powers in the intermediate fields become fourth powers in the top field. In the second case, the local conjectures provide the extra factor of two. In both cases, there is a Stark unit for K/k which evaluates all the L-functions for the extension. The abelian condition is satisfied in the multi-quadratic example, but not in the totally real cubic examples.

We also mention that a higher order version of the extended conjecture has been formulated by Popescu and has been proven in similar situations as in the current paper (unramified 1-subcovers, multi-quadratic extension) by Emmons [**6**] and Emmons and Popescu [**7**].

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