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GENERALIZED HILL LEMMA, KAPLANSKY THEOREM FOR COTORSION PAIRS AND SOME APPLICATIONS

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Dedicated to Luigi Salce on his 60th birthday.

ABSTRACT. We generalize Hill's lemma in order to obtain a large family of $\mathcal{C}$-filtered submodules from a single $\mathcal{C}$-filtration of a module. We use this to prove the following generalization of Kaplansky's structure theorem for projective modules: for any ring R, a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in Mod-R is of countable type if and only if every module $M \in \mathcal{A}$ is $\mathcal{A}^{\leq \omega}$-filtered. We also prove rank versions of these results for torsion-free modules over commutative domains.

As an application, we solve a problem of Bazzoni and Salce [**3**] by showing that strongly flat modules over any valuation domain coincide with the extensions of free modules by divisible torsion-free modules. Another application yields a short proof of the structure of Matlis localizations of commutative rings.

1. Introduction. In [**11**], Hill invented an ingenious method of constructing a large family of subgroups from a single infinite continuous chain of abelian p-groups. Later on, Fuchs and Lee extended the method to the general setting of arbitrary modules over arbitrary rings (including a rank version for torsion-free modules over commutative domains), [**7**], [**9**, XVI.Section 8]. Similar constructions were used in connection with Shelah's singular compactness theorem in [**4, 5**].

More recently, Šaroch and the second author [**14**] noticed an extra property of the Hill method, see property (H3) below. In Theorems 6 and 7 of Section 1, we discover an additional feature: the family is always a complete sublattice of the submodule lattice.

Hill's method provides a powerful tool for extending structure theory of various classes of modules from the countable (rank) case to the

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arbitrary one. It is applied either directly or in conjunction with Shelah's singular compactness theorem, see e.g., [**7**], [**9**, XVI.Section 8], [**14**].

Here, we first apply Hill's method to extend a theorem of Kaplansky on projective modules to the setting of cotorsion pairs. Kaplansky's theorem says that any projective module is a direct sum of countably generated modules. Considering the cotorsion pair (Proj-R, Mod-R) cogenerated by R, we can rephrase the theorem by saying that each module $M \in$ Proj-R is (Proj-$R)^{\leq\omega}$-filtered (see below for unexplained terminology).

Theorem 10 in Section 2 shows that the same holds for an arbitrary cotorsion pair $(\mathcal{A}, \mathcal{B})$ in Mod-R cogenerated by a set of $< \kappa$ presented modules (where κ is a regular uncountable cardinal): each module $M \in \mathcal{A}$ is $\mathcal{A}^{<\kappa}$-filtered. We also prove a rank version of this result in Lemma 16.

Section 3 deals with applications to strongly flat modules over valuation domains. Bazzoni and Salce [**3**] proved that any countable rank strongly flat module M has the following property: M contains a free submodule F such that M/F is torsion-free divisible. Theorem 17 shows that the property characterizes arbitrary strongly flat modules. This answers in the positive a question raised in [**3**]. (The property was known to hold by [**3**, Theorem 3.15] for any valuation domain, but restricted to strongly flat modules M of rank $\leq \aleph_1$ and, by [**10**, Theorem 3.3], for all strongly flat modules M, but restricted to Matlis valuation domains.)

The application in Section 4 yields a short proof of the fact that the localization, $Q = RS^{-1}$, of a commutative ring R in a set S of regular elements is a Matlis localization if and only if Q/R decomposes (as an R-module) into a direct sum of countably presented modules. This result, first proved in [**1**], extends Lee's characterization of Matlis domains [**12**] as well as its generalization to localizations of commutative domains by Fuchs and Salce [**8**].

Let R be a (unital associative) ring. Denote by Mod-R the category of all (right R-) modules, and by Proj-R the full subcategory of all projective modules.

A *filtration* is a continuous well-ordered chain of modules $(M_\alpha \mid \alpha \leq \sigma)$ with $M_0 = 0$. A filtration is called a $\mathcal{C}$-*filtration* for a class of modules $\mathcal{C}$ if in addition $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathcal{C}$ for each $\alpha < \sigma$. A module M is *$\mathcal{C}$-filtered* if there is a $\mathcal{C}$-filtration $(M_\alpha \mid \alpha \leq \sigma)$ such that $M = M_\sigma$.

For an infinite cardinal κ and a class of modules $\mathcal{A}$, denote by $\mathcal{A}^{<\kappa}$ and $\mathcal{A}^{\leq\kappa}$ the subclass of all $< \kappa$-presented, and $\leq \kappa$-presented, respectively, modules from $\mathcal{A}$.

A pair of classes of modules $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a *cotorsion pair* provided that $\mathcal{A}$ and $\mathcal{B}$ are orthogonal with respect to the Ext^1-functor, and they are maximal with this property, that is, $\mathcal{A} = \{A \in \text{Mod-}R \mid \mathrm{Ext}^1_R(A, B) = 0 \text{ for all } B \in \mathcal{B}\}$ and $\mathcal{B} = \{B \in \text{Mod-}R \mid \mathrm{Ext}^1_R(A, B) = 0 \text{ for all } A \in \mathcal{A}\}$. Cotorsion pairs were introduced by Salce in his pioneering work [**13**].

A cotorsion pair $\mathfrak{C}$ is *cogenerated* by a class of modules $\mathcal{C}$ provided that $\mathcal{B} = \{B \in \text{Mod-}R \mid \mathrm{Ext}^1_R(C, B) = 0 \text{ for all } C \in \mathcal{C}\}$. Moreover, $\mathfrak{C}$ is of *countable type* if $\mathfrak{C}$ is cogenerated by a set of countably presented modules.

1. Generalized Hill lemma. We start by recalling Hill's notion of a closed subset with respect to a filtration.

Definition 1. Let $\mathcal{M}$ be a filtration $(M_\alpha \mid \alpha \leq \sigma)$ together with a family of modules $(A_\alpha \mid \alpha < \sigma)$ such that $M_{\alpha+1} = M_\alpha + A_\alpha$ for each $\alpha < \sigma$. A subset S of σ is *closed* if every $\beta \in S$ satisfies

$$M_\beta \cap A_\beta \subseteq \sum_{\substack{\alpha \in S \\ \alpha < \beta}} A_\alpha.$$

The *height*, $\mathrm{ht}\,(x)$, of an element $x \in M_\sigma$ is defined as the least ordinal $\alpha < \sigma$ such that $x \in M_{\alpha+1}$. For any subset S of σ, we define

$$M(S) = \sum_{\alpha \in S} A_\alpha.$$

For each ordinal $\alpha \leq \sigma$, we have $M_\alpha = \sum_{\beta<\alpha} A_\beta$, so α $(= \{\beta < \sigma \mid \beta < \alpha\})$ is a closed subset of σ. The following lemma is inspired by the proof of [**14**, Lemma 1.4]:

Lemma 2. *Let $\mathcal{M}$ be as in Definition 1, let S be a closed subset of σ and $x \in M(S)$. Let $S' = \{\alpha \in S \mid \alpha \leq \operatorname{ht}(x)\}$. Then $x \in M(S')$.*

Proof. Let $x \in M(S)$. Then $x = x_1 + \cdots + x_k$ where $x_i \in A_{\alpha_i}$ for some $\alpha_i \in S$, $1 \leq i \leq k$. Without loss of generality, $\alpha_1 < \cdots < \alpha_k$, and α_k is the least possible.

If $\alpha_k > \operatorname{ht}(x)$, then $x_k = x - x_1 - \cdots - x_{k-1} \in M_{\alpha_k} \cap A_{\alpha_k} \subseteq \sum_{\alpha \in S, \alpha < \alpha_k} A_\alpha$ since S is closed, in contradiction with the minimality of α_k. □

As an immediate consequence, we get

Corollary 3. *Let $\mathcal{M}$ be as in Definition 1, let S be a closed subset of σ and $x \in M(S)$. Then $\operatorname{ht}(x) \in S$.*

An important implication is the following lemma.

Lemma 4. *Let $\mathcal{M}$ be as in Definition 1, and let S_i, $i \in I$, be a family of closed subsets of σ. Then*

$$M\left(\bigcap_{i \in I} S_i\right) = \bigcap_{i \in I} M(S_i).$$

Proof. Let $T = \cap_{i \in I} S_i$. Clearly, $M(T) \subseteq \cap_{i \in I} M(S_i)$. Suppose there is an $x \in \cap_{i \in I} M(S_i)$ such that $x \notin M(T)$, and choose such an x of minimal height. Then $x = y + z$ for some $y \in A_{\operatorname{ht}(x)}$ and $z \in M_{\operatorname{ht}(x)}$. By Corollary 3, $\operatorname{ht}(x) \in S_i$ for all $i \in I$, so $\operatorname{ht}(x) \in T$, and $y \in M(T)$. Then $z \in \cap_{i \in I} M(S_i)$, $z \notin M(T)$ and $\operatorname{ht}(z) < \operatorname{ht}(x)$, in contradiction with the minimality. □

Now, we can prove the additional property of closed subsets mentioned in the introduction.

Proposition 5. *Let $\mathcal{M}$ be as in Definition 1, and let S_i, $i \in I$, be a family of closed subsets of σ. Then both the union and the intersection* *of this family are again closed in σ. That is, closed subsets of σ form a complete sublattice of 2^σ.*

Proof. As for the union, if $\beta \in S = \cup_{i\in I} S_i$, then $\beta \in S_i$ for some $i \in I$, and $M_\beta \cap A_\beta \subseteq \sum_{\alpha\in S_i, \alpha<\beta} A_\alpha \subseteq \sum_{\alpha \in S, \alpha<\beta} A_\alpha$.

For the intersection, let $\beta \in T = \cap_{i\in I} S_i$. Then $M_\beta \cap A_\beta \subseteq M(S_i \cap \beta)$ for each $i \in I$. Therefore, Lemma 4 implies that

$$M_\beta \cap A_\beta \subseteq \bigcap_{i\in I} M(S_i \cap \beta) = M(T \cap \beta)$$

which says exactly that T is closed. $\square$

The following is the main result of this section.

Theorem 6 (Generalized Hill lemma). *Let R be a ring, κ an infinite regular cardinal and $\mathcal{C}$ a set of $< \kappa$-presented modules. Let M be a union of a $\mathcal{C}$-filtration*

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_\alpha \subseteq \cdots \subseteq M_\sigma = M$$

for some ordinal σ. Then there is a family $\mathcal{F}$ of submodules of M such that:

(H1) $M_\alpha \in \mathcal{F}$ for all $\alpha \leq \sigma$.

(H2) *$\mathcal{F}$ is closed under arbitrary sums and intersections* (*that is, $\mathcal{F}$ is a complete sublattice of the lattice of submodules of M*).

(H3) *Let $N, P \in \mathcal{F}$ be such that $N \subseteq P$. Then there exists a $\mathcal{C}$-filtration $(\overline{P}_\gamma \mid \gamma \leq \tau)$ of the module $\overline{P} = P/N$ such that $\tau \leq \sigma$, and for each $\gamma < \tau$ there is a $\beta < \sigma$ with $\overline{P}_{\gamma+1}/\overline{P}_\gamma$ isomorphic to $M_{\beta+1}/M_\beta$.*

(H4) *Let $N \in \mathcal{F}$ and X be a subset of M of cardinality $< \kappa$. Then there is a $P \in \mathcal{F}$ such that $N \cup X \subseteq P$ and P/N is $< \kappa$-presented.*

Proof. Let $\mathcal{M}$ denote the filtration $(M_\alpha \mid \alpha \leq \sigma)$ together with an arbitrary family of $< \kappa$-generated modules $(A_\alpha \mid \alpha < \sigma)$ such that, for each $\alpha < \sigma$:

$$M_{\alpha+1} = M_\alpha + A_\alpha,$$

as in Definition 1. We claim that

$$\mathcal{F} = \{M(S) \mid S \text{ a closed subset of } \sigma\}$$

is the desired family $\mathcal{F}$.

Property (H1) is clear, since each ordinal $\alpha \leq \sigma$ is a closed subset of σ. Property (H2) follows by Proposition 5 and Lemma 4.

Property (H3) is proved as in [**14**]: we have $N = M(S)$ and $P = M(T)$ for some closed subsets S, T. Since $S \cup T$ is closed, we can assume that $S \subseteq T$. For each $\beta \leq \sigma$, put

$$F_\beta = N + \sum_{\substack{\alpha \in T \setminus S \\ \alpha < \beta}} A_\alpha = M(S \cup (T \cap \beta)) \quad \text{and} \quad \overline{F}_\beta = F_\beta / N.$$

Clearly, $(\bar{F}_\beta \mid \beta \leq \sigma)$ is a filtration of $\overline{P} = P/N$ such that $\overline{F}_{\beta+1} = \overline{F}_\beta + (A_\beta + N)/N$ for $\beta \in T \setminus S$ and $\overline{F}_{\beta+1} = \overline{F}_\beta$ otherwise. Let $\beta \in T \setminus S$. Then,

$$\overline{F}_{\beta+1}/\overline{F}_\beta \cong F_{\beta+1}/F_\beta \cong A_\beta/(F_\beta \cap A_\beta),$$

and

$$F_\beta \cap A_\beta \supseteq \left(\sum_{\substack{\alpha \in T \\ \alpha < \beta}} A_\alpha \right) \cap A_\beta = M_\beta \cap A_\beta.$$

On the other hand, if $x \in F_\beta \cap A_\beta$ then $\operatorname{ht}(x) \leq \beta$, so $x \in M(T')$ by Lemma 2, where $T' = \{\alpha \in S \cup (T \cap \beta) \mid \alpha \leq \beta\}$. By Proposition 5, we get $x \in M_\beta$ because $\beta \notin S$. Hence, $F_\beta \cap A_\beta = M_\beta \cap A_\beta$ and $\overline{F}_{\beta+1}/\overline{F}_\beta \cong A_\beta/(M_\beta \cap A_\beta) \cong M_{\beta+1}/M_\beta$. The filtration $(\overline{P}_\gamma \mid \gamma \leq \tau)$ is obtained from $(\overline{F}_\beta \mid \beta \leq \sigma)$ by removing possible repetitions and (H3) follows. Denote by τ' the ordinal type of the well-ordered set $(T \setminus S, <)$. Notice that the length τ of the filtration can be taken as $1 + \tau'$ (the ordinal sum, hence $\tau = \tau'$ for τ' infinite).

For property (H4), we first prove that every subset of σ of cardinality $< \kappa$ is contained in a closed subset of cardinality $< \kappa$. Because κ is an infinite regular cardinal, by Proposition 5, it is enough to prove this only for one-element subsets of σ. That is, to prove that every $\beta < \sigma$ is contained in a closed subset of cardinality $< \kappa$. We induct on β. For $\beta < \kappa$, just take $S = \beta + 1$. Otherwise, the short exact sequence

$$0 \longrightarrow M_\beta \cap A_\beta \longrightarrow A_\beta \longrightarrow M_{\beta+1}/M_\beta \longrightarrow 0$$

shows that $M_\beta \cap A_\beta$ is $< \kappa$ generated. Thus, $M_\beta \cap A_\beta \subseteq \sum_{\alpha \in S_0} A_\alpha$ for a subset $S_0 \subseteq \beta$ of cardinality $< \kappa$. Moreover, we can assume that S_0 is closed in σ by inductive premise, and put $S = S_0 \cup \{\beta\}$. To show that S is closed, it suffices to check the definition for β. But $M_\beta \cap A_\beta \subseteq M(S_0) = \sum_{\alpha \in S, \alpha < \beta} A_\alpha$.

Finally, let $N = M(S)$ where S is closed in σ, and let X be a subset of M of cardinality $< \kappa$. Then $X \subseteq \sum_{\alpha \in T} A_\alpha$ for a subset T of σ of cardinality $< \kappa$. By the preceding paragraph, we can assume that T is closed in σ. Let $P = M(S \cup T)$. Then P/N is $\mathcal{C}$-filtered by property (H3), and the filtration can be chosen indexed by $1+$ the ordinal type of $T \setminus S$, which is less than κ. In particular, P/N is $< \kappa$-presented. □

Remark (cf. [**7**, Remark 2.2]). The proof of property (H3) for the family $\mathcal{F}$ in Theorem 6 yields the following additional property: if for $\beta \in T \setminus S$, A_β can be chosen as a complement to M_β in $M_{\beta+1}$, then $(A_\beta + N)/N$ will be a complement of $\overline{P}_\gamma$ in $\overline{P}_{\gamma+1}$ in the filtration of $\overline{P}$. This follows from the fact that in this case (in the proof of (H3)) $F_\beta \cap A_\beta = M_\beta \cap A_\beta = 0$, so $F_\beta/N \cap (A_\beta + N)/N = \bar{0}$.

We will also need a rank version of the generalized Hill lemma for torsion-free modules over commutative domains.

Let R be a commutative domain and M a torsion-free module. We define the *rank*, $\operatorname{rk} X$, of a subset $X \subseteq M$ as the torsion-free rank of the submodule $\langle X \rangle$ of M generated by X. Note that $\operatorname{rk} X \leq \operatorname{card}(X)$.

Theorem 7 (Rank version of the generalized Hill lemma). *Let R be a commutative domain, κ an infinite regular cardinal and $\mathcal{C}$ a set of torsion-free R-modules. Let M be a union of a $\mathcal{C}$-filtration*

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_\alpha \subseteq \cdots \subseteq M_\sigma = M$$

for some ordinal σ. Assume, moreover, that for each $\alpha < \sigma$ there is a submodule A_α of M of rank $< \kappa$ such that $M_{\alpha+1} = M_\alpha + A_\alpha$. Then there is a family $\mathcal{F}$ of submodules of M such that the properties (H1), (H2) *and* (H3) *from Theorem* 6 *hold true. Moreover, the following rank version of property* (H4) *holds*:

(H4*) *Let $N \in \mathcal{F}$ and X be a subset of M with $\operatorname{rk} X < \kappa$. Then there are $P \in \mathcal{F}$ and a submodule $A \subseteq M$ of rank $< \kappa$ such that $N \cup X \subseteq P$ and $P = N + A$.*

Proof. Denote by $\mathcal{M}$ the filtration $(M_\alpha \mid \alpha \leq \sigma)$ together with the family $(A_\alpha \mid \alpha < \sigma)$ as in Definition 1. Put

$$\mathcal{F} = \{M(S) \mid S \text{ a closed subset of } \sigma\}.$$

The properties (H1), (H2) and (H3) are proved exactly as in Theorem 6. For (H4*), consider $N \in \mathcal{F}$ and $X \subseteq M$ with $\operatorname{rk} X < \kappa$. Note first that we can without loss of generality assume that the cardinality of X is $< \kappa$. To see this, take a maximal R-independent subset B of $\langle X \rangle$. Then B has cardinality $< \kappa$ and $\langle B \rangle$ is an essential submodule of $\langle X \rangle$. Then, for a module $P \in \mathcal{F}$ containing B, the inclusion $\langle X \rangle \hookrightarrow M$ induces a map $f : \langle X \rangle / \langle B \rangle \to M/P$. Then $f = 0$ since $\langle X \rangle / \langle B \rangle$ is torsion, but M/P is torsion-free by property (H3). Hence, also, $X \subseteq P$.

Now, we continue as in the proof of property (H4) in Theorem 6. We prove that every subset of σ of cardinality $< \kappa$ is contained in a closed subset of cardinality $< \kappa$. It is again enough to prove that every $\beta < \sigma$ is contained in a closed subset T of cardinality $< \kappa$. We induct on β. For $\beta < \kappa$, we take $T = \beta + 1$. Otherwise, $A_\beta \cap M_\beta$ has rank $< \kappa$, so we can find by inductive premise a closed subset $T' \subseteq \beta$ of cardinality $< \kappa$ such that $M_\beta \cap A_\beta \subseteq \mathcal{M}(T')$. Then it suffices to take $T = T' \cup \{\beta\}$.

Finally, if $N = M(S)$ and $X \subseteq M(T)$ where S, T are closed and T is of cardinality $< \kappa$, we put $A = M(T \setminus S)$ and $P = N + A$. Clearly, $P = M(S \cup T)$ and A satisfy the claim of (H4*). □

Remark 8. Notice the following difference between the assumptions of the two versions of the generalized Hill lemma. The assumption of $\mathcal{C}$ consisting of $< \kappa$-presented modules in Theorem 6 already guarantees existence of a family of $< \kappa$-generated modules $\mathcal{A} = (A_\alpha \mid \alpha < \sigma)$ such that $M_{\alpha+1} = M_\alpha + A_\alpha$ for each $\alpha < \sigma$ (in fact, in the proof of Theorem 6, and in its applications, the particular choice of $\mathcal{A}$ does not really matter).

On the other hand, if we just assume that $M_{\alpha+1}/M_\alpha$ has rank $< \kappa$ for each $\alpha < \sigma$ in Theorem 7, there need not exist any family of modules $\mathcal{A} = (A_\alpha \mid \alpha < \sigma)$ such that $M_{\alpha+1} = M_\alpha + A_\alpha$ and A_α has rank $< \kappa$ for each $\alpha < \sigma$.

Indeed, assume that $\kappa > \aleph_0$ and the minimal number of R-generators of Q is $\lambda \geq \kappa$. So there is an exact sequence $0 \to K \subseteq F \to Q \to 0$ where F is free of rank λ. Since K is torsion-free, there is a filtration $(M_\alpha \mid \alpha \leq \sigma)$ of K such that $M_{\alpha+1}/M_\alpha$ is torsion-free of rank 1 for each $\alpha < \sigma$. Define $M_{\sigma+1} = F$.

Assume that $A_\sigma \subseteq F$ has rank $< \lambda$. Then A_σ is contained in a free direct summand G of F of rank $< \lambda$, so $(A_\sigma + K)/K \subseteq (G+K)/K \subsetneqq F/K$ because $Q \cong F/K$ is not $< \lambda$-generated. So certainly there is no A_σ of rank $< \kappa$ such that $M_{\sigma+1} = M_\sigma + A_\sigma$.

2. Kaplansky theorem for cotorsion pairs. Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ a cotorsion pair in Mod-R cogenerated by a set $\mathcal{C}$ containing R. Then $\mathcal{A}$ coincides with the class of all direct summands of $\mathcal{C}$-filtered modules, cf. [**16**, Theorem 2.2]. Our goal is to remove the term 'direct summands' in this characterization of $\mathcal{A}$ on account of replacing the set $\mathcal{C}$ by a suitable small subset of $\mathcal{A}$.

The following application of Theorem 6 is crucial:

Lemma 9. *Let κ be an uncountable regular cardinal and $\mathcal{C}$ a set of $< \kappa$-presented modules. Denote by $\mathcal{A}$ the class of all direct summands of $\mathcal{C}$-filtered modules. Then every module in $\mathcal{A}$ is $\mathcal{A}^{<\kappa}$-filtered.*

Proof. Let $K \in \mathcal{A}$, so there is a $\mathcal{C}$-filtered module M such that $M = K \oplus L$ for some $L \subseteq M$. Denote by $\pi_K : M \to K$ and $\pi_L : M \to L$ the corresponding projections. Let $\mathcal{F}$ be the family of submodules of M as in Theorem 6. We proceed in two steps:

Step I. By induction, we construct a filtration $(N_\alpha \mid \alpha \leq \tau)$ of M such that

(1) $N_\alpha \in \mathcal{F}$,

(2) $N_\alpha = \pi_K(N_\alpha) + \pi_L(N_\alpha)$, and

(3) $N_{\alpha+1}/N_\alpha$ is $< \kappa$-presented for all $\alpha < \tau$.

First, $N_0 = 0$, and $N_\beta = \cup_{\alpha<\beta} N_\alpha$ for all limit ordinals $\beta \leq \tau$. Suppose we have $N_\alpha \subsetneqq M$ and we wish to construct $N_{\alpha+1}$. Take $x \in M \setminus N_\alpha$; by property (H4), there is a $Q_0 \in \mathcal{F}$ such that $N_\alpha \cup \{x\} \subseteq Q_0$ and Q_0/N_α is $< \kappa$-presented. Let X_0 be a subset of Q_0 of cardinality $< \kappa$ such that the set $\{x + N_\alpha \mid x \in X_0\}$ generates Q_0/N_α. Put $Z_0 = \pi_K(Q_0) \oplus \pi_L(Q_0)$. Clearly, $Q_0/N_\alpha \subseteq Z_0/N_\alpha$. Since $\pi_K(N_\alpha), \pi_L(N_\alpha) \subseteq N_\alpha$, the module Z_0/N_α is generated by the set

$$\{x + N_\alpha \mid x \in \pi_K(X_0) \cup \pi_L(X_0)\}.$$

Thus, we can find $Q_1 \in \mathcal{F}$ such that $Z_0 \subseteq Q_1$ and Q_1/N_α is $< \kappa$-presented. Similarly, we infer that Z_1/N_α is $< \kappa$-generated for $Z_1 = \pi_K(Q_1) \oplus \pi_L(Q_1)$ and find $Q_2 \in \mathcal{F}$ with $Z_1 \subseteq Q_2$ and Q_2/N_α a $< \kappa$-presented module. In this way, we obtain a chain $Q_0 \subseteq Q_1 \subseteq \cdots$ such that for all $i < \omega$: $Q_i \in \mathcal{F}$, Q_i/N_α is $< \kappa$-presented, and $\pi_K(Q_i) + \pi_L(Q_i) \subseteq Q_{i+1}$. It is easy to see that $N_{\alpha+1} = \cup_{i<\omega} Q_i$ satisfies properties (1)–(3).

Step II. By condition (2), we have

$$\pi_K(N_{\alpha+1}) + N_\alpha = \pi_K(N_{\alpha+1}) \oplus \pi_L(N_\alpha)$$

and similarly for L. Hence,

$$\begin{aligned}(\pi_K(N_{\alpha+1}) + N_\alpha) &\cap (\pi_L(N_{\alpha+1}) + N_\alpha)\\ &= (\pi_K(N_{\alpha+1}) \oplus \pi_L(N_\alpha)) \cap (\pi_L(N_{\alpha+1}) \oplus \pi_K(N_\alpha))\\ &= \big(\pi_K(N_{\alpha+1}) \cap (\pi_L(N_{\alpha+1}) \oplus \pi_K(N_\alpha))\big) \oplus \pi_L(N_\alpha)\\ &= \pi_K(N_\alpha) \oplus \pi_L(N_\alpha) = N_\alpha\end{aligned}$$

and

$$N_{\alpha+1}/N_\alpha = (\pi_K(N_{\alpha+1}) + N_\alpha)/N_\alpha \oplus (\pi_L(N_{\alpha+1}) + N_\alpha)/N_\alpha.$$

By condition (1), $N_{\alpha+1}/N_\alpha$ is $\mathcal{C}$-filtered. Since

$$(\pi_K(N_{\alpha+1}) + N_\alpha)/N_\alpha \cong \pi_K(N_{\alpha+1})/\pi_K(N_\alpha),$$

$\pi_K(N_{\alpha+1})/\pi_K(N_\alpha)$ is isomorphic to a direct summand of a $\mathcal{C}$-filtered module, so $\pi_K(N_{\alpha+1})/\pi_K(N_\alpha) \in \mathcal{A}$. By condition (3), $\pi_K(N_{\alpha+1})/\pi_K(N_\alpha)$ is $< \kappa$-presented. We conclude that $(\pi_K(N_{\alpha+1}) \mid \alpha \leq \tau)$ is the desired $\mathcal{A}^{<\kappa}$-filtration of $K = \pi_K(N_\tau)$. □

Now, we can easily prove the main result of this section:

Theorem 10. *Let R be a ring, κ an uncountable regular cardinal, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ a cotorsion pair of R-modules. Then the following statements are equivalent*:

(1) *$\mathfrak{C}$ is cogenerated by a class of $< \kappa$-presented modules.*

(2) *Every module in $\mathcal{A}$ is $\mathcal{A}^{<\kappa}$-filtered.*

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{C}$ be a class of $< \kappa$-presented modules cogenerating $\mathfrak{C}$. Without loss of generality, $\mathcal{C}$ is a set, and $R \in \mathcal{C}$. Then, by [**16**, Theorem 2.2], $\mathcal{A}$ consists of all direct summands of $\mathcal{C}$-filtered modules. So statement (2) follows by Lemma 9.

(2) $\Rightarrow$ (1). It is well known that every $\mathcal{A}$-filtered module is again in $\mathcal{A}$, see e.g., [**6**, Lemma 1]. Thus, (2) implies that $\mathfrak{C}$ is cogenerated by the class $\mathcal{A}^{<\kappa}$. □

In particular, for $\kappa = \aleph_1$, we get

Corollary 11. *Let R be a ring. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is of countable type if and only if every module $M \in \mathcal{A}$ is $\mathcal{A}^{\leq\omega}$-filtered.*

As another immediate corollary, for the cotorsion pair (Proj-R, Mod-R) cogenerated by R, we obtain the Kaplansky theorem on the structure of projective modules:

Corollary 12. *Every projective module over an arbitrary ring is a direct sum of countably generated projective modules.*

Remark. In general, it is not possible to extend the results in this section to $\kappa = \aleph_0$, since there are rings which admit countably generated projective modules that are not direct sums of finitely generated projective modules.

3. Strongly flat modules. In this section, R is a commutative domain with the quotient field Q. We denote by $(\mathcal{SF}, \mathcal{MC})$ the cotorsion pair in Mod-R cogenerated by Q. The modules in $\mathcal{SF}$ are called *strongly flat*. They are flat (since Q is flat), hence torsion-free.

A (torsion-free) module M is called *free-by-divisible* provided there exist cardinals κ, λ and an exact sequence $0 \to R^{(\kappa)} \to M \to Q^{(\lambda)} \to 0$. In [**16**, Proposition 2.8], strongly flat modules were characterized as the direct summands of free-by-divisible modules. Our goal is to remove the term 'direct summand' in this characterization in the case when R is a valuation domain.

First, we need a characterization of free-by-divisible modules:

Lemma 13. *Let R be a domain and M a module. Then M is free-by-divisible if and only if M is $\{R, Q\}$-filtered.*

Proof. The only if-part is clear. For the if-part, let $(M_\alpha \mid \alpha \leq \sigma)$ be an $\{R, Q\}$-filtration of M.

By induction on $\alpha \leq \sigma$, we define ordinals μ_α and ν_α, and a well-ordered direct system of exact sequences $0 \to R^{(\mu_\alpha)} \stackrel{i_\alpha}{\to} M_\alpha \stackrel{\pi_\alpha}{\to} Q^{(\nu_\alpha)} \to 0$ and embeddings $(f_\alpha, g_\alpha, h_\alpha)$ $(\alpha \leq \sigma)$, as follows. First, $\mu_0 = \nu_0 = 0$.

If $M_{\alpha+1}/M_\alpha \cong R$, then $M_{\alpha+1} = M_\alpha \oplus x_\alpha R$ where $\operatorname{Ann}_R(x_\alpha) = 0$, and we take $\mu_{\alpha+1} = \mu_\alpha + 1$, $\nu_{\alpha+1} = \nu_\alpha$, letting $f_\alpha : R^{(\mu_\alpha)} \hookrightarrow R^{(\mu_{\alpha+1})}$ and $g_\alpha : M_\alpha \hookrightarrow M_{\alpha+1}$ be the inclusions, $i_{\alpha+1}$ the extension of i_α mapping the extra free generator to x_α, and putting $h_\alpha = \mathrm{id}_{Q^{(\nu_\alpha)}}$.

If $M_{\alpha+1}/M_\alpha \cong Q$, we consider the pushout of the embedding $g_\alpha : M_\alpha \hookrightarrow M_{\alpha+1}$ and of π_α (see the following commutative diagram).

Since $\operatorname{Ext}^1_R(Q, Q^{(\nu_\alpha)}) = 0$, the righthand column splits, so without loss of generality $X = Q^{(\nu_\alpha+1)}$, and we take $\mu_{\alpha+1} = \mu_\alpha$, $\nu_{\alpha+1} = \nu_\alpha + 1$.

If α is a limit ordinal, we take the direct limit of the direct system of exact sequences $0 \to R^{(\mu_\beta)} \stackrel{i_\beta}{\to} M_\beta \to Q^{(\nu_\beta)} \to 0$ with the embeddings $(f_\beta, g_\beta, h_\beta)$, $\beta < \alpha$, so $\mu_\alpha = \sup_{\beta<\alpha} \mu_\beta$ and $\nu_\alpha = \sup_{\beta<\alpha} \nu_\beta$.

Finally, the sequence $0 \to R^{(\mu_\sigma)} \stackrel{i_\sigma}{\to} M_\sigma \stackrel{\pi_\sigma}{\to} Q^{(\nu_\sigma)} \to 0$ shows that $M = M_\sigma$ is free-by-divisible. $\square$

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & R^{(\mu_\alpha)} \arrow[r, "i_\alpha"] \arrow[d, equal, "f_\alpha"] & M_\alpha \arrow[r, "\pi_\alpha"] \arrow[d, "g_\alpha"] & Q^{(\nu_\alpha)} \arrow[r] \arrow[d, "h_\alpha"] & 0 \\
0 \arrow[r] & R^{(\mu_\alpha)} \arrow[r, "i_{\alpha+1}"] & M_{\alpha+1} \arrow[r, "\pi_{\alpha+1}"] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
 & & Q \arrow[r, equal] \arrow[d] & Q \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Lemma 13 does not guarantee the validity of the rank version assumptions of the generalized Hill lemma, see Remark 8. However, in our particular setting, we have:

Lemma 14. *Let R be a valuation domain, and let P be a free-by-divisible module. Then there are an $\{R, Q\}$-filtration $\mathcal{P} = (P_\alpha \mid \alpha \leq \sigma)$ of P and a sequence of submodules $(A_\alpha \mid \alpha < \sigma)$ of P, such that A_α has countable rank and $P_{\alpha+1} = P_\alpha + A_\alpha$ for each $\alpha < \sigma$.*

Proof. We will prove the lemma in three steps:

Step I. By assumption, there is an exact sequence $0 \to R^{(\kappa)} \overset{\subseteq}{\to} P \to Q^{(\lambda)} \to 0$ for some cardinals κ and λ. We put $\sigma = \kappa + \lambda$ (the ordinal sum). By induction on α, we will construct the sequence $(A_\alpha \mid \alpha < \sigma)$ together with the filtration $\mathcal{P}$, the latter simply by taking $P_\alpha = \sum_{\beta<\alpha} A_\beta$. This is easy in the cases where $\kappa = 0$ or $\lambda = 0$, so we will assume that $\kappa > 0$ and $\lambda > 0$.

For $\alpha < \kappa$, we take A_α as the αth copy of R in the canonical direct sum decomposition of $R^{(\kappa)}$. For $\alpha \geq \kappa$, we need some preparation first.

Step II. Take any submodule $R^{(\kappa)} \subseteq N \subseteq P$ such that $N/R^{(\kappa)} \cong Q$. We claim that there is a countable rank submodule $A \subseteq N$ such that

$R^{(\kappa)} + A = N$. Consider the pushout of the inclusions $i : R^{(\kappa)} \hookrightarrow N$ and $j : R^{(\kappa)} \hookrightarrow Q^{(\kappa)}$:

$$\begin{tikzcd} 0 \arrow[r] & R^{(\kappa)} \arrow[r, "i"] \arrow[d, "j"] & N \arrow[r, "p \restriction N"] \arrow[d, "\subseteq"] & Q \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Q^{(\kappa)} \arrow[r, "\subseteq"] & X \arrow[r, "p"] & Q \arrow[r] & 0 \end{tikzcd}$$

Since $\operatorname{Ext}^1_R(Q, Q^{(\kappa)}) = 0$, the second row splits. Let $k : Q \to X$ be the splitting monomorphism with $pk = \mathrm{id}_Q$. Let $Y = \operatorname{Im}(k)$. Then $X = Q^{(\kappa)} \oplus Y$.

If Q is countably generated, we take any countable subset S of N such that $R^{(\kappa)} + \langle S \rangle = N$ and put $A = \langle S \rangle$.

If Q is not countably generated, then (since R is a valuation domain) there are a regular uncountable cardinal ρ and a set $\{r_\gamma \mid \gamma < \rho\} \subseteq R$ with the following two properties:

(1) $\{r_\gamma^{-1} \mid \gamma < \rho\}$ generates Q as an R-module, and

(2) r_γ is divisible by r_δ, but r_γ does not divide r_δ, for each $\delta < \gamma$.

That is, $(r_\gamma R \mid \gamma < \rho)$ is a strictly descending chain of principal right ideals with zero intersection.

For each $\gamma < \rho$, let $n_\gamma \in N \subseteq X$ be such that $p(n_\gamma) = r_\gamma^{-1}$. Then $n_\gamma \in X$ decomposes as $n_\gamma = q_\gamma + k(r_\gamma^{-1})$ where $q_\gamma \in Q^{(\kappa)} = \operatorname{Ker}(p)$. By property (1), $R^{(\kappa)} + \langle n_\gamma \mid \gamma < \rho \rangle = N$. Since $R^{(\kappa)} \subseteq \operatorname{Ker}(p \restriction N)$, we can without loss of generality assume that

$(*)$ all the (finitely many) nonzero components of q_γ in the direct sum $Q^{(\kappa)}$ belong to $Q \setminus R$.

Denote by I_γ ($\subseteq \kappa$) the support of q_γ. By property (2), for each $\delta < \gamma$, there is a (noninvertible) element $r_{\gamma\delta} \in R$ such that $r_{\gamma\delta} \cdot r_\gamma^{-1} = r_\delta^{-1}$, and hence $r_{\gamma\delta} q_\gamma - q_\delta \in \operatorname{Ker}(p \restriction N) = R^{(\kappa)}$. By $(*)$, it follows that $I_\delta \subseteq I_\gamma$.

We claim that there is a finite set $I \subseteq \kappa$ such that $I_\gamma \subseteq I$ for all $\gamma < \rho$. If not, there is a countably infinite set $\{x_n \mid n < \omega\} \subseteq \kappa$ such that for each $n < \omega$ there is a $\gamma_n < \rho$ with $x_n \in I_{\gamma_n}$. Since ρ is regular and uncountable, there exists a $\gamma < \rho$ such that $\gamma_n < \gamma$ for all $n < \omega$. But then $I_\gamma \supseteq \cup_{n<\omega} I_{\gamma_n} \supseteq \{x_n \mid n < \omega\}$ is infinite, a contradiction.

This proves that $n_\gamma \in Q^{(I)} \oplus Y$ for each $\gamma < \rho$. Let $A = \langle n_\gamma \mid \gamma < \rho \rangle$. Then A is a submodule of N of finite rank, and $R^{(\kappa)} + A = N$.

Step III. We enumerate the copies of Q in $Q^{(\lambda)} = P/R^{(\kappa)}$ by ordinals $< \lambda$. Then for each $\tau < \lambda$, there is a unique module N_τ such that $R^{(\kappa)} \subseteq N_\tau \subseteq P$ and $N_\tau / R^{(\kappa)}$ is the τth copy of Q in $P/R^{(\kappa)}$. The modules $A_{\kappa+\tau}$ $(\tau < \lambda)$ are defined by induction on $\tau < \lambda$ as follows.

First, for $\tau = 0$, we take $N = N_0$, construct A as in Step II for this choice of N, and put $A_\kappa = A$. Then $R^{(\kappa)} + A_\kappa = N_0$.

If $\alpha = \kappa + \tau$ for an ordinal $0 < \tau < \lambda$ then, by induction, we already have an exact sequence $0 \to R^{(\kappa)} \to P_\alpha \to Q^{(\tau)} \to 0$ where $P_\alpha = \sum_{\beta<\alpha} A_\beta$. Moreover, $N_\tau \cap P_\alpha = R^{(\kappa)}$. We take $N = N_\tau$, construct A as in Step II for this choice of N, and put $A_\alpha = A$. Then $R^{(\kappa)} + A_\alpha = N_\tau$, and $P_{\alpha+1} = P_\alpha + N_\tau$. So $P_{\alpha+1}/P_\alpha \cong N/(N \cap P_\alpha) = N/R^{(\kappa)} \cong Q$, and we have the exact sequence $0 \to R^{(\kappa)} \to P_{\alpha+1} \to Q^{(\tau+1)} \to 0$.

Finally, by construction, $P = \cup_{\alpha<\sigma} P_\alpha$. $\square$

The following result was proved in [**3**, Theorem 3.13]:

Lemma 15. *Let R be a valuation domain and M a module of countable rank. Then M is strongly flat if and only if M is free-by-divisible.*

Before characterizing strongly flat modules of any rank over valuation domains, we will apply the rank version of the generalized Hill lemma in order to obtain a rank version of Lemma 9:

Lemma 16. *Let R be a commutative domain, κ an uncountable regular cardinal and $\mathcal{C}$ a set of torsion-free R-modules. Denote by $\mathcal{A}$ the class of all direct summands of the modules M satisfying*:

($\dagger$) *there is a $\mathcal{C}$-filtration $(M_\alpha \mid \alpha \leq \sigma)$ of M and a family of modules $(A_\alpha \mid \alpha < \sigma)$ of rank $< \kappa$ such that $M_{\alpha+1} = M_\alpha + A_\alpha$ for each $\alpha < \sigma$.*

Then every module in $\mathcal{A}$ is filtered by modules from $\mathcal{A}$ of rank $< \kappa$.

Proof. The proof is very similar to the one for Lemma 9. Let $K \in \mathcal{A}$; that is, there is a module $M = K \oplus L$ with a $\mathcal{C}$-filtration $(M_\alpha \mid \alpha \leq \sigma)$ and a family of modules $(A_\alpha \mid \alpha < \sigma)$ as above. Denote by $\pi_K : M \to K$ and $\pi_L : M \to L$ the projections.

Let $\mathcal{F}$ be a family of submodules of M given by Theorem 7. By induction, we will construct a filtration $(N_\alpha \mid \alpha \leq \tau)$ of M such that

(1) $N_\alpha \in \mathcal{F}$,

(2) $N_\alpha = \pi_K(N_\alpha) + \pi_L(N_\alpha)$, and

(3) $N_{\alpha+1}/N_\alpha$ has rank $< \kappa$

for all $\alpha < \tau$; the rest of the proof then follows as in Step II of Lemma 9.

By definition, $N_0 = 0$ and $N_\beta = \cup_{\alpha<\beta} N_\alpha$ for limit ordinals β. Suppose we have constructed $N_\alpha \subsetneq M$ for some α, and let $x \in M \setminus N_\alpha$. Let $A_0 \subseteq M$ be a submodule of rank $< \kappa$ such that $A_0 \in \mathcal{F}$ and $x \in A_0$. Then the module $\pi_K(A_0) + \pi_L(A_0)$ also has rank $< \kappa$, so there is a module $A_1 \in \mathcal{F}$ of rank $< \kappa$ such that $\pi_K(A_0) + \pi_L(A_0) \subseteq A_1$. Iterating this process, we obtain a chain

$$x \in A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$$

of submodules of M with rank $< \kappa$ such that $\pi_K(A_i) + \pi_L(A_i) \subseteq A_{i+1}$ for $i < \omega$. Put $A = \cup_{i<\omega} A_i$. Then clearly A has rank $< \kappa$ and $A = \pi_K(A) + \pi_L(A)$. Hence, $N_{\alpha+1} = N_\alpha + A$ has the required properties. □

Now, we can extend Lemma 15 to modules of arbitrary rank, giving a positive answer to the problem of Bazzoni and Salce.

Theorem 17. *Let R be a valuation ring and M a module. Then M is strongly flat if and only if M is free-by-divisible.*

Proof. The if-part is clear since both R and Q are strongly flat, and strongly flat modules are closed under arbitrary direct sums and extensions.

For the only-if part, let M be strongly flat. By [**16**, Proposition 2.8], M is a direct summand in a free-by-divisible module P. By Lemma 14, strongly flat modules form a class $\mathcal{A}$ as in Lemma 16 for $\mathcal{C} = \{R, Q\}$ and

$\kappa = \aleph_1$. Thus, M is filtered by countable rank strongly flat modules. But such modules are $\{R, Q\}$-filtered by Lemma 15. Hence, M is free-by-divisible by Lemma 13. □

4. Matlis localizations of commutative rings. In this section, R denotes a commutative ring, S a multiplicative subset in R consisting of regular elements (that is, not zero divisors) and Q the localization RS^{-1}.

Q is a *Matlis localization* provided that Q has projective dimension ≤ 1 (as an R-module). For example, if R is a domain and $S = R \setminus \{0\}$, then the quotient field $Q = RS^{-1}$ is a Matlis localization if and only if R is a Matlis domain in the sense of [**9**, IV.Section 4].

Our goal here is to apply the generalized Hill lemma to a simple proof of a characterization of Matlis localizations given in [**1**].

We will first need some preliminary definitions and results. We start with Hamsher's notion of a restriction, and Griffith's of a $G(\aleph_0)$-family:

A submodule N of a module M is a *restriction* if, for each prime (equivalently, maximal) ideal p of R, the localization N_p of N at p satisfies $N_p = 0$ or $N_p = M_p$.

A family $\mathcal{S}$ of submodules of a module M is a $G(\aleph_0)$*-family* provided that $0, M \in \mathcal{S}$, $\mathcal{S}$ is closed under unions of chains, and if $N \in \mathcal{S}$ and X is a countable subset of M then there exists an $N' \in \mathcal{S}$ such that $N \cup X \subseteq N'$ and N'/N are countably generated.

Lemma 18. *Let R be a commutative ring, S a multiplicative subset in R consisting of regular elements and $Q = RS^{-1}$.*

(1) *The set $\mathcal{S}$ of all restrictions of the R-module Q/R is a $G(\aleph_0)$-family of submodules of Q/R.*

(2) *If N is a restriction of Q/R such that $(Q/R)/N$ has projective dimension ≤ 1, then N is a direct summand in Q/R.*

Proof. (1) is proved in [**1**, page 543] and (2) in [**1**, Proposition 3.10]. □

Another ingredient is the notion of an (infinitely generated) tilting module. Recall that an R-module T is *tilting* provided that

(T1) T has projective dimension ≤ 1,

(T2) $\mathrm{Ext}^1_R(T, T^{(\kappa)}) = 0$ for all cardinals κ, and

(T3) there is an exact sequence $0 \to R \to T_0 \to T_1 \to 0$ such that T_0 and T_1 are direct summands in (possibly infinite) direct sums of copies of T.

We arrive at the main result of this section:

Theorem 19 [**1**, Theorem 1.1]. *Let R be a commutative ring, S a multiplicative subset in R consisting of regular elements and $Q = RS^{-1}$. Then the following conditions are equivalent*:

(1) *Q is a Matlis localization.*

(2) *$T = Q \oplus Q/R$ is a tilting R-module.*

(3) *Q/R decomposes into a direct sum of countably presented R-submodules.*

Proof. Assume (1). We will verify conditions (T1)–(T3) for T. First, the projective dimension of Q, Q/R, and hence of T, is ≤ 1 by the assumption, so (T1) holds. (T3) holds since there is the exact sequence $0 \to R \to Q \to Q/R \to 0$. In order to prove (T2), in view of (T1), it suffices to show that $\mathrm{Ext}^1_R(Q/R, Q^{(\kappa)}) = 0$ for each cardinal κ. However, $\mathrm{Ext}^1_R(Q, Q^{(\kappa)}) \cong \mathrm{Ext}^1_Q(Q, Q^{(\kappa)}) = 0$ since Q is a localization of R. So in order to prove that $\mathrm{Ext}^1_R(Q/R, Q^{(\kappa)}) = 0$, it suffices to show that any $f \in \mathrm{Hom}_R(R, Q^{(\kappa)})$ extends to some $g \in \mathrm{Hom}_R(Q, Q^{(\kappa)}) = \mathrm{Hom}_Q(Q, Q^{(\kappa)})$. But we can simply define $g(q) = f(1)q$ for all $q \in Q$.

Assume (2). Consider the cotorsion pair $(\mathcal{A}, \mathcal{B})$ cogenerated by T. By [**2**, Theorem 15], each module in $\mathcal{A}$ is $\mathcal{A}^{\leq \omega}$-filtered. In particular, this holds for $Q/R \in \mathcal{A}$. Let $\mathcal{F}$ be a family corresponding to a $\mathcal{A}^{\leq \omega}$-filtration of Q/R by Theorem 6 (for $\kappa = \aleph_1$). Let $\mathcal{G} = \mathcal{F} \cap \mathcal{S}$ where $\mathcal{S}$ is the $G(\aleph_0)$-family of restrictions of Q/R coming from Lemma 18 (1).

We claim that there is a filtration $(G_\alpha \mid \alpha \leq \sigma)$ of Q/R such that $G_\alpha \in \mathcal{G}$ for all $\alpha \leq \sigma$ and $G_{\alpha+1}/G_\alpha$ is countably presented. Indeed, let $G_0 = 0$ and $G_\alpha = \cup_{\beta<\alpha} G_\beta$ for limit ordinals α. Assume $G_\alpha \in \mathcal{G}$ is defined and there is an $x \in Q/R \setminus G_\alpha$. Let $F_0 = S_0 = G_\alpha$. By

Theorem 6, there is an $F_1 \in \mathcal{F}$ such that $F_0 \cup \{x\} \subseteq F_1$, and F_1/F_0 is countably presented. Clearly, $S_0 \subseteq F_1$. Let C_1 be a countable subset of F_1 such that $F_0 + \langle C_1 \rangle = F_1$. Since $\mathcal{S}$ is a $G(\aleph_0)$-family, there is an $S_1 \in \mathcal{S}$ such that $S_0 \cup C_1 \subseteq S_1$, and S_1/S_0 is countably generated. Then $F_1 \subseteq S_1$. Let D_1 be a countable subset of S_1 such that $S_0 + \langle D_1 \rangle = S_1$. Then there is an $F_2 \in \mathcal{F}$ such that $F_1 \cup D_1 \subseteq F_2$, and F_2/F_1 is countably presented. Then $S_1 \subseteq F_2$. Proceeding in this way, we obtain a chain

$$G_\alpha = F_0 = S_0 \subseteq F_1 \subseteq S_1 \subseteq F_2 \subseteq \cdots \subseteq S_n \subseteq F_{n+1} \subseteq S_{n+1} \subseteq \cdots .$$

We define $G_{\alpha+1} = \cup_{n<\omega} F_n = \cup_{n<\omega} S_n$. Then $G_{\alpha+1} \in \mathcal{G}$ and, since F_{n+1}/F_n is countably presented for each $n < \omega$, so is $G_{\alpha+1}/G_\alpha$. This proves the claim.

Now, each G_α is a restriction of $Q/R = G_\sigma$ such that $(Q/R)/G_\alpha \in \mathcal{A}$, so $(Q/R)/G_\alpha$ has projective dimension ≤ 1. By Lemma 18 (2), G_α is a direct summand in Q/R, and hence in $G_{\alpha+1}$ for each $\alpha < \sigma$. This yields a decomposition of Q/R into a direct sum of copies of the countably presented modules $G_{\alpha+1}/G_\alpha$, $\alpha < \sigma$.

The implication (3) $\Rightarrow$ (1) is well known, cf. [**1**, Proposition 7.1]. $\square$

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