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LOCAL SOLVABILITY FOR $\Box_b$ ON DEGENERATE CR MANIFOLDS AND RELATED SYSTEMS

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ABSTRACT. We show that the Kohn Laplacian is locally solvable in Sobolev spaces H^k, $k \geq 0$, on any degenerate CR manifold whose Levi form has a kernel of constant dimension. A similar result is indeed proved for a more general class of systems of linear partial differential operators.

1. Introduction and discussion of the results. Let M be an abstract CR manifold of CR-dimension n and real codimension h (so that $\dim M = 2n+h$), and denote by $N^*(M)$ its characteristic bundle (of rank h). Consider the tangential Cauchy-Riemann complex $\overline{\partial}_b$ on M and the corresponding Kohn Laplacian $\Box_b^{(q)} = \overline{\partial}_b\overline{\partial}_b^* + \overline{\partial}_b^*\overline{\partial}_b$ acting on $(0,q)$-forms on M, see e.g., Shaw and Wang [**21**] or Section 3 below for terminology). In this paper we carry on the investigation in [**15**] of the following L^2 estimate:

For every $x_0 \in M$ and every $\delta > 0$, there exists an open neighborhood

$$\begin{aligned} &\Omega_\delta \subset M \textit{ of } x_0 \textit{ such that } \|u\|_0 \leq \delta(\Box_b^{(q)}u, u), \\ &\qquad \textit{for all } u \in \mathcal{D}_{(0,q)}(\Omega_\delta), \end{aligned} \tag{1.1}$$

where $(\cdot,\cdot)$ denotes the L^2 inner product, $\|\cdot\|_0$ the L^2-norm, and $\mathcal{D}_{(0,q)}(\Omega_\delta)$ is the space of smooth $(0,q)$-forms with compact support in Ω_δ. Such an estimate is weaker than both the subelliptic ones and the semi-maximal estimates of Derridj and Tartakoff [**3, 4**]. It is however strong enough to guarantee local solvability of $\Box_b$ in the Sobolev spaces $H^k_{(0,q)}$, $k \geq 0$, of $(0,q)$-forms.

In [**15**] we proved that, if (1.1) holds, then the Levi form $\mathcal{L}(\rho)$ of M at any characteristic point $\rho \in N^*(M) \setminus 0$ cannot have q positive and $n-q$ negative eigenvalues. Of course, this necessary condition is clearly

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satisfied at those points where the Levi form is degenerate. This paper is devoted to prove that, in fact, when the Levi form $\mathcal{L}(\rho)$ is degenerate and has locally constant dimension (when ρ varies in $N^*(M)\setminus 0$), there are no further obstructions for (1.1) to hold.

Theorem 1.1. *Let M be an abstract CR manifold of CR-dimension n and real codimension h; let $\Box_b^{(q)}$ be the Kohn Laplacian with respect to a fixed Riemannian metric on M, acting on $(0,q)$-forms. Let us suppose that the Levi form $\mathcal{L}(\rho)$ is degenerate and has a kernel of locally constant dimension when ρ varies in the characteristic bundle $N^*(M)$ with the 0-section removed. Then*

(a) (1.1) *holds true*;

(b) *for every point $x_0 \in M$ and any integer $k \geq 0$ there exists a neighborhood Ω_k of x_0 such that the system $\Box_b^{(q)} u = f$ has a solution $u \in H^k_{(0,q)}(\Omega_k)$ for every $f \in H^k_{(0,q)}(\Omega_k)$, $q = 0, \dots, n$.*

Notice that under the hypotheses of Theorem 1.1 condition $Y(q)$ may of course be violated. For example, condition $Y(n)$ does not hold on any pseudoconvex manifold of CR-dimension n, see e.g., [**2**]. The condition given is therefore sufficient for local solvability in the absence of hypoellipticity (in general).

It is interesting to observe that Theorem 1.1 highlights the different behavior of the $\overline{\partial}_b$-complex and the Kohn Laplacian $\Box_b$ as regards local solvability in *degenerate* CR manifolds. In fact, a recent result by Hill and Nacinovich [**9**] shows that, under the assumptions of Theorem 1.1 (and M embeddable), if the Levi form $\mathcal{L}(\rho)$, $\rho = (x_0, \xi_0)$, $\xi_0 \neq 0$, has q positive eigenvalues, with the other being ≤ 0, the Poincaré lemma does not hold in degree q at x_0, i.e., there is a smooth $(0,q)$-cocycle f in an open neighborhood V of x_0 such that, for every open neighborhood $U \subset V$ of x_0 there are no distributions $u \in \mathcal{D}'_{(0,q-1)}(U)$ which solve $\overline{\partial}_b u = f$ in U. Instead, Theorem 1.1 shows that $\Box_b^{(q)}$ is always locally solvable.

In the special case when M is a quadratic CR manifold, i.e., $M = \{(z, t+iu) \in \mathbf{C}^n \times \mathbf{C}^m : u = \Phi(z,z)\}$ for some Hermitian map $\Phi : \mathbf{C}^n \times \mathbf{C}^n \to \mathbf{C}^m$, Theorem 1.1 was already proved without the constant rank assumption by Peloso and Ricci [**18, 19**], see also Nagel, Ricci and Stein [**14**], Treves [**23**], by taking advantage of the natural group structure carried by such a CR manifold where techniques from harmonic analysis then apply. In particular, they make use of explicit representations.

Theorem 1.1 will be shown to follow from a more general result we are going to establish, concerning a class of systems with double characteristics whose principal symbol is a scalar multiple of the identity matrix.

More precisely, let X be an open subset of $\mathbf{R}^n$, and consider an $N\times N$ system P of linear partial differential operators in X of order m. We will assume the following:

(H_1) *The principal symbol of P has the form $p_m \mathrm{Id}_{N\times N}$, where $p_m(x,\xi)$ is homogeneous of degree m with respect to ξ and vanishes exactly to second order on a manifold $\Sigma \subset T^*X \setminus 0$ (transversal ellipticity), namely* $\operatorname{Ker} F_{p_m}(\rho) = T_\rho\Sigma$, *for all $\rho \in \Sigma$.*

Here we used standard notation for the fundamental matrix (or symplectic Hessian) $F_{p_m}(\rho)$ associated with p_m, defined by

$$\sigma(v, F_{p_m}(\rho)w) = \frac{1}{2}\langle \operatorname{Hess} p_m(\rho)\, v, w\rangle, \qquad \text{for all} \quad v, w \in T_\rho T^*X, \tag{1.2}$$

where $\sigma = \sum_{j=1}^n d\xi_j \wedge dx_j$ is the canonical symplectic 2-form on T^*X. The definition (1.2) has an invariant meaning at points of Σ, because p_m vanishes to second order there, see [**10**, subsection 21.5]. Moreover, since p_m is nonnegative, the spectrum of F_{p_m} consists of the eigenvalue 0 and of eigenvalues $\pm i\mu_j$, with $\mu_j > 0$. One then sets $\mathrm{Tr}^+ F_{p_m} = \sum_j \mu_j$.

We moreover suppose that

(H_2) *Σ is a manifold with locally constant symplectic rank, namely* $\dim(T_\rho\Sigma \cap T_\rho\Sigma^\sigma) = \text{const}$, *when ρ varies in the connected components of Σ where $T_\rho\Sigma^\sigma$ is the symplectic orthogonal of $T_\rho\Sigma$;*

(H_3) *$\sigma|_\Sigma$ is degenerate, namely* $\dim(T_\rho\Sigma \cap T_\rho\Sigma^\sigma) \geq 1$, *for all $\rho \in \Sigma$;*

(H_4) *for every $\rho \in \Sigma$, $T_\rho\Sigma \cap T_\rho\Sigma^\sigma \not\subset \operatorname{Ker} d\pi(\rho)$, where $\pi : T^*X \to X$ is the projection.*

Before stating the result, we recall that the subprincipal symbol p^s_{m-1} of P is defined by $p^s_{m-1} = p_{m-1} + (i/2)\langle\partial_x, \partial_\xi\rangle p_m \,\mathrm{Id}$, ($p_{m-1}$ denotes the matrix of the homogeneous terms of degree $m-1$), and is invariantly defined at points of Σ as well. Moreover, for any given complex matrix Q we set $\mathrm{Re}\, Q := (Q + Q^*)/2$.

Theorem 1.2. *Let P be an $N \times N$ system of linear partial differential operators in X of order m, satisfying (H_1)–(H_4). Let us suppose, in addition, that*

$$\mathrm{Re}\, p^s_{m-1}(\rho) + \mathrm{Tr}^+ F_{p_m}(\rho)\mathrm{Id} \geq 0, \quad \text{for all } \rho \in \Sigma, \tag{1.3}$$

as a Hermitian matrix. Then

(a) *for every $x_0 \in X$ and every $\delta > 0$, there exists a neighborhood Ω_δ of x_0 such that*

$$\|u\|_0^2 \leq \delta\, \mathrm{Re}\,(Pu, u), \quad \text{for all } u \in C_0^\infty(\Omega_\delta; \mathcal{C}^N); \tag{1.4}$$

(b) *for every $x_0 \in X$ and any integer $k \geq 0$, there exists a neighborhood Ω_k of x_0 such that the system $Pu = f$ has a solution $u \in H^{k+m-2}(\Omega_k; \mathcal{C}^N)$ for every $f \in H^k(\Omega_k; \mathcal{C}^N)$.*

As a model in the scalar case ($N = 1$), the reader may think of the Baouendi-Goulaouic type operator

$$P = D_1^2 + x_1^2 D_2^2 + D_3^2 - D_2$$

in $\mathbf{R}^3$ (as usual $D_j = -i\partial/\partial x_j$). We can write $P = MM^* + D_3^2$, where $M = D_1 + ix_1 D_2$ is the Mizohata operator, and therefore solvability in $L^2(B(0,\delta))$, $\delta > 0$, immediately follows from the estimate

$$(Pu, u) \geq \|D_3 u\|_0^2 \geq \frac{1}{2\delta^2}\|u\|_0^2,$$

for a smooth u with $\mathrm{supp}\, u \subset B(0,\delta)$.

When $N = 1$, $k = 0$, and $T_\rho\Sigma^\sigma \subset T_\rho\Sigma$ for every $\rho \in \Sigma$, namely, Σ is involutive, Theorem 1.2 reduces to Theorem 3 (2D) by Popivanov [**20**]. Notice that $\mathrm{Tr}^+ F_{p_m} \equiv 0$ in that case.

Let us now discuss the hypotheses of Theorem 1.2 in more detail.

For a formally self-adjoint second order system satisfying (H_1) and (H_2), condition (1.3) is actually equivalent to saying that P is bounded from below in L^2, see Theorem 3.1 below, i.e., $(Pu,u) \geq -C_K\|u\|_0^2$, for any compact subset $K \subset X$ and every $u \in C_0^\infty(K;\mathbf{C}^N)$, which is of course true for $\Box_b$.

As regards Hypothesis (H_4), it is easy to see that it is really essential; for example, the operator $P = (x_2D_1 - x_1D_2)^2$ satisfies all the remaining hypotheses (with $N = 1$), but it is not locally solvable at the origin, because $P^*u = 0$ for every $u \in C_0^\infty(\mathbf{R}^2)$ which is rotation invariant.

Also, if the symplectic form is nondegenerate, then P may not be locally solvable, as $\Box_b^{(0)}$ on the Heisenberg group $\mathbf{H}_n$, see Lewy [**11**], Folland and Stein [**5**], or the operator $P = D_1^1 + x_1^2D_2^2 - D_2$ in $\mathbf{R}^2$, see Grushin [**8**], Gilioli and Treves [**6**], cf. also Müller [**13**]. However, if we replace (1.3) by a strict inequality, then the system is known to be hypoelliptic and locally solvable with the loss of one derivative (without assuming (H_2)–(H_4), and the fact that the principal symbol is diagonal), as shown by Boutet de Monvel, et al. [**1**]. For the Kohn Laplacian on a CR manifold, this corresponds to the case in which the condition $Y(q)$ is satisfied, as shown by Grigis [**7**] and Nicola [**15**], see also Parmeggiani [**17**]. Instead, under the hypotheses of Theorem 1.2, system P is not hypoelliptic in general; one may think, for example, of the operator $P = D_1^2$ in $\mathbf{R}^n$, $n \geq 2$.

Finally, we observe that in Theorem 1.2 the radial vector field is allowed to be symplectically orthogonal to the tangent space to Σ at some characteristic point.

Theorem 1.2 is proved in Section 2. Then, in Section 3 we will prove Theorem 1.1 by verifying that, under the assumptions in that theorem, $\Box_b$ satisfies the hypotheses of Theorem 1.2.

2. Proof of Theorem 1.2.

Proof of Theorem 1.2 (a). We will use the following result, due to Hörmander that the reader can extract from the proof of Theorem 22.3.2 of [**10**].

Lemma 2.1. *Consider a smooth nonnegative function* $p_2(x,\xi)$ *which is positively homogeneous of degree* 2, *with respect to* ξ, *and vanishes exactly to second order on a manifold* $\Sigma \subset T^*X \setminus 0$ *satisfying* (H_2). *Fix any* $\rho_0 \in \Sigma$, *and let* $l = \dim(T_\rho\Sigma \cap T_\rho\Sigma^\sigma)$ *and* $2\nu = \dim T_\rho\Sigma^\sigma/(T_\rho\Sigma \cap T_\rho\Sigma^\sigma)$ *for* ρ *near* ρ_0. *Then there exist independent real functions* f_j, $j = 1,\dots,2\nu+l$, *defined in a conic neighborhood* V *of* ρ_0, *and homogeneous of degree* 1, *such that*

$$p_2(x,\xi) = \sum_{j=1}^{2\nu+l} f_j(x,\xi)^2,$$

and, upon defining

$$\begin{cases} g_j = f_{2j-1} + i f_{2j} & j = 1,\dots,\nu, \\ g_{\nu+j} = f_{2\nu+j} & j = 1,\dots,l, \end{cases} \tag{2.1}$$

satisfying $\sum_{j=1}^{\nu}\{\overline{g}_j, g_j\} = -2i\mathrm{Tr}^+F_{p_2}$ *at* $\Sigma \cap V$ *and*

$$\begin{gathered} \{g_j, g_k\} = 0 \quad \text{at} \quad \Sigma \cap V, \\ \textit{for all} \quad j = \nu+1,\dots,\nu+l, \\ \textit{for all} \quad k = 1,\dots,\nu+l. \end{gathered} \tag{2.2}$$

Consider then a point $x_0 \in X$, and apply Lemma 2.1 to any characteristic point $\rho_0 = (x_0,\xi)$ in the fibre over x_0. Let $X_j \in \mathrm{OP}S^1(X;\mathbf{C}^N)$ be pseudodifferential systems whose principal symbols coincide with $g_j\,\mathrm{Id}$ in V. Microlocally near ρ_0 we have

$$Q := P - \sum_{j=1}^{\nu+l} X_j^* X_j \in \mathrm{OP}\,S^1.$$

Moreover, the principal symbol of Q is given, at $\Sigma \cap V$, by

$$q_1 = p_1^s + \frac{i}{2}\sum_{j=1}^{\nu}\{\overline{g}_j, g_j\}\mathrm{Id} = p_1^s + \mathrm{Tr}^+F_{p_2}\mathrm{Id},$$

so that $\operatorname{Re} sq_1 \geq 0$ on $\Sigma \cap V$ by (1.3). Now we take any Hermitian matrix $\tilde{q}(x,\xi)$, defined in V, nonnegative and positively homogeneous of degree 1, such that $\tilde{q} = \operatorname{Re} q_1$ on $\Sigma \cap V$. Then

$$\operatorname{Re} q_1 = \tilde{q} + \sum_{j=1}^{\nu+l} r_j^* g_j + \overline{g}_j r_j \quad \text{in } V,$$

for convenient $N \times N$ matrices $r_j(x,\xi)$ homogeneous of degree 0. Let $\widetilde{Q} \in \operatorname{OP} S^1(X;\mathbf{C}^N)$ and $R_j \in \operatorname{OP} S^0(X;\mathbf{C}^N)$ be pseudodifferential systems whose principal symbols coincide with $\tilde{q}$ and r_j in V, respectively, and let $\psi \in S^0(X \times \mathbf{R}^n)$ be any real symbol supported in V. Upon setting $\Psi = \psi(x,D) \otimes \mathrm{Id}$, by the Sharp Gårding inequality for systems applied to $\widetilde{Q}$, see, e.g., [**10**, Theorem 18.6.14], we have

(2.3)
$$\begin{aligned}\operatorname{Re}(P\Psi u, \Psi u) &\geq \sum_{j=1}^{\nu+l} \|(X_j + R_j)\Psi u\|_0^2 - C\|u\|_0^2 \\ &\geq \sum_{j=\nu+1}^{\nu+l} \|(X_j + R_j)\Psi u\|_0^2 - C\|u\|_0^2 \\ &\geq \sum_{j=\nu+1}^{\nu+l} \|X_j \Psi u\|_0^2 - C'\|u\|_0^2, \quad \forall u \in C_0^\infty(X;\mathbf{C}^N).\end{aligned}$$

Now we are going to prove the following estimate:

For some $j_0 \in \{\nu+1, \dots, \nu+l\}$ and every $\delta > 0$ there exists a neighborhood Ω_δ of x_0 such that

$$\begin{gathered}\|X_{j_0}\Psi u\|_0 \geq \frac{1}{\delta}\|\Psi u\|_0^2 - C_\delta \|u\|_{-1}^2, \\ \text{for all } u \in C_0^\infty(\Omega_\delta;\mathcal{C}^N).\end{gathered} \tag{2.4}$$

To this end, we observe that, by (2.2), the vector bundle $T\Sigma \cap T\Sigma^\sigma$ on Σ of rank l is generated by the Hamiltonian vector fields $H_{g_{\nu+1}}, \dots, H_{g_{\nu+l}}$, on $\Sigma \cap V$. By Hypothesis (H_4) it follows that for some $j_0 \in \{\nu+1, \dots, \nu+l\}$ one has $d_\xi g_{j_0} \neq 0$ at ρ_0. Possibly after shrinking V, by the implicit function theorem and Taylor's formula, we can then write, say, $g_{j_0}(x,\xi) = e(x,\xi)(\xi_1 - \lambda(x,\xi'))$ in V, with $\xi' = (\xi_2, \dots, \xi_n)$, for suitable real functions λ and e homogeneous of degree 1, and $e \neq 0$.

The elliptic factor e is of course irrelevant, so that we can assume that $g_{j_0}(x,\xi) = \xi_1 - \lambda(x,\xi')$ in V. Now, there exists a canonical transformation $(y,\eta) = \chi(x,\xi)$ from V into a conic neighborhood of (x_0,ε_n), $\varepsilon_n = (0,\dots,0,1) \in \mathbf{R}^n$, with $y_1 = x_1$ and $\eta_1 = g_{j_0}$. Let F be any properly supported unitary Fourier integral operator associated with χ. By Egorov's theorem, see e.g. [**22**, Theorem 6.2], we have

$$\|X_{j_0}\Psi u\|_0 = \|D_{y_1}F\Psi u\|_0 + O(\|\Psi u\|_0). \tag{2.5}$$

Now, for any given δ, we take $\phi_1 \in C_0^\infty(B(x_0,\delta))$, $\phi_2 \in C_0^\infty(B(x_0,\delta/3))$, with $\phi_2 = 1$ on $B(x_0,\delta/4)$, and $\phi_1 = 1$ in $B(x_0,\delta/2)$, so that $\phi_2 u = u$ if $\operatorname{supp} u \subset B(x_0,\delta/4)$. Then the operator $(1-\phi_1)F\Psi\phi_2$ is regularizing, i.e., it maps $\mathcal{D}'(X) \to C^\infty(X)$, because of the choice of χ, see for example [**12**, subsection 4.1]. It follows from the Poincaré inequality that

$$\begin{aligned} \|D_{y_1}F\Psi u\|_0 &\geq \frac{1}{\sqrt{2}\delta}\|\phi_1 F\Psi\phi_2 u\|_0 - C_\delta\|u\|_{-1} \\ &\geq \frac{1}{\sqrt{2}\delta}\|\Psi u\|_0 - C'_\delta\|u\|_{-1}, \end{aligned} \tag{2.6}$$

for all $u \in C_0^\infty(B(x_0,\delta/4))$. From (2.5) and (2.6) we obtain (2.4). In view of (2.3) we therefore deduce the following estimate:

Any given point $\rho_0 = (x_0,\xi)$ has a conic neighborhood V such that, for every real symbol $\psi \in S^0(X\times\mathbf{R}^n)$ supported in V and any $\delta > 0$ there exists a neighborhood Ω_δ of x_0 for which

$$\operatorname{Re}(P\Psi u,\Psi u) \geq \frac{1}{\delta}\|\Psi u\|_0^2 - C_\delta\|u\|_{-1}^2, \quad \text{for all } u \in C_0^\infty(\Omega_\delta;\mathbf{C}^N). \tag{2.7}$$

Indeed, when $\rho_0 \notin \Sigma$, this estimate is of course satisfied.

Now we are going to patch together the microlocal estimates (2.7). We take real symbols $\psi_j \in S^0(X\times\mathbf{R}^n)$, $j = 1,\dots,J$, with so small support such that (2.7) holds for each of them, and $\sum_{j=1}^J \psi_j(x,\xi)^2 = 1$ for x in a neighborhood Ω of x_0. Then we observe that, with $\Psi_j = \psi_j(x,D)\otimes \mathrm{Id}$, we have $\operatorname{Re}(Pu,u) = \sum_{j=1}^J \operatorname{Re}(P\Psi_j u,\Psi_j u) + O(\|u\|_0^2)$ and $\|u\|_0^2 = \sum_{j=1}^J \|\Psi_j u\|^2 + O(\|u\|_{-2}^2)$ for $u \in C_0^\infty(\Omega;\mathbf{C}^N)$. Hence, for any $\delta' > 0$ we can find an open neighborhood $\Omega'_{\delta'}$ such that

$$\operatorname{Re}(Pu,u) \geq \frac{1}{\delta'}\|u\|_0^2 - C\|u\|_0^2 - C_{\delta'}\|u\|_{-1}^2, \quad \text{for all } u \in C_0^\infty(\Omega'_{\delta'};\mathbf{C}^N),$$

where the constant $C > 0$ is independent of δ'. Given $\delta > 0$, we therefore choose $\delta' \leq \min\{C/2, \delta/4\}$, and we take any neighborhood $\Omega_\delta \subset \Omega'_{\delta'}$ such that $C'_{\delta'}\|u\|^2_{-1} \leq (1/4\delta)\|u\|^2_0$ for every $u \in C_0^\infty(\Omega_\delta; \mathbf{C}^N)$. Then (1.4) is verified.

Proof of Theorem 1.2 (b). We first prove the following estimate:

For every $\delta > 0$, $s \in \mathbf{R}$, and any point $x_0 \in X$, there exists an open neighborhood $\Omega_{\delta,s}$ of x_0 such that

$$(2.8)\quad \|u\|_s \leq \delta\|Pu\|_{s+2-m} + C_{\delta,s}\|u\|_{s-1}, \quad \text{for all } u \in C_0^\infty(\Omega_{\delta,s}; \mathbf{C}^N).$$

After that, we will show that (2.8) implies statement (b) of Theorem 1.2. This second step indeed follows by classical arguments from functional analysis as in the proofs of Theorems 1 and 5 of [**16**]. In any case we will give a detailed proof for the convenience of the reader.

It suffices to prove (2.8), with $m = 2$, for any second order classical and properly supported pseudodifferential system $P \in \mathrm{OP}\, S^2(X; \mathbf{C}^N)$ satisfying the hypotheses of Theorem 1.2. Hence, by Cauchy-Schwarz's inequality, it is enough to prove that

$$(2.9)\quad \|u\|^2_s \leq \delta \mathrm{Re}\,(Pu, u)_s + C'_{\delta,s}\|u\|^2_{s-1}, \quad \text{for all } u \in C_0^\infty(\Omega_{\delta,s}; \mathbf{C}^N),$$

where $(\cdot,\cdot)_s$ denotes the scalar product in $H^s(\mathbf{R}^n; \mathbf{C}^n)$.

Let $\Lambda(\xi) = (1 + |\xi|^2)^{1/2}$ and $\Lambda^s = \Lambda(D)^s \otimes \mathrm{Id}$, $s \in \mathbf{R}$. We have

$$(Pu, u)_s = (\Lambda^s Pu, \Lambda^s u) = ((P + [\Lambda^s, P]\Lambda^{-s})\Lambda^s u, \Lambda^s u).$$

Now the operator $[\Lambda^s, P]\Lambda^{-s} \in \mathrm{OP}\, S^1(X; \mathbf{C}^N)$ has the principal symbol

$$-\frac{i}{2}\{|\xi|^s, p_2(x,\xi)\}|\xi|^{-s}\mathrm{Id},$$

which is conjugate-Hermitian. It follows that

$$\mathrm{Re}\,(Pu, u)_s = \mathrm{Re}\,(P\Lambda^s u, \Lambda^s u) + O(\|u\|^2_s),$$

for smooth u supported in a fixed compact subset of X. Then (2.9) easily follows from (1.4), and (2.8) is proved.

We now observe that (2.8) implies the following a priori estimate:

For every $s \in \mathbf{R}$ and any point $x_0 \in X$, there exist an open neighborhood Ω_s of x_0 and a constant $C > 0$ such that

$$\|u\|_s \leq C\|Pu\|_{s+2-m}, \quad \text{for all } u \in C_0^\infty(\Omega_s; \mathbf{C}^N). \tag{2.10}$$

In fact, suppose that for some $x_0 \in X$ there is no such open neighborhood, namely, there is a sequence $u_j \in C_0^\infty(X; \mathbf{C}^N)$ such that $\|u_j\|_s = 1$, $\|Pu_j\|_{s+2-m} \to 0$ and $\operatorname{supp} u_j \subset B(x_0, 1/j)$. Then a subsequence converges in H^{s-1} to $u \in H^{s-1}(\mathbf{R}^n; \mathbf{C}^N)$, with $Pu = 0$ and $\operatorname{supp} u \subset \{x_0\}$. Moreover, by (2.8) we have $\|u\|_{s-1} \geq C_{\delta,s}^{-1}$. Hence, we have found a nontrivial solution to $Pu = 0$ whose support is reduced to $\{x_0\}$. By Lemma 1 of [**16**, page 469], this is only possible if $p_m(x, \xi)$ vanishes identically at $x = x_0$, but this is not the case by our assumptions (precisely, see the second sentence after (2.4)). This proves (2.10).

Finally we observe that also the formal adjoint P^* satisfies (2.10); namely, one has

$$\|u\|_s \leq C\|P^*u\|_{s+2-m}, \quad \text{for all } u \in C_0^\infty(\Omega_s; \mathbf{C}^N). \tag{2.11}$$

Indeed, P^* satisfies the same hypotheses as P, having the same principal symbol and subprincipal symbol given by the adjoint $p_{m-1}^s(x, \xi)^*$ (as matrix) of that of P.

We can now prove statement (b) of Theorem 1.2. Precisely, given any integer $k \geq 0$, consider the open neighborhood Ω_{-k} for which (2.11) holds with $s = -k$. Possibly after shrinking this open set, we can suppose that it has a smooth boundary, so that every $f \in H^k(\Omega_{-k}; \mathbf{C}^N)$ extends to a function $\tilde{f} \in H^k(\mathbf{R}^n; \mathbf{C}^N)$ (see [**2**, page 339, Theorem A.6]). Then, for such an f we consider the linear functional

$$H^{-k-m+2}(\mathbf{R}^n; \mathbf{C}^N) \supset P^*(C_0^\infty(\Omega_{-k}; \mathbf{C}^N)) \longrightarrow \mathbf{C}$$

given by $P^*v \mapsto (v, f)$. It is well defined in view of (2.11), and also continuous since, by (2.11), with $s = -k$,

$$|(v, f)| = |(v, \tilde{f})| \leq \|v\|_{-k}\|\tilde{f}\|_k \leq C\|P^*v\|_{-k-m+2}\|\tilde{f}\|_k,$$
$$\text{for all } v \in C_0^\infty(\Omega_{-k}; \mathbf{C}^N).$$

It follows by the Hahn-Banach theorem that a $u \in H^{k+m-2}(\mathbf{R}^n; \mathbf{C}^N)$ exists such that

$$(v, f) = (P^* v, u), \quad \text{for all } v \in C_0^\infty(\Omega_{-k}; \mathbf{C}^N),$$

namely, $Pu = f$ in Ω_{-k}. This concludes the proof. $\square$

3. Proof of Theorem 1.1. We have to verify that the hypotheses of Theorem 1.2 are satisfied. To this end, we briefly recall the definition of the Kohn Laplacian $\Box_b^{(q)}$ on a CR manifold of codimension $h \geq 1$; see [**21**] for details.

Let $T^{0,1}M$ be the CR structure of M and $\mathcal{E}^q(M) := \Gamma(M, \Lambda^q M)$, and let $\mathcal{E}^{0,q}(M) := \Gamma(M, \Lambda^{0,q} M)$ be the spaces of q-forms and $(0,q)$-forms respectively, on M. Let $\pi_q : \mathcal{E}^q(M) \to \mathcal{E}^{0,q}(M)$ be the projection and define $\overline{\partial}_b^{(q)} := \pi_{q+1} \circ d : \mathcal{E}^{0,q}(M) \to \mathcal{E}^{0,q+1}(M)$. Fix a Hermitian metric on $\mathbf{C}TM$ such that $T^{0,1}M \perp T^{1,0}M$, and define

$$\Box_b^{(q)} = \overline{\partial}_b^{(q-1)} \overline{\partial}_b^{(q-1)^*} + \overline{\partial}_b^{(q)^*} \overline{\partial}_b^{(q)} : \mathcal{E}^{0,q}(M) \to \mathcal{E}^{0,q}(M),$$

(for $q = 0$, $\Box_b^{(0)} = \overline{\partial}_b^{(0)^*} \overline{\partial}_b^{(0)}$) where the adjoint is taken with respect to the induced inner product. Consider a local basis $\overline{L}_1, \dots, \overline{L}_n$ of sections of $T^{0,1}M$ and real vector fields $T_1, \dots, T_h$ such that $L_1, \dots, L_n, \overline{L}_1, \dots, \overline{L}_n, T_1, \dots, T_h$ is a local orthonormal basis of $\mathbf{C}TM$. We recall that the characteristic bundle $N^*(M) \subset T^*M$ consists of the covectors which are conormal to $T^{1,0}M \oplus T^{0,1}M$.

Let $\omega^1, \dots, \omega^n$, $\overline{\omega}^1, \dots, \overline{\omega}^n$, $\tau_1, \dots, \tau_h$, be the dual basis to $L_1, \dots,$ L_n, $\overline{L}_1, \dots, \overline{L}_n$, $T_1, \dots, T_h$, and let $\phi = \sum_I \phi_I \overline{\omega}^I$ be a $(0,q)$-form, with $I = (i_1, \dots, i_q)$, $1 \leq i_1 < \cdots < i_q \leq n$, and $\overline{\omega}^I = \overline{\omega}^{i_1} \wedge \cdots \wedge \overline{\omega}^{i_q}$. An easy computation shows that the Kohn Laplacian $\Box_b^{(q)}$ reads as

$$\Box_b^{(q)} \phi = -\frac{1}{2} \sum_I \sum_{j=1}^n (L_j \overline{L}_j + \overline{L}_j L_j) \phi_I \overline{\omega}^I + \text{lower order terms.}$$

We therefore have

$$\Sigma = \{\rho \in T^*M \setminus 0 : \ v_j := \sigma_1(L_j)(\rho) = 0, \ j = 1, \dots, n\},$$

which is exactly $N^*(M)$ with the 0-section removed.

Since the vector fields $L_j, \overline{L}_j$, $j = 1, \dots, n$, are linearly independent, it follows that Hypothesis (H_1) is satisfied.

For $\rho \in N^*(M)$ the Levi form $\mathcal{L}(\rho)$ is then defined as the Hermitian matrix whose entries are

$$\mathcal{L}(\rho)_{jk} := i\langle \rho, [L_j, \overline{L}_k]\rangle = \sigma_1([L_j, \overline{L}_k])(\rho) = i\{v_j, \overline{v}_k\}(\rho),\quad j,k = 1,\dots,n.$$

Let $\rho \in \Sigma = N^*(M) \setminus 0$; the map

$$\operatorname{Ker}\mathcal{L}(\rho) \ni \alpha \longmapsto u = \sum_{j=1}^{n} \overline{\alpha}_j H_{v_j} + \alpha_j H_{\overline{v}_j} \in T_\rho\Sigma \cap T_\rho\Sigma^\sigma \tag{3.1}$$

is easily seen to be an isomorphism of real vector spaces, see e.g., the proof of Proposition 3.6 of [**15**]. It follows that

$$\dim(T_\rho\Sigma \cap T_\rho\Sigma^\sigma) = 2\dim_{\mathbf{C}}\operatorname{Ker}\mathcal{L}(\rho), \tag{3.2}$$

and therefore Hypotheses (H_2) and (H_3) are fulfilled as well.

Moreover, we observe that, if $\alpha \neq 0$, the vector u in (3.1) satisfies

$$d\pi(u) = i\sum_{j=1}^{n}(\overline{\alpha}_j L_j - \alpha_j \overline{L}_j) \neq 0,$$

so that (H_4) is verified.

We now come to (1.3). Since $\Box_b^{(q)}$ is formally self-adjoint, its subprincipal symbol p_1^s is indeed a Hermitian matrix. We are going to use the necessity part of the following lower bound.

Theorem 3.1. *Let X be an open subset of $\mathbf{R}^n$, and let $P = P^* \in \mathrm{OP}\,S^m(X;\mathbf{C}^N)$ be an $N\times N$ matrix of classical pseudodifferential operators satisfying (H_1) and (H_2). Then the following properties are equivalent*:

$$p^s_{m-1}(\rho) + \mathrm{Tr}^+ F_{p_m}(\rho)\mathrm{Id} \geq 0, \quad \text{for all } \rho \in \Sigma; \tag{3.3}$$

for every compact subset $K \subset X$, there exists a constant $C_K > 0$ such that

$$(Pu, u) \geq -C_K\|u\|^2_{(m-2)/2}, \quad \text{for all } u \in C_0^\infty(K;\mathbf{C}^n). \tag{3.4}$$

This theorem is a slight generalization of Theorem 22.3.2 and Proposition 22.4.1 of [**10**], to systems with a principal symbol which is a scalar multiple of the identity matrix. We do not repeat the proof here, which goes exactly as the one given in [**10**].

Instead, we observe that the Kohn Laplacian certainly satisfies (3.4) ($m = 2$), because $(\Box_b^{(q)}\phi, \phi) \geq 0$ for every smooth $(0, q)$-form ϕ with compact support. It follows from Theorem 3.1 that (3.2) is verified by $\Box_b^{(q)}$, and this concludes the proof. $\square$

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