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A MOUNTAIN PASS THEOREM FOR A SUITABLE CLASS OF FUNCTIONS

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ABSTRACT. The main purpose of this paper is to establish a three critical points result without assuming coercivity of the involved functional. To this end, a mountain-pass theorem, where the usual Palais-Smale condition is not requested, is presented. These results are then applied to prove the existence of three solutions for a two-point boundary value problem with no asymptotic conditions.

1. Introduction. It is well known that the mountain-pass theorem of Ambrosetti and Rabinowitz [**1**, Theorem 2.1] and its variants or generalizations, as for instance Theorem 1 of [**17**], is successfully used to find critical points of real-valued C^1 functions J defined on an infinite-dimensional Banach space X. One of the key assumptions in this result is a compactness hypothesis, usually called the Palais-Smale condition.

The present paper deals with the case

$$J(x) = \Phi(x) - \Psi(x), \qquad x \in X,$$

which often occurs in the variational formulation of both ordinary and partial differential problems. We first introduce a new type of Palais-Smale condition, see Section 3. It is mutually independent from the usual condition and holds true every time Φ and Ψ turn out sufficiently smooth and Φ is coercive, see Theorem 3.1. A mountain pass-like result, which also provides a more precise localization of the obtained critical point obtained in regards to the function Φ, is then established, see Theorem 4.3. Moreover, putting this result together with Theorem 2.1 in [**5**] yields the main result of the paper, which is a three critical points theorem for the functional

$$J_\lambda(x) = \Phi(x) - \lambda\Psi(x), \quad x \in X,$$

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where $\lambda > 0$, whose norms are uniformly bounded with respect to λ, see Theorem 5.2. Let us point out that, contrary to the basic result of Ricceri [**20**, Theorem 1] on this topic, no coercivity of J_λ is assumed. To prove our results we employ tools from the generalized gradient theory for locally Lipschitz continuous functions, which has been introduced and developed by Clarke [**12**]. We make further use of a critical point theorem in this framework, namely Theorem 2.2 of [**15**]. The critical point theory for locally Lipschitz continuous functionals, including applications to elliptic problems with discontinuous nonlinearities, has been introduced and extensively studied by Chang in [**11**], see also [**16**] for an in-depth account in this field. The results of the present paper deal with smooth functions in a natural way, using the nonsmooth theory.

As an application of these results to nonlinear differential problems, we present an existence theorem of three solutions for a two-point boundary value problem (see Theorem 6.1). The assumptions of Theorem 6.2, which are a consequence of Theorem 6.1, are that there is a growth which is greater than the quadratic of the antiderivative for the function in a suitable interval, see assumption (k), and a growth less than the quadratic of the same antiderivative in a following, suitable interval, see assumption (kk). By way of example, here we present the following result, which is a particular case of Theorem 6.2.

Theorem 1.1. *Let $g : [0,1] \to \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ be two continuous, nonnegative and nonzero functions. Put $F(x) = \int_0^x f(t)\,dt$ for all $x \in \mathbf{R}$ and $g_0 = (\int_{1/4}^{3/4} g(t)\,dt)/(\int_0^1 g(t)\,dt)$, and assume that there are three positive constants c, d and p, with $c < d < (p/2)$, such that*

(k) $F(c)/c^2 < (g_0/3)(F(d)/d^2)$;

(kk) $F(p)/p^2 < (g_0/6)(F(d)/d^2)$.

Then, for each

$$\lambda \in \left] \frac{6d^2}{F(d)\int_{1/4}^{3/4} g(t)\,dt}, \min\left\{ \frac{2c^2}{F(c)\int_0^1 g(t)\,dt}; \frac{p^2}{F(p)\int_0^1 g(t)\,dt} \right\} \right[,$$

the problem

$$(AP_\lambda) \qquad \begin{cases} -u'' = \lambda g(t) f(u) \\ u(0) = u(1) = 0, \end{cases}$$

admits at least three nonnegative (positive, if $f(0) \neq 0$) classical solutions u_i, $i = 1, 2, 3$, such that

$$\|u_i\|_\infty < p, \quad i = 1, 2, 3.$$

We also observe that in Theorems 6.1 and 6.2 no asymptotic condition is assumed, see Example 6.1. Moreover, again as a consequence of Theorem 6.1, we obtain a result, Theorem 6.3, similar to that given by Ambrosetti and Rabinowitz [**18**, Theorem 2.32] and, in addition, we obtain an upper bound involving $\overline{\lambda}$, where $]\overline{\lambda}, +\infty[$ is the interval for which the problem (AP_λ) admits at least two positive solutions, see Remark 6.6.

Finally, we recall that three solutions for two-point boundary value problems were ensured by Avery and Henderson [**4**], and Henderson and Thompson [**13, 14**] by using different methods, such as lower and upper solutions or fixed-point theorems. It is easy to verify that their results are mutually independent from ours, see also [**2**, Remark 3.7].

2. Preliminaries. Let $(X, \|\cdot\|)$ be a real Banach space. As usual, X^* is the dual space and $\langle\cdot,\cdot\rangle$ stands for the duality pairing between X^* and X. A function $I : X \to \mathbf{R}$ is called Lipschitz near a given point u if there exists a neighborhood U of u and a constant $L \geq 0$ such that

$$|I(v) - I(w)| \leq L\|v - w\|$$

for all $v, w \in U$. I is called locally Lipschitz in X if it is Lipschitz near u for each $u \in X$.

The generalized directional derivative of a locally Lipschitz function I at point u along direction v is defined as follows:

$$I^0(u; v) = \limsup_{\substack{w \to u \\ t \to 0^+}} \frac{I(w + tv) - I(w)}{t}.$$

Moreover, the generalized gradient of I at u is the following set:

$$\partial I(u) = \{u^* \in X^* : \langle u^*, v\rangle \leq I^0(u; v) \text{ for all } v \in X\}.$$

We recall that if I is continuously Gâteaux differentiable at u, then I is Lipschitz near u and $\partial I(u) = \{I'(u)\}$. Further, a point $u \in X$ is called a (generalized) critical point of the locally Lipschitz function I if $0_{X^*} \in \partial I(u)$, namely,

$$I^0(u; v) \geq 0$$

for all $v \in X$. Clearly, if I is a continuously Gâteaux differentiable at u, then u becomes a (classical) critical point of I, that is,

$$I'(u) = 0_{X^*}.$$

Finally, we say that a locally Lipschitz function I satisfies the Palais-Smale condition $(P.S.)_c$, $c \in \mathbf{R}$, if each sequence $\{u_n\}$ such that

(1) $I(u_n) \to c$,

(2) $\min_{v \in \partial I(u_n)} \|v\|_{X^*} \to 0$

possesses a convergent subsequence.

It is well known that condition (2) and

(2′) $I^0(u_n, v - u_n) \geq -\varepsilon_n \|v - u_n\|$ for all $v \in X$, where $\varepsilon_n \to 0^+$,

are equivalent. Moreover, when I is a continuously Gâteaux differentiable function, the $(P.S.)_c$ condition is reduced to the classical one, namely, each sequence $\{u_n\}$ such that

(1) $I(u_n) \to c$,

(2) $\|I'(u_n)\|_{X^*} \to 0$

possesses a convergent subsequence.

For a thorough treatment on these topics, we refer to [**11, 12, 18**].

Now, given two functions $\Phi, \Psi : X \to \mathbf{R}$, we define the following functions:

$$\varphi_1(r) := \inf_{x \in \Phi^{-1}(]-\infty, r[)} \frac{\sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^w} \Psi(x) - \Psi(x)}{r - \Phi(x)} \tag{2.1}$$

$$\varphi_1(r_1, r_2) := \max\{\varphi_1(r_1); \varphi_1(r_2)\} \tag{2.2}$$

$$\varphi_2(r_1, r_2) = \inf_{x \in \Phi^{-1}(]-\infty, r_1[)} \sup_{y \in \Phi^{-1}([r_1, r_2[)} \frac{\Psi(y) - \Psi(x)}{\Phi(y) - \Phi(x)} \tag{2.3}$$

for all $r, r_1, r_2 > \inf_X \Phi$, with $r_1 < r_2$ and where $\overline{\Phi^{-1}(]-\infty, r[)}^w$ is the closure of $\Phi^{-1}(]-\infty, r[)$ in the weak topology. For the reader's convenience we recall below the theorem obtained in [**5**] which ensures the existence of a precise open interval $\Lambda \subseteq]0, +\infty[$ such that for each $\lambda \in \Lambda$ the function $J = \Phi - \lambda\Psi$ admits two local minima which are uniformly bounded in norm with respect to λ.

Theorem 2.1 [**5**, Theorem 2.1]. *Let X be a reflexive real Banach space, and let $\Phi, \Psi : X \to \mathbf{R}$ be two functions. Assume that Φ is sequentially weakly lower semi-continuous,* (*strongly*) *continuous and coercive, and Ψ is sequentially weakly upper semi-continuous. Assume also that two constants r_1 and r_2 exist such that*

(B_1) $\inf_X \Phi < r_1 < r_2$;

(B_2) $\varphi_1(r_1, r_2) < \varphi_2(r_1, r_2)$;

Then, for each $\lambda \in](1/\varphi_2(r_1, r_2)), (1/\varphi_1(r_1, r_2))[$ the restriction of $\Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r_1[)$ admits a global minimum v_1, and the restriction of $\Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r_2[)$ admits a global minimum $v_2 \in \Phi^{-1}([r_1, r_2[)$, which are two local minima for $\Phi - \lambda\Psi$.

We also recall that the proof of Theorem 2.1 is based on the variational principle of Ricceri in [**19**].

3. A new type of Palais-Smale condition. Let $\Phi, \Psi : X \to \mathbf{R}$ be two locally Lipschitz functions. Of course, the function

$$J = \Phi - \Psi$$

is a locally Lipschitz function. Fix $M \in \mathbf{R}$ and put

$$\Psi_M(u) = \begin{cases} \Psi(u) & \text{if } \Psi(u) \leq M \\ M & \text{if } \Psi(u) > M. \end{cases}$$

It is simple to show that also Ψ_M is a locally Lipschitz function.

We now give the following definition.

Definition 3.1. We say that the function $J = \Phi - \Psi$ satisfies the Palais-Smale condition cut off at M (briefly $(P.S.)_c^M$) if the function $J_M = \Phi - \Psi_M$ satisfies the Palais-Smale condition.

It is easy to verify that the $(P.S.)_c^M$ condition and the $(P.S.)_c$ condition on J are mutually independent. For instance, it is enough to pick $X = \mathbf{R}$, $\Phi = 0$, $\Psi(x) = -1/2x^2$; it is clear that $J(x) = 1/2x^2$ satisfies the $(P.S.)_c$ condition, while J_1 does not satisfy the $(P.S.)_c$ condition. On the contrary, by choosing $X = \mathbf{R}$, $\Phi = 1/2x^2$, $\Psi(x) = 1/2x^2$, J_M (with $M > 0$) satisfies the $(P.S.)_c$ condition, while J does not satisfy the $(P.S.)_c$ condition.

Now, we emphasize the following theorem that guarantees the $(P.S.)_c^M$ condition for all $M \in \mathbf{R}$ for functions of class C^1.

Theorem 3.1. *Let $\Phi, \Psi : X \to \mathbf{R}$ be two continuously Gâteaux differentiable functions, with Φ coercive. Assume that $\Phi' : X \to X^*$ admits a continuous inverse operator on X^*, and $\Psi' : X \to X^*$ is compact. Then, the function $\Phi - \Psi$ satisfies the $(P.S.)_c^M$ condition for all $M \in \mathbf{R}$.*

Proof. Fix $M \in \mathbf{R}$, and let $\{u_n\}$ be a sequence such that

(1) $\Phi(u_n) - \Psi_M(u_n) \to c$, $c \in \mathbf{R}$,

(2) $\min_{\xi \in \partial(\Phi - \Psi_M)(u_n)} \|\xi\|_{X^*} \to 0$.

Since Φ is coercive, also $\Phi - \Psi_M$ is coercive and, hence, from (1) we obtain that $\{u_n\}$ is bounded. Moreover, taking into account that

$$\partial(\Phi - \Psi_M)(u_n) = \partial(\Phi)(u_n) - \partial(\Psi_M)(u_n) = \Phi'(u_n) - \partial(\Psi_M)(u_n)$$

and

$$\partial(\Psi_M)(u_n) = \begin{cases} \Psi'(u_n) & \text{if } \Psi(u_n) < M \\ 0 & \text{if } \Psi(u_n) > M \\ \{r\Psi'(u_n) : r \in [0,1]\} & \text{if } \Psi(u_n) = M \end{cases}$$

(see [**12**, Corollary 2, page 39; Proposition 2.2.4, page 33; Proposition 2.3.12, page 47]), one has

$$\begin{aligned} \min_{\xi \in \partial(\Phi - \Psi_M)(u_n)} \|\xi\|_{X^*} &= \min_{\xi \in \partial(\Phi)(u_n) - \partial(\Psi_M)(u_n)} \|\xi\|_{X^*} \\ &= \min_{\eta \in \partial(\Psi_M)(u_n)} \|\Phi'(u_n) - \eta\|_{X^*} \end{aligned}$$

$$= \min_{r\in[0,1]} \|\Phi'(u_n) - r\Psi'(u_n)\|_{X^*}$$
$$= \|\Phi'(u_n) - r_n\Psi'(u_n)\|_{X^*}.$$

From the compactness of Ψ' and the fact that $\{r_n\} \subseteq [0,1]$, there are two sequences $\{r_{n_k}\}$ and $\{u_{n_k}\}$ such that $\{r_{n_k}\Psi'(u_{n_k})\}$ converges. From (2) and the previous equalities, one has $\|\Phi'(u_{n_k}) - r_{n_k}\Psi'(u_{n_k})\|_{X^*} \to 0$. Hence, $\{\Phi'(u_{n_k})\}$ also converges and, due to our assumption on Φ', $\{u_{n_k}\}$ converges. □

Remark 3.1. We explicitly observe that if, in the previous theorem, instead of Φ coercive, we assume $\Phi - \Psi$ coercive, then the function $\Phi - \Psi$ satisfies the classical $(P.S.)_c$ condition (see, for instance, [**22**, Example 38.25, page 162]).

4. A mountain pass theorem. In this section we assume that X is a reflexive real Banach space and $\Phi, \Psi : X \to \mathbf{R}$ are two continuously Gâteaux differentiable functions. Moreover, we assume that

(A_1) Φ is convex;

(A_2) for every x_1, x_2 such that $\Psi(x_1) \geq 0$ and $\Psi(x_2) \geq 0$, one has

$$\inf_{t\in[0,1]} \Psi(tx_1 + (1-t)x_2) \geq 0.$$

Now, we present a version of the classical mountain pass theorem for functions of the type $J = \Phi - \Psi$ where a localization of the (classical) critical point is also guaranteed.

Theorem 4.1. *Assume that there is a positive real number s and two points x_1, $x_2 \in X$, with $\|x_2 - x_1\| > s$ and $\overline{B}(x_1, s) \subseteq \Phi^{-1}(]-\infty, \rho[)$, where $\rho > \max\{\Phi(x_1), \Phi(x_2)\}$, such that*

$$J(x) \geq \max\{J(x_1), J(x_2)\} \tag{4.1}$$

for all $x \in \partial B(x_1, s)$. Assume also that $\min\{\Psi(x_1), \Psi(x_2)\} \geq 0$. Further, assume that

(C) *there is an $M > 0$ such that*

(C_1) $\sup_{x\in\Phi^{-1}(]-\infty,\rho+M[)} \Psi(x) < M$.

and

(C_2) *J satisfies the $(P.S.)_c^M$ condition for all $c \in \mathbf{R}$.*

Then, the function J has a (classical) critical point x_3 distinct from x_1 and x_2 such that

$$J(x_3) \geq \max\{J(x_1); J(x_2)\}$$

and

$$\Phi(x_3) < \rho + M.$$

Proof. Put

$$\Psi_M(u) = \begin{cases} \Psi(u) & \text{if } \Psi(u) \leq M \\ M & \text{if } \Psi(u) > M, \end{cases}$$

$J_M(u) = \Phi(u) - \Psi_M(u)$ and $I_M(u) = J_M(u+x_1)$ for all $u \in X$. Clearly, I_M is a locally Lipschitz function and, taking into account (C_2), satisfies the Palais-Smale condition $(P.S.)_c$, $c \in \mathbf{R}$. Now, put $e = x_2 - x_1$ and $a = \max\{I_M(0); I_M(e)\}$. One has $I_M(0) \leq a$, $I_M(e) \leq a$ and $\|e\| > s$. Moreover, taking into account (C_1), assumption (4.1) ensures that $I_M(u) \geq a$ for all $u \in \partial B(0,s)$. Hence, all the assumptions of the mountain pass theorem for nondifferentiable functions (see, for instance, Theorem 2.2 of [**15**]) are verified. Therefore, there exists a (generalized) critical point y_3 of I_M such that

$$I_M(y_3) = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I_M(\gamma(t)),$$

where $\Gamma = \{\gamma \in C^0([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}$. Moreover, one has

$$I_M(y_3) \geq \max\{I_M(0); I_M(e)\}. \tag{4.2}$$

We claim that y_3 is a (classical) critical point of I_M. In fact, by choosing the segment with endpoints 0 and e as γ, and from (A_1) and (A_2), one has $I_M(y_3) \leq \sup_{t\in[0,1]} I_M((1-t)e) = \sup_{t\in[0,1]} I_M((1-t)(x_2-x_1)) = \sup_{t\in[0,1]} J_M(tx_1 + (1-t)x_2) = \sup_{t\in[0,1]}[\Phi(tx_1 + (1-t)x_2) - \Psi_M(tx_1+(1-t)x_2)] \leq \sup_{t\in[0,1]}[t\Phi(x_1) + (1-t)\Phi(x_2)] - \inf_{t\in[0,1]} \Psi_M(tx_1 + (1-t)x_2) < \rho$, namely,

$$I_M(y_3) < \rho. \tag{4.3}$$

On the other hand, setting $x_3 = y_3 + x_1$, one has $I_M(y_3) = J_M(y_3 + x_1) = J_M(x_3) = \Phi(x_3) - \Psi_M(x_3)$. Hence, from (4.3) one has

$$\Phi(x_3) < \rho + M.$$

Therefore, due to (C_1) one has $\Psi(x_3) < M$. So, it follows immediately that I_M is a continuously Gâteaux differentiable at y_3 and our claim is proved.

Finally, it is easy to verify that x_3 is a classical critical point of J and that, from (4.2), we obtain $J(x_3) \geq \max\{J(x_1); J(x_2)\}$. Hence, the proof is complete. □

We also obtain the result below.

Theorem 4.2. *Assume that J admits two distinct local minima x_1, $x_2 \in X$. Put $\rho \in \mathbf{R}$ such that $\max\{\Phi(x_1), \Phi(x_2)\} < \rho$, and assume that $\min\{\Psi(x_1), \Psi(x_2)\} \geq 0$. Further, assume that*

(C) there is an $M > 0$ such that

(C_1) $\sup_{x \in \Phi^{-1}(]-\infty, \rho+M[)} \Psi(x) < M$.

and

(C_2) J satisfies the $(P.S.)_c^M$ condition for all $c \in \mathbf{R}$.

Then, the function J has a third (classical) critical point x_3 distinct from x_1 and x_2 such that

$$\Phi(x_3) < \rho + M.$$

Proof. Without loss of generality, we can assume $J(x_2) \leq J(x_1)$. Since $x_1 \in \Phi^{-1}(] - \infty, \rho[)$, which is an open set, there is an $r > 0$ such that $B(x_1, r) \subseteq \Phi^{-1}(] - \infty, \rho[)$. By choosing $s > 0$ such that $s < \min\{\|x_2 - x_1\|; r\}$, condition (4.1) is easily verified. Hence, the conclusion follows directly from Theorem 4.1. □

Finally, we present the main result of this section.

Theorem 4.3. *Assume that Φ is coercive, $\Phi' : X \to X^*$ admits a continuous inverse operator on X^*, $\Psi' : X \to X^*$ is compact, and*

$J=\Phi-\Psi$ admits two distinct local minima x_1, $x_2 \in X$. Put $\rho \in \mathbf{R}$ such that $\max\{\Phi(x_1), \Phi(x_2)\} < \rho$, *and assume that* $\min\{\Psi(x_1), \Psi(x_2)\} \geq 0$ *and there is an $M > 0$ such that*

(C_1) $\sup_{x \in \Phi^{-1}(]-\infty, \rho+M[)} \Psi(x) < M$.

Then, the function J has a third critical point x_3 distinct from x_1 and x_2 such that

$$\Phi(x_3) < \rho + M.$$

Proof. It follows directly from Theorem 4.2 and Theorem 3.1. □

5. Multiple critical points theorems. In this section we assume that X is a reflexive real Banach space and $\Phi, \Psi : X \to \mathbf{R}$ are two continuously Gâteaux differentiable functions. Moreover, we assume that Φ is coercive, and Ψ is sequentially weakly upper semi-continuous. Finally, we also assume that (A_1) and (A_2) in Section 4 hold, and

(A_3) $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$.

Let r_1, r_2 be two constants and $\varphi_1(r_1, r_2)$, $\varphi_2(r_1, r_2)$ as given in Section 2. Moreover, given $M > 0$, we define

$$\varphi_3(r_2, M) = \frac{\sup_{x \in \Phi^{-1}(]-\infty, r_2+M[)} \Psi(x)}{M}$$

and

$$\varphi_4(r_1, r_2, M) = \max\{\varphi_1(r_1, r_2); \varphi_3(r_2, M)\}.$$

Now, we present the following three critical points theorem.

Theorem 5.1. *Assume that there exist two constants r_1 and r_2 such that*

(B_1) $0 < r_1 < r_2$;

(B_2) $\varphi_1(r_1, r_2) < \varphi_2(r_1, r_2)$;

and

(C) *there is an $M > 0$ such that*

(C_1') $\varphi_3(r_2, M) < \varphi_2(r_1, r_2)$;

and

(C_2') *for each* $\lambda \in](1/(\varphi_2(r_1,r_2))),(1/(\varphi_4(r_1,r_2,M)))[$ *the function* $\Phi - \lambda\Psi$ *satisfies the* $(P.S.)_c^M$ *condition for all* $c \in \mathbf{R}$.

Then, for each $\lambda \in](1/(\varphi_2(r_1,r_2))),(1/(\varphi_4(r_1,r_2,M)))[$ *the function* $\Phi - \lambda\Psi$ *admits three distinct critical points* x_i, $i = 1,2,3$, *such that* $x_1 \in \Phi^{-1}(]-\infty,r_1[)$, $x_2 \in \Phi^{-1}([r_1,r_2[)$, $x_3 \in \Phi^{-1}(]-\infty,r_2+M[)$.

Proof. Fix $\lambda \in](1/(\varphi_2(r_1,r_2))),(1/(\varphi_4(r_1,r_2,M)))[$. Taking into account that Φ is also sequentially weakly lower semi-continuous and since $\lambda \in](1/(\varphi_2(r_1,r_2))),(1/(\varphi_1(r_1,r_2)))[$, from Theorem 2.1 one has that the restriction of $\Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty,r_1[)$ admits a global minimum x_1 and the restriction of $\Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty,r_2[)$ admits a global minimum x_2, which are two distinct local minima for $\Phi - \lambda\Psi$. Therefore, from (A_3) one has $\min\{\lambda\Psi(x_1),\lambda\Psi(x_2)\} \geq 0$. Further, since $\lambda < (1/(\varphi_3(r_2,M)))$, one has

$$\frac{\sup_{x\in\Phi^{-1}(]-\infty,r_2+M[)}\Psi(x)}{M} < \frac{1}{\lambda},$$

from which

$$\sup_{x\in\Phi^{-1}(]-\infty,r_2+M[)} \lambda\Psi(x) < M,$$

which is (C_1) of Theorem 4.2 applied to the function $\Phi - (\lambda\Psi)$.

Clearly, from (C_2') it follows that the function $\Phi - (\lambda\Psi)_M$ satisfies the $(PS)_c$ condition, which is (C_2) of Theorem 4.2 applied to the function $\Phi - (\lambda\Psi)$.

Hence, Theorem 4.2 ensures the conclusion. □

Finally, we present the main result of the paper.

Theorem 5.2. *Assume that* $\Phi' : X \to X^*$ *admits a continuous inverse operator on* X^* *and* $\Psi' : X \to X^*$ *is compact. Assume also that there exist three positive constants* r_1, r_2 *and* M, *with* $r_1 < r_2$, *such that*

(D) $\varphi_4(r_1,r_2,M) < \varphi_2(r_1,r_2)$;

Then, for each $\lambda \in](1/(\varphi_2(r_1,r_2))),(1/(\varphi_4(r_1,r_2,M)))[$ *the function* $\Phi - \lambda\Psi$ *admits three distinct critical points* x_i, $i = 1,2,3$, *such that* $x_1 \in \Phi^{-1}(]-\infty,r_1[)$, $x_2 \in \Phi^{-1}([r_1,r_2[)$, $x_3 \in \Phi^{-1}(]-\infty,r_2+M[)$.

Proof. It follows from Theorems 5.1 and 3.1. □

Remark 5.1. Since Φ is coercive, there exists a $\sigma > 0$ such that

$$\Phi^{-1}(]-\infty, r_2 + M[) \subseteq B(0_X, \sigma).$$

Therefore, the conclusion of the previous theorems ensures that the three critical points are uniformly bounded in norm with respect to λ, that is,

$$\|x_i\| < \sigma,$$

$i = 1, 2, 3$, for all $\lambda \in](1/(\varphi_2(r_1, r_2))), (1/(\varphi_4(r_1, r_2, M)))[$.

Remark 5.2. Recently, three critical point theorems have been established in [**2, 5, 6**]. In [**6**, Theorem 2.1], which is based on [**20**, Theorem 3], it was established that the existence of an interval $\Lambda \subseteq [0, +\infty[$ such that, for each $\lambda \in \Lambda$, the function $\Phi - \lambda\Psi$ has three critical points which are uniformly bounded in norm with respect to λ. However, in Theorem 2.1 of [**6**], only an upper bound of the interval Λ was guaranteed. On the other hand, in Theorem B of [**2**], which is based on the variational principle of Ricceri in [**19**], a precise interval of parameters, λ, for which the function $\Phi - \lambda\Psi$ has three critical points was established, losing however the uniform boundedness in norm. For a more precise comparison between Theorem 2.1 of [**6**] and Theorem B of [**2**], we refer to [**7**]. Here, we explicitly observe that the conclusion of previous theorems ensures both a precise interval of parameters, λ, for which the function $\Phi - \lambda\Psi$ has three critical points and the uniform boundedness in norm of the three critical points with respect to λ. Further, we point out that one of the key assumptions, both of [**6**, Theorem 2.1] and of [**2**, Theorem B] ,

$$\lim_{\|x\| \to +\infty} (\Phi(x) - \lambda\Psi(x)) = +\infty$$

is not requested in Theorems 5.2 and 5.2. On the other hand, the assumption (C_1') is not requested in Theorem 2.1 of [**6**] and Theorem B of [**2**]. We also observe that the assumptions of Theorem 2.3 of [**5**] (which is also based on Theorem 2.1) are mutually independent from those of Theorem 5.1 and, in this case, the third critical point is actually the third local minimum point.

6. A two-point boundary value problem. The multiple critical point theorems established in [**2, 5, 6**] have been applied in several nonlinear differential problems, see, for instance, [**2, 3, 5–10, 21**]. Our aim is to apply the three critical points theorem (Theorem 5.2) to these types of nonlinear differential problems. The novel situation with respect to previously cited papers is expressed by the assumption

there is an $M > 0$ *such that*

(C_1) $\sup_{x\in\Phi^{-1}(]-\infty,\rho+M[)} \Psi(x) < M$

and how it can be translated in differential problems. By way of example, here we consider a two-point boundary value problem to give an application of the results in Section 5.

In Theorems 6.1 and 6.2 below, the function $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ is an L^1-Carathéodory function which is nonnegative in $[0,1] \times [0,+\infty[$, namely,

(a) $t \to f(t,x)$ is measurable for every $x \in \mathbf{R}$;

(b) $x \to f(t,x)$ is continuous for almost every $t \in [0,1]$;

(c) for every $\rho > 0$ there exists a function $l_\rho \in L^1([0,1])$ such that

$$\sup_{|x|\le\rho} |f(t,x)| \le l_\rho(t)$$

for almost every $t \in [0,1]$,

(d) $f(t,x) \ge 0$ for almost every $t \in [0,1]$ and all $x \ge 0$.

Consider the following problem

$$(P_\lambda) \qquad \begin{cases} -u'' = \lambda f(t,u) \\ u(0) = u(1) = 0, \end{cases}$$

where λ is a positive real parameter, and put

$$F(t,\xi) = \int_0^\xi f(t,x)\,dx$$

for all $(t,\xi) \in [0,1] \times \mathbf{R}$.

We have the following theorem.

Theorem 6.1. *Assume that there exist four positive constants c_1, d, c_2 and k, with $c_1 < d < (\sqrt{2}/2)c_2 < (\sqrt{2}/2)k$, such that*

(i) $(\int_0^1 F(t,c_1)\,dt)/c_1^2 < (1/3)(\int_{1/4}^{3/4} F(t,d)\,dt)/d^2$;

(ii) $(\int_0^1 F(t,c_2)\,dt)/c_2^2 < (1/3)(\int_{1/4}^{3/4} F(t,d)\,dt)/d^2$;

(iii) $(\int_0^1 F(t,k)\,dt)/(k^2 - c_2^2) < (1/3)(\int_{1/4}^{3/4} F(t,d)\,dt)/d^2$.

Then, for each $\lambda \in \left]6d^2/(\int_{1/4}^{3/4} F(t,d)\,dt), \min\{2c_1^2/(\int_0^1 F(t,c_1)\,dt); 2c_2^2/(\int_0^1 F(t,c_2)\,dt); (2k^2 - 2c_2^2)/(\int_0^1 F(t,k)\,dt)\}\right[$ *the problem (P_λ) admits three nonnegative generalized solutions u_i, $i = 1,2,3$, such that*

$$\|u_i\|_\infty < k, \qquad i = 1,2,3.$$

Proof. Without loss of generality, we can assume $f(t,x) \geq 0$ for almost every $t \in [0,1]$ and for all $x \in \mathbf{R}$. Let X be the Sobolev space $W_0^{1,2}([0,1])$ endowed with the norm $\|u\| := (\int_0^1 |u'(t)|^2\,dt)^{1/2}$. For each $u \in X$, put:

$$\Phi(u) := \frac{1}{2}\|u\|^2, \qquad \Psi(u) := \int_0^1 F(t,u(t))\,dt.$$

It is well known that the critical points in X of the functional $\Phi - \lambda\Psi$ are precisely the generalized solutions of problem (P_λ) and that Φ and Ψ are as in Theorem 5.2 (see, for instance, [**3**, Section 2]). Our aim is to verify that there exist three positive constants r_1, r_2, with $r_1 < r_2$, and M such that (D) of Theorem 5.2 holds. First, we claim that r_1 and r_2 exist such that $\varphi_1(r_1,r_2) < \varphi_2(r_1,r_2)$. We argue as in Theorem 3.1 of [**5**]. By choosing $r_1 = 2c_1^2$, $r_2 = 2c_2^2$, and

$$y_0(t) := \begin{cases} 4d(t-1) & \text{if } t \in [0,1/4[\\ d & \text{if } t \in [1/4,3/4] \\ 4d(1-t) & \text{if } t \in]3/4,1], \end{cases}$$

we have $0 < r_1 < r_2$, $y_0 \in X$, $\|y_0\|^2 = 8d^2$. Moreover, taking into account that

$$|x(t)| \leq \frac{1}{2}\|x\| \tag{6.1}$$

for all $t \in [0,1]$ and for all $x \in X$, one has

$$\begin{aligned}\varphi_2(r_1,r_2) \geq{}& \frac{\int_{1/4}^{3/4} F(t,d)\,dt + \int_0^{1/4} F(t,4dt)\,dt}{4d^2} \\ &+ \frac{\int_{3/4}^{1} F(t,4d(1-t))\,dt - \int_0^1 F(t,c_1)\,dt}{4d^2}\end{aligned} \tag{6.2}$$

$$\varphi_1(r_1,r_2) \leq \max\left\{\frac{\int_0^1 F(t,c_1)\,dt}{2c_1^2}; \frac{\int_0^1 F(t,c_2)\,dt}{2c_2^2}\right\}. \tag{6.3}$$

Since f is nonnegative and (i) holds, in particular, one has

$$\begin{aligned}\varphi_2(r_1,r_2) &\geq \frac{\int_{1/4}^{3/4} F(t,d)\,dt - \int_0^1 F(t,c_1)\,dt}{4d^2} \\ &\geq \frac{\int_{1/4}^{3/4} F(t,d)\,dt}{4d^2} - \frac{\int_0^1 F(t,c_1)\,dt}{4c_1^2} \\ &> \left(\frac{1}{4} - \frac{1}{12}\right)\frac{\int_{1/4}^{3/4} F(t,d)\,dt}{d^2},\end{aligned}$$

that is,

$$\varphi_2(r_1,r_2) \geq \frac{1}{6}\frac{\int_{1/4}^{3/4} F(t,d)\,dt}{d^2}. \tag{6.4}$$

Due to (i) and (ii), from (6.3) and (6.4) our claim is proved.

Now, we prove that there is an $M > 0$ such that $\varphi_3(r_2,M) < \varphi_2(r_1,r_2)$. To this end, fix $M = (2k^2 - 2c_2^2)$. Taking (6.1) again into account, one has $\varphi_3(r_2,M) = (\sup_{x\in\Phi^{-1}(]-\infty,r_2+M[)} \Psi(x))/M \leq (\int_0^1 F(t,k)\,dt)/(2k^2 - 2c_2^2)$, that is,

$$\varphi_3(r_2, M) \leq \frac{\int_0^1 F(t,k)\,dt}{(2k^2 - 2c_2^2)}. \tag{6.5}$$

Therefore, due to (iii) from (6.4) and (6.5), we obtain the proof as we claimed.

Hence, from Theorem 5.2 we obtain that for each

$$\lambda \in \left] \frac{6d^2}{\int_{1/4}^{3/4} F(t,d)\,dt}, \min\left\{ \frac{2c_1^2}{\int_0^1 F(t,c_1)\,dt}; \frac{2c_2^2}{\int_0^1 F(t,c_2)\,dt}; \frac{2k^2 - 2c_2^2}{\int_0^1 F(t,k)\,dt} \right\} \right[,$$

the problem (P_λ) admits three generalized solutions x_i, $i = 1, 2, 3$, which, due to the maximum principle, are nonnegative. Further, one has $\Phi(x_i) < r_2 + M$, that is, $\|x_i\|_\infty < k$, and the proof is complete. □

Remark 6.1. We observe that the conclusion of Theorem 6.1 can be more precise. In fact, the three solutions satisfy the following conditions: $\|x_1\|_\infty < c_1$, $\|x_2\|_\infty < c_2$, $\|x_2\| \geq 2c_1$ and $\|x_3\|_\infty < k$.

Remark 6.2. We observe that in Theorem 6.1 assumptions (i)–(iii) can be expressed in the following, more general, form

$$\begin{aligned} \frac{1}{\lambda_2} &:= \max\left\{ \frac{\int_0^1 F(t,c_1)\,dt}{2c_1^2}; \frac{\int_0^1 F(t,c_2)\,dt}{2c_2^2}; \frac{\int_0^1 F(t,k)\,dt}{2k^2 - 2c_2^2} \right\} \\ &< \frac{\int_{1/4}^{3/4} F(t,d)\,dt + \int_0^{1/4} F(t,4dt)\,dt}{4d^2} \\ &\quad + \frac{\int_{3/4}^1 F(t,4d(1-t))\,dt - \int_0^1 F(t,c_1)\,dt}{4d^2} \\ &=: \frac{1}{\lambda_1}, \end{aligned}$$

as can easily be deduced from the proof itself of the theorem (see (6.2), (6.3) and (6.5)). Clearly, in this case, the interval for which the problem admits at least three solutions, whose norms in $C^0([0,1])$ are uniformly bounded with respect to λ from k, is $]\lambda_1, \lambda_2[$.

Finally, we observe that, in a similar way and taking into account the techniques used in [**5**], see also [**3**], we can study the more general problem

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' = \lambda f(t,u)h(u') \\ u(a) = u(b) = 0. \end{cases}$$

Remark 6.3. If f is a continuous function we explicitly observe that the three solutions are classical. We also observe that, if $f(t,0) \neq 0$ for some $t \in [0,1]$, then the three solutions are positive; while, on the contrary, Theorem 6.1 ensures at least two positive solutions for the problem considered.

Now, we highlight the following consequence of Theorem 6.1.

Theorem 6.2. *Assume that there exist three positive constants c_1, d, p, with $c_1 < d < (p/2)$, such that*

(k) $(\int_0^1 F(t,c_1)\,dt)/c_1^2 < (1/3)(\int_{1/4}^{3/4} F(t,d)\,dt)/d^2$;

(kk) $(\int_0^1 F(t,p)\,dt)/p^2 < (1/6)(\int_{1/4}^{3/4} F(t,d)\,dt)/d^2$.

Then, for each $\lambda \in]6d^2/(\int_{1/4}^{3/4} F(t,d)\,dt), \min\{(2c_1^2)/(\int_0^1 F(t,c_1)\,dt); p^2/(\int_0^1 F(t,p)\,dt)\}[$, the problem (P_λ) admits three nonnegative generalized solutions u_i, $i = 1,2,3$, such that

$$\|u_i\|_\infty < p, \quad i = 1,2,3.$$

Proof. It is enough to pick $c_2 = (1/\sqrt{2})p$ and $k = p$ and to apply Theorem 6.1. In fact, one has

$$\begin{aligned} \frac{\int_0^1 F(t,c_2)\,dt}{c_2^2} &= 2\frac{\int_0^1 F(t,(1/\sqrt{2})p)\,dt}{p^2} \\ &\leq 2\frac{\int_0^1 F(t,p)\,dt}{p^2} < \frac{1}{3}\frac{\int_{1/4}^{3/4} F(t,d)\,dt}{d^2} \end{aligned}$$

and

$$\frac{\int_0^1 F(t,k)\,dt}{k^2-c_2^2} = \frac{\int_0^1 F(t,p)\,dt}{p^2} < \frac{1}{3}\frac{\int_{1/4}^{3/4} F(t,d)\,dt}{d^2}. \qquad \square$$

Remark 6.4. In Theorem 3.1 of [**3**] we assumed the condition

(jjj) $F(t,\xi) \le \mu(1+|\xi|^s)$

for all $(t,\xi) \in [0,1] \times \mathbf{R}$, for some $s < 2$ and for some $\mu > 0$. A simple computation shows that, by choosing p big enough, (jjj) implies (kk) of Theorem 6.2. Hence, when f is a nonnegative function, Theorem 6.2 (with (k) expressed as seen in Remark 6.2) improves Theorem 3.1 of [**3**].

Remark 6.5. Theorem 1.1 in the introduction follows directly from Theorem 6.2.

Now, we present another consequence of Theorem 6.1.

Theorem 6.3. *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function. Assume that there exist three positive constants c, d and $\overline{c}$, with $c < d < \overline{c}$ such that*

(α) $(\int_0^c f(x)\,dx)/c^2 < (1/6)(\int_0^d f(x)\,dx)/d^2$;

(β) $f(\overline{c}) = 0$ *and* $f(x) > 0$ *for all* $x \in]0,\overline{c}[$;

Then, for each $\lambda \in](12d^2)/(\int_0^d f(t)\,dt), (2c^2)/(\int_0^c f(t)\,dt)[$, *the problem* (AP_λ) *admits three positive classical solutions whose norms in* $C^0([0,1])$ *are less than or equal to* $\overline{c}$.

Proof. Define $f^* : \mathbf{R} \to \mathbf{R}$ as follows

$$f^*(x) := \begin{cases} f(0) & \text{if } x \in]-\infty, 0[\\ f(x) & \text{if } t \in [0,\overline{c}] \\ 0 & \text{if } t \in]\overline{c}, +\infty], \end{cases}$$

and fix $\lambda \in]12d^2)/(\int_0^d f(t)\,dt), (2c^2)/(\int_0^c f(t)\,dt)[$.

By choosing $c_1 = c$ and

$$p > \max\left\{2d; \bar{c}; \sqrt{12(F(\bar{c}))/(F(d))}d; \sqrt{2(F(\bar{c}))/(F(c))}c\right\},$$

it is a simple computation to show that (k)–(kk) of Theorem 6.2 hold and $(2c^2)/(\int_0^c f(t)\,dt) = \min\{(2c_1^2)/(F^*(c_1)); p^2/(F^*(p))\}$. Therefore, the problem

$$\begin{cases} -u'' = \lambda f^*(u) \\ u(0) = u(1) = 0, \end{cases}$$

admits three nonnegative solutions u_i, $i = 1, 2, 3$, such that $|u_i(t)| < p$, $t \in [0, 1]$, $i = 1, 2, 3$. We claim that $|u_i(t)| \leq \bar{c}$, $t \in [0, 1]$, $i = 1, 2, 3$. In fact, arguing by a contradiction, there exists $[a, b] \subseteq [0, 1]$ such that $u_i(t) > \bar{c}$ for all $t \in [a, b]$, $u_i(a) = u_i(b) = 0$ and hence, being a solution of the previous problem, must be $u_i(t) = 0$ in $[a, b]$; this is a contradiction and our claim is proved. The conclusion follows since u_i are also solutions of (AP_λ). $\square$

Remark 6.6. In Theorem 2.32 of [**18**], under the assumptions

(α') $f(0) = 0$;

and (β) of Theorem 6.3, the existence of $\overline{\lambda} > 0$ such that, for each $\lambda > \overline{\lambda}$, the problem (AP_λ) admits two positive solutions and is ensured (the result for elliptic equations also holds). We explicitly observe that in Theorem 6.3 $f(0)$ may be different from zero and that three positive solutions are obtained. Moreover, if we assume

(α'') $\lim_{x\to 0^+}(f(x)/x) = 0$;

(which is stronger than both (α') and (α)) we obtain that the problem (AP_λ) admits at least two positive solutions for each $\lambda > \inf_{d\in]0,\bar{c}[}(12d^2)/(\int_0^d f(t)\,dt)$. In other words, Theorem 6.3 ensures (under the stronger condition (α'')) an upper bound of the $\overline{\lambda}$ established in Theorem 2.32 of [**18**], namely,

$$\overline{\lambda} \leq \inf_{d\in]0,\bar{c}[} \frac{12d^2}{\int_0^d f(t)\,dt}.$$

Remark 6.7. We explicitly observe that in Theorem 6.1 no condition at infinity is requested, as the following easy example shows.

Example 6.1. Let $g : [0,1] \to \mathbf{R}$ be defined by $g(t) = t$ for every $t \in [0,1]$, and let $h : \mathbf{R} \to \mathbf{R}$ be the function defined as follows

$$h(x) := \begin{cases} 1 & \text{if } x \in \,]-\infty, 1] \\ x^{10} & \text{if } x \in \,]1, 2] \\ 2^{10} & \text{if } x \in \,]2, 1002] \\ \overline{h}(x) & \text{if } x \in \,]1002, +\infty[, \end{cases}$$

where $\overline{h} :]1002, +\infty[\,\to \mathbf{R}$ is an arbitrary function.

The function $f(t,x) = g(t)h^*(x)$, where h^* is a continuous function which coincides with h in $]-\infty, 1002]$, satisfies all the assumptions of Theorem 6.1 by choosing, for instance, $c_1 = 1$, $d = 2$, $c_3 = 500$, $k = 1002$. Then, for each $\lambda \in \,](6/10), (18/10)[$, the problem

$$\begin{cases} -u'' = \lambda t h(u) \\ u(0) = u(1) = 0, \end{cases}$$

admits at least three positive classical solutions whose norms in $C([0,1])$ are less than 1002.

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