ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 39, Number 1, 2009

WHICH WEIGHTS ON R ADMIT L_p JACKSON THEOREMS?

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ABSTRACT. Let $1 \le p \le \infty$ and $W : \mathbf{R} \to (0, \infty)$ be continuous. Does W admit a Jackson theorem in L_p? That is, does there exist a sequence $\{\eta_n\}_{n=1}^{\infty}$ of positive numbers with limit 0 such that

$$\inf_{\deg(P) \le n} \|(f-P)W\|_{L_p(\mathbf{R})} \le \eta_n \|f'W\|_{L_p(\mathbf{R})}$$

for all absolutely continuous f with $\|f'W\|_{L_p(\mathbf{R})}$ finite? We show that such a theorem is true if and only if

$$\lim_{x\to\infty} \|W^{-1}\|_{L_q[0,x]} \|W\|_{L_p[x,\infty)} = 0,$$

with an analogous limit at $-\infty$. Here q is the conjugate parameter of p. In an earlier paper, we considered weights admitting a Jackson theorem for all $1 \le p \le \infty$.

1. Introduction. Let $W : \mathbf{R} \to (0, \infty)$. Bernstein's approximation problem addresses the following question: when are the polynomials dense in the weighted space generated by W? That is, when is it true that for every continuous $f : \mathbf{R} \to \mathbf{R}$ with

$$\lim_{|x|\to\infty} (fW)(x) = 0,$$

there exist a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ with

$$\lim_{n\to\infty} \|(f - P_n) W\|_{L_\infty(\mathbf{R})} = 0?$$

This problem was resolved independently by Pollard, Mergelyan and Achieser in the 1950s [**6**]. If $W \le 1$ is even, and $\ln 1/W(e^x)$ is convex,

Keywords and phrases. Weighted approximation, polynomial approximation, Jackson-Bernstein theorems.
Research supported by NSF grant DMS-0400446 and Israel–US BSF grant 2004353.
Received by the editors on February 24, 2006, and in revised form on November 28, 2008.
DOI:10.1216/RMJ-2009-39-1-165

a necessary and sufficient condition for density of the polynomials is [**6**, page 170]

$$\int_0^\infty \frac{\ln 1/W(x)}{1+x^2}\,dx = \infty.$$

In particular, for $W_\alpha(x) = \exp(-|x|^\alpha)$, the polynomials are dense if and only if $\alpha \geq 1$.

In the 1950s the search began for a quantitative form of Bernstein's theorem. One obvious question is whether there are weighted analogues of classical theorems of Jackson and Bernstein, namely,

$$\inf_{\deg(P)\leq n} \|f-P\|_{L_\infty[-1,1]} \leq \frac{C}{n}\|f'\|_{L_\infty[-1,1]},$$

with C independent of f and n, and the inf being over (algebraic) polynomials of degree at most n. For the weights W_α, where $\alpha > 1$, it is known that if $1 \leq p \leq \infty$,

$$\inf_{\deg(P)\leq n} \|(f-P)W_\alpha\|_{L_p(\mathbf{R})} \leq Cn^{-1+(1/\alpha)}\|f'W\|_{L_p(\mathbf{R})}, \tag{1}$$

with C independent of f and n [**5**, page 185] and [**11**, page 81]. This inequality is also often formulated in Jackson-Favard form,

$$\inf_{\deg(P)\leq n} \|(f-P)W_\alpha\|_{L_p(\mathbf{R})} \leq Cn^{-1+(1/\alpha)} \inf_{\deg(P)\leq n-1} \|(f'-P)W_\alpha\|_{L_p(\mathbf{R})}.$$

More general Jackson type theorems involving weighted moduli of continuity for various classes of weights were proved in [**4, 5, 11**].

In a recent paper [**10**], the author showed that the weight W_1 does not admit a Jackson estimate like (1), even though the polynomials are dense in the weighted space generated by W_1. The author also characterized weights that admit Jackson theorems in L_p for all $1 \leq p \leq \infty$. The main result there was:

Theorem 1.1. *Let* $W : \mathbf{R} \to (0,\infty)$ *be continuous. The following are equivalent*:

(a) *There exists a sequence* $\{\eta_n\}_{n=1}^{\infty}$ *of positive numbers with limit* 0 *and with the following property. For each* $1 \le p \le \infty$, *and for all absolutely continuous* f *with* $\|f'W\|_{L_p(\mathbf{R})}$ *finite, we have*

$$\inf_{\deg(P)\le n} \|(f-P)W\|_{L_p(\mathbf{R})} \le \eta_n \|f'W\|_{L_p(\mathbf{R})}, \quad n \ge 1. \tag{2}$$

(b) *Both*

$$\lim_{x\to\infty} W(x) \int_0^x W^{-1} = 0 \tag{3}$$

and

$$\lim_{x\to\infty} W(x)^{-1} \int_x^{\infty} W = 0 \tag{4}$$

with analogous limits as $x \to -\infty$.

As a corollary it was shown that if $W = e^{-Q}$, where Q' exists for large $|x|$, then there is a Jackson theorem in L_p for all $1 \le p \le \infty$, when $\pm Q'(x) \to \infty$ as $x \to \pm\infty$ and there is no Jackson theorem if $Q'(x)$ is bounded for large $|x|$. In this paper, we focus on just a single L_p space and ask which weights admit Jackson theorems in that space. We prove:

Theorem 1.2. *Let* $W : \mathbf{R} \to (0,\infty)$ *be continuous. Let* $1 \le p \le \infty$ *and* $1/p + 1/q = 1$. *The following are equivalent*:

(a) *There exists a sequence* $\{\eta_n\}_{n=1}^{\infty}$ *of positive numbers with limit* 0 *such that for all absolutely continuous* f *with* $\|f'W\|_{L_p(\mathbf{R})}$ *finite, we have*

$$\inf_{\deg(P)\le n} \|(f-P)W\|_{L_p(\mathbf{R})} \le \eta_n \|f'W\|_{L_p(\mathbf{R})}, \quad n \ge 1. \tag{5}$$

(b)

$$\lim_{x\to\infty} \|W\|_{L_p[x,\infty]} \left\|W^{-1}\right\|_{L_q[0,x]} = 0, \tag{6}$$

with an analogous limit as $x \to -\infty$.

Remarks. (a) Thus, there is a Jackson type theorem in a specific L_p space if and only if (6) holds. In fact, we shall show in Section 3 that (6) is necessary and sufficient for the existence of a decreasing function $\eta : (0,\infty) \to (0,\infty)$ with limit 0 at ∞, such that

$$\|f'W\|_{L_p[a,\infty)} \leq \eta(a) \|fW\|_{L_p[0,\infty)}$$

for all absolutely continuous f with $f(0) = 0$. This is a "shifting" weighted Hardy inequality.

(b) Theorem 1.2 actually implies Theorem 1.1. For condition (6) for $p = 1$ is equivalent to (4) and for $p = \infty$ is equivalent to (3). Interpolation then gives (2) for $1 < p < \infty$. Of course, Theorem 1.1 does not imply Theorem 1.2.

(c) It was shown in [**10**] that there is a weight W admitting an L_1 Jackson theorem, but not an L_∞ one (and conversely). Here we show:

Theorem 1.3. *Let $1 \leq p, r \leq \infty$ with $p \neq r$. There exists $W : \mathbf{R} \to (0,\infty)$ such that*

$$\frac{1}{1+x^2} \leq W(x)/\exp\left(-x^2\right) \leq 1 + x^2, \quad x \in \mathbf{R},$$

and W admits an L_r Jackson theorem, but not an L_p Jackson theorem. That is, there exist $\{\eta_n\}_{n=1}^{\infty}$ with limit 0 at ∞ satisfying (5) in the L_r norm, but there does not exist such a sequence satisfying (5) in the L_p norm.

Theorem 1.3 shows that not only rate of decay, but also regularity, of W is necessary for a Jackson theorem. After all, the Hermite weight $\exp(-x^2)$ admits a Jackson theorem in L_p for all $1 \leq p \leq \infty$, but W is close to W_2, yet admits a Jackson theorem in L_r but not L_p.

This paper is organized as follows: we prove restricted range inequalities in the next section, and an estimate for the "tails" $\|fW\|_{L_p(|x|\geq\lambda)}$ in Section 3. In Section 4, we prove Theorem 1.2. In Section 5, we prove Theorem 1.3.

Throughout $C, C_1, C_2, \ldots$ denote constants independent of n and x and polynomials P of degree $\leq n$. The same symbol may denote different constants in different occurrences. If (c_n) and (d_n) are sequences of real numbers, we write

$$c_n \sim d_n$$

if there exist $C_1, C_2 > 0$ such that

$$C_1 \leq c_n/d_n \leq C_2, n \geq 1.$$

Similar notation is used for functions. The linear measure of a set $B \subset \mathbf{R}$ is denoted by meas (B). The set of all polynomials of degree $\leq n$ is denoted P_n.

2. Restricted range inequalities. Restricted range (or infinite-finite range) inequalities are a crucial ingredient in weighted approximation on the real line [**8, 11, 12, 14**]. However, none of the standard ones cover our class of weights. The methods used to prove the form we need, are similar to, but not the same, as in [**10**]. In this section, we fix $1 \leq p \leq \infty$, and let

$$\widetilde{W}(x) = \left\| W^{-1} \right\|^{-1}_{L_q[0,x]}, \quad x \in (0,\infty), \tag{7}$$

where $1/q + 1/p = 1$.

Theorem 2.1. *Assume that, for* $x \in [0,\infty)$,

$$\|W\|_{L_p[x,\infty)} \left\| W^{-1} \right\|_{L_q[0,x]} \leq \psi(x), \tag{8}$$

where ψ *is decreasing in* $[0,\infty)$ *and*

$$\lim_{x\to\infty} \psi(x) = 0, \tag{9}$$

with a similar relation in $(-\infty, 0]$. *There exists* $q_n > 0$, $n \geq 1$, *such that*

$$q_n = o(n), \quad n \to \infty, \tag{10}$$

and for $n \geq 1$, *and all polynomials* P *of degree* $\leq n$,

$$\|PW\|_{L_p(|x| \geq q_n)} \leq C4^{-n} \|PW\|_{L_p(\mathbf{R})}. \tag{11}$$

Here C *is independent of* n *and* P.

In the rest of this section, ψ is the function specified in Theorem 2.1. For $n \geq 1$, we choose $A_n > 0$ such that

$$\|x^n W(x)\|_{L_p[A_n, 2A_n]} = \max_{u \geq 1} \|x^n W(x)\|_{L_p[u,2u]} =: \Lambda_n.$$

(We show below that A_n exists).

Lemma 2.2. (i) *For* $n \geq 0$,

$$\|x^n W(x)\|_{L_p[1,\infty)}$$

is finite.

(ii) *For* $n \geq 1$, A_n *exists, is finite and positive, and*

$$\lim_{n \to \infty} A_n = \infty. \tag{12}$$

(iii) *For* $n \geq 1$,

$$\begin{aligned} (2A_{n+2})^{-2} \Lambda_{n+2} &\leq \|x^n W(x)\|_{L_p[1,\infty)} \\ &\leq (2A_{n+2}^{-2p} + 2^{2p+1})^{1/p} \Lambda_{n+2}. \end{aligned} \tag{13}$$

(iv)

$$A_n = o(n), \quad n \to \infty. \tag{14}$$

(v) *If* $\mathcal{B} \subset [0, 2A_{n+2}]$ *has linear Lebesgue measure at least* 1, *then*

$$\|W\|_{L_p(\mathcal{B})} \geq \psi(1)^{-1} (2A_{2n+2})^{-(2n+2)} \Lambda_{2n+2}.$$

Proof. Observe that (8) implies

$$\|W\|_{L_p[x,\infty)} \leq \psi(x) \widetilde{W}(x), \quad x > 0, \tag{15}$$

and by Hölder's inequality, for $x \geq 1$,

$$1 \leq \|W\|_{L_p[x-1,x]} \left\|W^{-1}\right\|_{L_q[x-1,x]} \leq \|W\|_{L_p[x-1,x]} \left\|W^{-1}\right\|_{L_q[0,x]},$$

so that

$$\widetilde{W}(x) \leq \|W\|_{L_p[x-1,x]}, \quad x \geq 1. \tag{16}$$

(i) If $p = \infty$, this was established in Lemma 2.3 (a) in [**10**]. Suppose now $p < \infty$. Let $0 \leq a < b < \infty$. We see using (15) and (16) that

$$\begin{aligned}\int_a^b x^{np}\left(\int_x^\infty W^p(t)\,dt\right)dx &\leq \int_a^b x^{np}\psi^p(x)\widetilde{W}^p(x)\,dx\\ &\Longrightarrow \int_a^\infty W^p(t)\left[\int_a^{\min\{t,b\}} x^{np}\,dx\right]dt\\ &\leq \psi^p(a)\int_a^b x^{np}\left[\int_{x-1}^x W^p(t)\,dt\right]dx\end{aligned}$$

so

$$\begin{aligned}&\int_a^b W^p(t)\frac{t^{np+1}-a^{np+1}}{np+1}\,dt\\ &\qquad\leq \psi^p(a)\int_{a-1}^b W^p(t)\left[\int_{\max\{t,a\}}^{\min\{t+1,b\}} x^{np}\,dx\right]dt\\ &\qquad\leq \psi^p(a)\int_{a-1}^b (t+1)^{np}W^p(t)\,dt.\end{aligned} \tag{17}$$

Here, if $a \geq 2$, in the integral on the righthand side, $t \geq a-1 \geq 1$, so

$$(t+1)^{np} = t^{np}\left(1+\frac{1}{t}\right)^{np} \leq t^{np}\left(1+\frac{2}{a}\right)^{np} \leq t^{np}e^{(2np)/a}.$$

Moreover, if $t \geq a2^{1/(np+1)}$, then $t^{np+1} - a^{np+1} \geq (1/2)t^{np+1}$. Thus, (17) implies

$$\begin{aligned}&\int_{a2^{1/(np+1)}}^b t^{np+1}W^p(t)\,dt\\ &\qquad\leq 2\psi^p(a)(np+1)e^{(2np)/a}\int_{a-1}^b t^{np}W^p(t)\,dt.\end{aligned} \tag{18}$$

As $a \geq 2$, $t^{np} \leq t^{np+1}$ in the integral on the right, so

$$\int_{a2^{1/(np+1)}}^{b} t^{np+1} W^p(t)\, dt \left[1 - 2\psi^p(a)(np+1)e^{(2np)/a}\right] \\ \leq 2\psi^p(a)(np+1)e^{(2np)/a} \int_{a-1}^{a2^{1/(np+1)}} x^{np} W^p(x)\, dx.$$

If a is so large that $a \geq 2np$ and

$$2\psi^p(a)(np+1)e \leq \frac{1}{2},$$

this gives

$$\int_{a2^{1/(np+1)}}^{b} t^{np+1} W^p(t)\, dt \leq \int_{a-1}^{a2^{1/(np+1)}} x^{np} W^p(x)\, dx.$$

Letting $b \to \infty$ gives the finiteness of the norm $\|x^n W(x)\|_{L_p[1,\infty)}$.

(ii) The existence of $A_n \in (0, \infty)$ follows as the norm in (i) is finite, and $u \to \|x^n W(x)\|_{L_p[u,2u]}$ is a continuous function of u, with limit 0 as $u \to 0+$ and $u \to \infty$. (In the case $p = \infty$, this follows from the finiteness of $\|x^{n+1} W(x)\|_{L_p[1,\infty)}$). Next, for fixed $u > 0$,

$$\Lambda_n \geq \|x^n W(x)\|_{L_p[u,2u]} \geq u^n \|W\|_{L_p[u,2u]}$$

so

$$\liminf_{n\to\infty} \Lambda_n^{1/n} \geq u,$$

and hence

$$\lim_{n\to\infty} \Lambda_n^{1/n} = \infty.$$

If a subsequence of $\{A_n\}$ remained bounded, we see that the corresponding subsequence of $\{\Lambda_n\}$ cannot admit the growth just proven.

(iii) If $p = \infty$, the righthand inequality in (13) is immediate. Suppose now that $p < \infty$. Choose j_0 such that

$$2^{j_0} \leq A_{n+2} \leq 2^{j_0+1}.$$

We see that

$$\int_1^{A_{n+2}} x^{np} W^p(x)\,dx$$

$$\leq \sum_{j=0}^{j_0} \int_{A_{n+2}/2^{j+1}}^{A_{n+2}/2^j} x^{np} \left(\frac{x}{A_{n+2}/2^{j+1}}\right)^{2p} W^p(x)\,dx$$

$$\leq A_{n+2}^{-2p} \sum_{j=0}^{j_0} 2^{(j+1)2p} \Lambda_{n+2}^p$$

$$\leq A_{n+2}^{-2p} 2^{(j_0+1)2p+1} \Lambda_{n+2}^p \leq 2^{2p+1} \Lambda_{n+2}^p.$$

Also,

(19)

$$\int_{A_{n+2}}^{\infty} x^{np} W^p(x)\,dx \leq \sum_{j=0}^{\infty} \int_{A_{n+2}2^j}^{A_{n+2}2^{j+1}} x^{np} \left(\frac{x}{A_{n+2}2^j}\right)^{2p} W^p(x)\,dx$$

$$\leq A_{n+2}^{-2p} \left(\sum_{j=0}^{\infty} 2^{-2jp}\right) \Lambda_{n+2}^p \leq 2A_{n+2}^{-2p} \Lambda_{n+2}^p.$$

Then the upper bound in (13) follows. The lower bound follows from

$$\begin{aligned} \|x^n W(x)\|_{L_p[1,\infty)} &\geq \|x^n W(x)\|_{L_p[A_{n+2}, 2A_{n+2}]} \\ &\geq (2A_{n+2})^{-2} \left\|x^{n+2} W(x)\right\|_{L_p[A_{n+2}, 2A_{n+2}]} \\ &= (2A_{n+2})^{-2} \Lambda_{n+2}. \end{aligned}$$

(iv) If $p = \infty$, this follows from (19) of Lemma 2.3 (a) in [**10**]. (There $\ell(n)$ plays a role similar to A_n). Suppose now $p < \infty$. If we choose $a = a_n := A_{n+2} 2^{-1/(np+1)}$, and $b = 2A_{n+2}$, (18) gives for large enough n,

$$\int_{A_{n+2}}^{2A_{n+2}} t^{np+1} W^p(t)\,dt$$

$$\leq 2\psi^p(a_n)(np+1)e^{(2np)/a_n} \int_{a_n-1}^{b} t^{np} W^p(t)\,dt.$$

Here by (iii), and choice of a_n,

$$\begin{aligned}\int_{a_n-1}^{b} t^{np}W^p(t)\,dt &\le (a_n-1)^{-2p}\int_{a_n-1}^{b} t^{(n+2)p}W^p(t)\,dt\\ &\le CA_{n+2}^{-2p}\left(\int_{a_n-1}^{A_{n+2}}+\int_{A_{n+2}}^{2A_{n+2}}\right)t^{(n+2)p}W^p(t)\,dt\\ &\le CA_{n+2}^{-2p}2\Lambda_{n+2}^p,\end{aligned}$$

with C independent of n. Combining the above two inequalities gives

$$\begin{aligned}\Lambda_{n+2}^p &= \int_{A_{n+2}}^{2A_{n+2}} t^{(n+2)p}W^p(t)\,dt\\ &\le (2A_{n+2})^{2p-1}\int_{A_{n+2}}^{2A_{n+2}} t^{np+1}W^p(t)\,dt\\ &\le (2A_{n+2})^{2p-1}2\psi^p(a_n)(np+1)e^{(2np)/a_n}CA_{n+2}^{-2p}\Lambda_{n+2}^p\\ &\le C_1\frac{n\psi^p(a_n)}{a_n}e^{(2np)/a_n}\Lambda_{n+2}^p.\end{aligned}$$

Here C_1 is independent of n. If we write $a_n=\delta_n n$, we can recast this as

$$\frac{1}{\psi^p(a_n)}\le C_1\frac{1}{\delta_n}e^{(2p)/\delta_n}.$$

Since ψ has limit 0 at ∞, and $a_n=A_{n+2}2^{-1/(np+1)}\to\infty$, $n\to\infty$, it follows that necessarily $\delta_n=o(1)$ and so $a_n=o(n)$. That is,

$$A_{n+2}=o(n).$$

(v) Exactly as above, Hölder's inequality gives

$$1\le \|W\|_{L_p(\mathcal{B})}\left\|W^{-1}\right\|_{L_q(\mathcal{B})}\le \|W\|_{L_p(\mathcal{B})}\left\|W^{-1}\right\|_{L_q[0,A_{2n+2}]}.$$

Using (15), we can continue this as

$$\begin{aligned}&\|W\|_{L_p(\mathcal{B})}\\ &\qquad\ge \widetilde{W}(A_{2n+2})\\ &\qquad\ge \psi(A_{2n+2})^{-1}\|W\|_{L_p[A_{2n+2},\infty)}\\ &\qquad\ge \psi(1)^{-1}(2A_{2n+2})^{-(2n+2)}\left\|x^{2n+2}W(x)\right\|_{L_p[A_{2n+2},2A_{2n+2}]}\\ &\qquad= \psi(1)^{-1}(2A_{2n+2})^{-(2n+2)}\Lambda_{2n+2}.\qquad\square\end{aligned}$$

Lemma 2.3. *There exists a* $C_2 > 0$ *such that for* $n \geq 1$ *and all polynomials* P *of degree* $\leq n$,

$$\|PW\|_{L_p[1600A_{2n+2},\infty)} \leq C_2 4^{-n} \|PW\|_{L_p[0,\infty)}.$$

Proof. Our approach is similar to that in [**9**]. Let P be a polynomial of degree $k \leq n$, say

$$P(z) = c\prod_{j=1}^{k}(z - x_j).$$

We assume $\rho > 8$, $c \neq 0$, and split the zeros into "small" and "large" zeros: we assume that

$$|x_j| \leq \rho, \quad j \leq i;$$
$$|x_j| > \rho, \quad j > i.$$

For $|u| \leq 1/2\rho$, $x \geq \rho$ and $i < j \leq k$,

$$\left|\frac{x - x_j}{u - x_j}\right| \leq \frac{1 + x/|x_j|}{1 - |u|/|x_j|} \leq 2\left(1 + \frac{x}{\rho}\right) \leq 4\frac{x}{\rho}.$$

Then for such x and u,

$$\left|\frac{P(x)}{P(u)}\right| \leq \left(\prod_{j=1}^{i}\frac{2x}{|u - x_j|}\right)\left(4\frac{x}{\rho}\right)^{k-i}.$$

We now apply a famous lemma of Cartan:

$$\left|\prod_{j=1}^{i}(u - x_j)\right| \geq \varepsilon^i$$

for u outside a set of linear measure at most $4e\varepsilon$ [**1**, page 175], [**2**, page 350]. Choosing $\varepsilon = \rho/100$, we obtain

$$\left|\frac{P(x)}{P(u)}\right| \leq \left(\frac{200x}{\rho}\right)^k \leq \left(\frac{200x}{\rho}\right)^n,$$

for $x \geq \rho$, $u \in [0, (1/2)\rho]\backslash\mathcal{S}$, where

$$\text{meas}\,(\mathcal{S}) \le \frac{4e}{100}\rho < \frac{1}{8}\rho.$$

Recall that meas denotes linear Lebesgue measure. Then, for such u,

$$(20) \qquad \|PW\|_{L_p[400\rho,\infty)} \le \left(\frac{200}{\rho}\right)^n |P(u)| \, \|x^n W(x)\|_{L_p[400\rho,\infty)} \, .$$

Moreover, $[0,(1/4)\rho]\backslash\mathcal{S}$ has measure at least $(1/8)\rho \ge 1$, so we may find $\mathcal{B} \subset [0,(1/4)\rho]\backslash\mathcal{S}$ with linear measure at least 1, and hence

$$\|PW\|_{L_p[400\rho,\infty)} \|W\|_{L_p(\mathcal{B})} \le \left(\frac{200}{\rho}\right)^n \|PW\|_{L_p(\mathcal{B})} \|x^n W(x)\|_{L_p[400\rho,\infty)} \, .$$

Now we choose $\rho = 4A_{2n+2}$, at least for n so large that $4A_{2n+2} > 8$. Then $[0,(1/4)\rho]\backslash\mathcal{S} \subset [0, A_{2n+2}]$. By (v) of the previous lemma,

$$\|W\|_{L_p(\mathcal{B})} \ge \psi(1)^{-1}(2A_{2n+2})^{-(2n+2)}\Lambda_{2n+2}.$$

Combining the above inequalities, we see that if P is not identically 0,

$$\begin{aligned}
&\|PW\|_{L_p[400\rho,\infty)} / \|PW\|_{L_p[0,\infty)} \\
&\le \left(\frac{200}{\rho}\right)^n \|x^n W(x)\|_{L_p[400\rho,\infty)} / \left[\psi(1)^{-1}\left(2A_{2n+2}\right)^{-(2n+2)} \Lambda_{2n+2}\right] \\
&\le \left(\frac{1}{2\rho^2}\right)^n \left\|x^{2n} W(x)\right\|_{L_p[400\rho,\infty)} / \left[\psi(1)^{-1}\left(2A_{2n+2}\right)^{-(2n+2)} \Lambda_{2n+2}\right] \\
&\le C8^{-n}A_{2n+2}^2,
\end{aligned}$$

by (iii) of the previous lemma. Here C is independent of n and P, and $A_{2n+2} = o(n)$, so the result follows. For the remaining finitely many n, for which $4A_{2n+2} < 8$, a simple compactness argument gives the result, if C_2 is large enough. □

Proof of Theorem 2.1. This follows from Lemma 2.3, its analogue in $(-\infty, 0]$, and the fact that $A_n = o(n)$. □

We also record:

Lemma 2.4. *Let $W : \mathbf{R} \to (0,\infty)$ be continuous, $1 \le p \le \infty$, and assume that for each $n \ge 0$,*

$$\|x^n W(x)\|_{L_p(\mathbf{R})} < \infty. \tag{21}$$

Then there exists an increasing sequence of positive numbers $\{\xi_n\}_{n=1}^{\infty}$ such that for $n \ge 1$ and all polynomials P of degree $\le n$,

$$\|PW\|_{L_p(|x|\ge\xi_n)} \le C_1 2^{-n} \|PW\|_{L_p(-1,1)}, \tag{22}$$

where C_1 is independent of n, p, P.

Proof. See Theorem 2.2 in [**10**]. □

Tail estimates. We prove a "shifting" weighted Hardy inequality, involving the function

$$\phi(x) = \|W\|_{L_p[x,\infty)} \left\|W^{-1}\right\|_{L_q[0,x]}, \quad x \ge 0.$$

Theorem 3.1. *Let $W : \mathbf{R} \to (0,\infty)$ be continuous. Let $1 \le p \le \infty$ and $1/q + 1/p = 1$. The following are equivalent*:

(I) *There exists a decreasing function $\eta : [0,\infty) \to (0,\infty)$ with limit 0 at ∞ such that*

$$\|fW\|_{L_p(|x|\ge a)} \le \eta(a) \|f'W\|_{L_p(\mathbf{R})}, \tag{23}$$

for all $a > 0$ and every absolutely continuous function $f : \mathbf{R} \to \mathbf{R}$ with $f(0) = 0$.

(II)

$$\lim_{a\to\infty} \phi(a) = \lim_{a\to\infty} \|W\|_{L_p[a,\infty)} \left\|W^{-1}\right\|_{L_q[0,a]} = 0, \tag{24}$$

with a similar limit as $a \to -\infty$.

Lemma 3.2. *Let $a > 0$. Then*

$$\|fW\|_{L_p[a,\infty)} \le p^{1/p} q^{1/q} \Big(\sup_{x\ge a} \phi(x)\Big) \|f'W\|_{L_p[a,\infty)},$$

for every absolutely continuous function $f : [a, \infty) \to \mathbf{R}$ *with* $f(a) = 0$. *Here, if* $p = \infty$ *or* $p = 1$, *we interpret* $p^{1/p} q^{1/q}$ *as* 1.

Proof. Let

$$B = \sup_{x \in (a,\infty)} \|W\|_{L_p[x,\infty)} \left\|W^{-1}\right\|_{L_q[a,x]}.$$

The classical weighted Hardy inequality asserts that, for every f as above,

$$\|fW\|_{L_p[a,\infty)} \le p^{1/p} q^{1/q} B \|f'W\|_{L_p[a,\infty)}.$$

(See [**13**, page 13, Theorem 1.14] for the proof when $1 < p < \infty$. Take $q = p$ there and $w = v = W^p$. For $p = 1$ or $p = \infty$, see [**13**, page 49, Lemma 5.4]. An alternative reference is [**7**].) Since

$$B \le \sup_{x \in (a,\infty)} \|W\|_{L_p[x,\infty)} \left\|W^{-1}\right\|_{L_q[0,x]} = \sup_{x \ge a} \phi(x),$$

the result follows. □

Lemma 3.3. *Let* $a > 0$. *Then*

$$\|fW\|_{L_p[a,\infty)} \le \left(1 + p^{1/p} q^{1/q}\right) \left(\sup_{x \ge a} \phi(x)\right) \|f'W\|_{L_p[0,\infty)},$$

for every absolutely continuous function $f : [0, \infty) \to \mathbf{R}$ *with* $f(0) = 0$.

Proof. Write for $x \ge a$,

$$f(x) = \int_0^a f' + \int_a^x f' =: C + f_1(x).$$

Then

$$\|fW\|_{L_p[a,\infty)} \le \|CW\|_{L_p[a,\infty)} + \|f_1 W\|_{L_p[a,\infty)}. \tag{25}$$

Here, by Hölder's inequality applied to C,

$$\begin{aligned} \|CW\|_{L_p[a,\infty)} &\le \|f'W\|_{L_p[0,a)} \left\|W^{-1}\right\|_{L_q[0,a]} \|W\|_{L_p[a,\infty)} \\ &= \|f'W\|_{L_p[0,a)} \phi(a). \end{aligned}$$

Moreover, by Lemma 3.2, as $f_1(a) = 0$,

$$\|f_1 W\|_{L_p[a,\infty)} \le p^{1/p} q^{1/q} \Big(\sup_{x \ge a} \phi(x) \Big) \|f' W\|_{L_p[a,\infty)} .$$

Combining the above three inequalities gives the result. □

Proof of Theorem 3.1. *Sufficiency of* (24) *and its analogous limit at* $-\infty$. This follows directly from Lemma 3.3. We can choose

$$\eta_+(a) = \Big(1 + p^{1/p} q^{1/q}\Big) \sup_{x \ge a} \phi(x), \quad a > 0,$$

with a similar function η_- to handle $(-\infty, 0)$, and then set $\eta = \max\{\eta_-, \eta_+\}$.

Necessity of (24) *and its analogous limit at* $-\infty$. For $p = 1$ and $p = \infty$, the necessity was established in the proof of Theorem 3.1 in [**10**]. Suppose now $1 < p < \infty$. Let $a > 0$ and

$$f(x) = \int_0^{\min\{x,a\}} W^{-q}, x \ge 0.$$

Then

$$\|f' W\|_{L_p[0,\infty)} = \left(\int_0^a W^{(1-q)p} \right)^{1/p} = \left\|W^{-1}\right\|_{L_q[0,a]}^{1/(p-1)},$$

so

$$\begin{aligned} \|f' W\|_{L_p[0,\infty)} \phi(a) &= \|f' W\|_{L_p[0,\infty)} \left\|W^{-1}\right\|_{L_q[0,a]} \|W\|_{L_p[a,\infty)} \\ &= \left\|W^{-1}\right\|_{L_q[0,a]}^{((1/(p-1))+1)} \|W\|_{L_p[a,\infty)} \\ &= \left(\int_0^a W^{-q} \right) \|W\|_{L_p[a,\infty)} = \|fW\|_{L_p[a,\infty)} . \end{aligned}$$

Our hypothesis gives

$$\eta(a) \ge \frac{\|fW\|_{L_p[a,\infty)}}{\|f' W\|_{L_p[0,\infty)}} = \phi(a).$$

So ϕ has limit 0 at ∞. Similarly, the analogous limit follows at $-\infty$. □

4. Weighted approximation. We begin with two lemmas, which are similar to corresponding lemmas in [**10**]. We shall use notation specific to this section: we use integers $n \geq 4$ and $1 \leq m \leq (n/4)$, as well as parameters

$$1 < \lambda \leq \frac{1}{2} q_m,$$

where $\{q_n\}_{n=1}^{\infty}$ are as in Theorem 2.1. We let $\rho(m)$ denote an increasing function that depends on m and W, while $\sigma(\lambda)$ denotes a function increasing in λ. These functions change in different occurrences. The essential feature is that σ is independent of m, n, p and functions f, while ρ is independent of λ, p and functions f. At the end, we choose m to grow slowly enough as a function of n, and then $\lambda \to \infty$ sufficiently slowly. We let $\mathcal{P}_m$ denote the set of polynomials of degree $\leq m$ with real coefficients.

Lemma 4.1. *Let $W : \mathbf{R} \to (0, \infty)$ be continuous and satisfy* (6), *with an analogous limit at $-\infty$.*

(a) *There exists an increasing function $\sigma : [0, \infty) \to [0, \infty)$ with the following properties*: *let $m, \lambda \geq 1$. For $1 \leq p \leq \infty$ and all absolutely continuous f with $f'W \in L_p(\mathbf{R})$, there exists $R_m \in \mathcal{P}_m$ such that*

$$\|(f - R_m) W\|_{L_p[-2\lambda, 2\lambda]} \leq \frac{\sigma(\lambda)}{m} \|f'W\|_{L_p(\mathbf{R})}.$$

(b) *There is an increasing function $\rho : \mathbf{Z}_+ \to (0, \infty)$ depending only on W such that*

$$\|R_m W\|_{L_p(\mathbf{R})} \leq \rho(m) \left(\|fW\|_{L_p(\mathbf{R})} + \|f'W\|_{L_p(\mathbf{R})} \right).$$

Proof. (a) By the classical Jackson's theorem [**3**, (6.4), Theorem 6.2, page 219], there exists $R_m \in \mathcal{P}_m$ such that

$$\|f - R_m\|_{L_p[-2\lambda, 2\lambda]} \leq \frac{\pi \lambda}{m+1} \|f'\|_{L_p[-2\lambda, 2\lambda]}.$$

Then

$$\|(f - R_m) W\|_{L_p[-2\lambda, 2\lambda]} \leq \frac{\pi \lambda}{m} \|W\|_{L_\infty[-2\lambda, 2\lambda]} \left\|W^{-1}\right\|_{L_\infty[-2\lambda, 2\lambda]} \|f'W\|_{L_p(\mathbf{R})}.$$

So we may take

$$\sigma(\lambda) = \pi\lambda \|W\|_{L_\infty[-2\lambda,2\lambda]} \left\|W^{-1}\right\|_{L_\infty[-2\lambda,2\lambda]}.$$

(b) From our restricted range of inequalities, and continuity of W,

$$\|R_m W\|_{L_p(\mathbf{R})} \le C \,\|R_m\|_{L_p[-q_m,q_m]} \,\|W\|_{L_\infty[-q_m,q_m]}.$$

Moreover, from the proof of (a),

$$\begin{aligned}
&\|R_m\|_{L_p[-2\lambda,2\lambda]} \\
&\quad \le \|f\|_{L_p[-2\lambda,2\lambda]} + \frac{\pi\lambda}{m}\,\|f'\|_{L_p[-2\lambda,2\lambda]} \\
&\quad \le \left\|W^{-1}\right\|_{L_\infty[-2\lambda,2\lambda]} \left[\|fW\|_{L_p[-2\lambda,2\lambda]} + \pi\lambda\,\|f'W\|_{L_p[-2\lambda,2\lambda]}\right].
\end{aligned}$$

We shall show that

$$\|R_m\|_{L_p[-q_m,q_m]} \le C m^{2/p} \left(\frac{q_m}{\lambda}\right)^{m+(1/p)} \|R_m\|_{L_p[-2\lambda,2\lambda]}, \tag{26}$$

where C is independent of m, λ, q_m, $\{R_m\}$. (Recall that $2\lambda \le q_m$). Then, on combining the above inequalities, we obtain

$$\|R_m W\|_{L_p(\mathbf{R})} \le \rho(m) \left[\|fW\|_{L_p[-2\lambda,2\lambda]} + \|f'W\|_{L_p[-2\lambda,2\lambda]}\right]$$

where

$$\begin{aligned}
\rho(m) = {} & C m^{2/p} q_m^{m+1/p} \|W\|_{L_\infty[-q_m,q_m]} \\
& \times \left\|W^{-1}\right\|_{L_\infty[-q_m,q_m]} (1+\pi q_m).
\end{aligned}$$

Now we proceed to establish (26). Recall the Chebyshev inequality [**3**, page 101, Proposition 2.3], valid for polynomials P of degree $\le m$:

$$|P(x)| \le |T_m(x)|\, \|P\|_{L_\infty[-1,1]}, \quad |x| > 1.$$

Here T_m is the classical Chebyshev polynomial of the first kind. By dilating this, and using the bound

$$|T_m(x)| \le (2|x|)^m, \quad |x| > 1,$$

we obtain

$$\|R_m\|_{L_\infty[-q_m,q_m]} \le \left(\frac{q_m}{\lambda}\right)^m \|R_m\|_{L_\infty[-2\lambda,2\lambda]}.$$

Using Nikolskii inequalities [**3**, page 102, Theorem 2.6], we continue this as

$$\begin{aligned}\|R_m\|_{L_p[-q_m,q_m]} &\le (2q_m)^{1/p}\|R_m\|_{L_\infty[-q_m,q_m]}\\ &\le (2q_m)^{1/p}\left(\frac{q_m}{\lambda}\right)^m\left(\frac{(p+1)m^2}{2\lambda}\right)^{1/p}\\ &\qquad\times\|R_m\|_{L_p[-2\lambda,2\lambda]},\end{aligned}$$

and then we have (26). □

Lemma 4.2. *There exists $C>0$ such that for large enough n, and for $1 \le \lambda \le (1/2)q_n$, there are nonnegative polynomials V_n of degree $\le 3n/4$ such that*

$$|1-V_n(x)| \le C\frac{q_n}{n\lambda}, \quad x\in[-\lambda,\lambda]; \tag{27}$$

$$0 \le V_n(x) \le C, \quad |x|\in[\lambda,2\lambda]; \tag{28}$$

$$0 \le V_n(x) \le C\left(\frac{q_n}{n\lambda}\right)^2, \quad |x|\in[2\lambda,q_n]. \tag{29}$$

Here C is independent of n, λ and x.

Proof. See Lemma 4.2 in [**10**]. □

Proof of the sufficiency part of Theorem 1.2. This is quite similar to that of Theorem 1.1 in [**10**], but there is an important difference: there we introduced estimates for R_mW in the uniform norm, while here we need to restrict ourselves to a given L_p norm. So we include all the details.

We may assume that $f(0)=0$. (If not, replace f by $f-f(0)$ and absorb the constant $f(0)$ into the approximating polynomial). We choose $n \geq 1$ and $1 \leq m \leq n/4$, and let λ satisfy $1 \leq \lambda \leq (1/2)q_m$. Let R_m and V_n denote the polynomials of Lemma 4.1 and 4.2 respectively, and let

$$P_n = R_m V_n.$$

Then P_n is a polynomial of degree $\leq n$, and

$$\begin{aligned}(30)\quad \inf_{\deg(P)\leq n} \|(f-P)W\|_{L_p(\mathbf{R})} &\leq \|(f-P_n)W\|_{L_p(\mathbf{R})} \\ &\leq \|(f-P_n)W\|_{L_p[-q_n,q_n]} + \|fW\|_{L_p(\mathbf{R}\setminus[-q_n,q_n])} \\ &\quad + \|P_nW\|_{L_p(\mathbf{R}\setminus[-q_n,q_n])} \\ &\leq \|(f-P_n)W\|_{L_p[-q_n,q_n]} + \|fW\|_{L_p(\mathbf{R}\setminus[-\lambda,\lambda])} \\ &\quad + C4^{-n}\|P_nW\|_{L_p[-q_n,q_n]},\end{aligned}$$

by Theorem 2.1 and as $q_n > \lambda$. Here,

$$\begin{aligned}(31)\quad \|(f-P_n)W\|_{L_p[-q_n,q_n]} &\leq \|(f-P_n)W\|_{L_p[-\lambda,\lambda]} + \|fW\|_{L_p(\mathbf{R}\setminus[-\lambda,\lambda])} \\ &\quad + \|P_nW\|_{L_p([-q_n,q_n]\setminus[-\lambda,\lambda])} \\ &=: T_1 + T_2 + T_3.\end{aligned}$$

Firstly,

(32)

$$\begin{aligned}T_1 &\leq \|(f-R_m)W\|_{L_p[-\lambda,\lambda]} + \|R_m(1-V_n)W\|_{L_p[-\lambda,\lambda]} \\ &\leq \|(f-R_m)W\|_{L_p[-\lambda,\lambda]} \\ &\quad + \|R_mW\|_{L_p[-\lambda,\lambda]}\|1-V_n\|_{L_\infty[-\lambda,\lambda]} \\ &\leq \frac{\sigma(\lambda)}{m}\|f'W\|_{L_p(\mathbf{R})} \\ &\quad + \rho(m)\left(\|fW\|_{L_p(\mathbf{R})} + \|f'W\|_{L_p(\mathbf{R})}\right)\|1-V_n\|_{L_\infty[-\lambda,\lambda]} \\ &\leq \frac{\sigma(\lambda)}{m}\|f'W\|_{L_p(\mathbf{R})} + \rho(m)\frac{q_n}{n}\|f'W\|_{L_p(\mathbf{R})}(\eta(0)+C),\end{aligned}$$

by Lemmas 4.1, 4.2 and Theorem 3.1. Note that since $f(0) = 0$, the latter gives

$$\|fW\|_{L_p(\mathbf{R})} \leq \eta(0)\|f'W\|_{L_p(\mathbf{R})}.$$

The crucial thing in (32) is that $C, \eta(0), \sigma$ and ρ are independent of f, n, p. Next, Theorem 3.1 gives

$$T_2 \leq \eta(\lambda) \|f'W\|_{L_p(\mathbf{R})}. \tag{33}$$

Of course this estimate also applies to the middle term in the righthand side of (30). Next,

$$\begin{aligned} T_3 &\leq \|P_n W\|_{L_p(\lambda \leq |x| \leq 2\lambda)} + \|P_n W\|_{L_p(2\lambda \leq |x| \leq q_n)} \\ &=: T_{31} + T_{32}. \end{aligned}$$

Here

(34)

$$\begin{aligned} T_{31} &\leq \|R_m W\|_{L_p(\lambda \leq |x| \leq 2\lambda)} \|V_n\|_{L_\infty(\lambda \leq |x| \leq 2\lambda)} \\ &\leq C(\|(R_m - f)W\|_{L_p(\lambda \leq |x| \leq 2\lambda)} + \|fW\|_{L_p(\lambda \leq |x| \leq 2\lambda)}) \\ &\leq C\left(\frac{\sigma(\lambda)}{m} \|f'W\|_{L_p(\mathbf{R})} + \eta(\lambda)\|f'W\|_{L_p(\mathbf{R})}\right), \end{aligned}$$

by Lemmas 4.1, 4.2 and Theorem 3.1. Also,

$$\begin{aligned} T_{32} &\leq \|R_m W\|_{L_p(2\lambda \leq |x| \leq q_n)} \|V_n\|_{L_\infty(2\lambda \leq |x| \leq q_n)} \\ &\leq \rho(m) \|f'W\|_{L_p(\mathbf{R})} C_1 \left(\frac{q_n}{n}\right)^2, \end{aligned}$$

by Lemmas 4.1, 4.2 and another application of Theorem 3.1. Combining this and the estimates in (31) to (34) gives

$$\begin{aligned} &\|(f - P_n) W\|_{L_p[-q_n, q_n]} \\ &\qquad \leq \|f'W\|_{L_p(\mathbf{R})} C \left\{ \frac{\sigma(\lambda)}{m} + \rho(m) \frac{q_n}{n} + \eta(\lambda) \right\}. \end{aligned} \tag{35}$$

Then using this estimate and Theorem 3.1, we deduce that

$$\|P_n W\|_{L_p[-q_n, q_n]} \leq \|f'W\|_{L_p(\mathbf{R})} C \left\{ \frac{\sigma(\lambda)}{m} + \rho(m) \frac{q_n}{n} + 1 \right\}.$$

Combining this estimate, (30) and (35) give

$$\begin{aligned} &\inf_{\deg(P) \leq n} \|(f - P)W\|_{L_p(\mathbf{R})} \\ &\qquad \leq \|f'W\|_{L_p(\mathbf{R})} C \left\{ \frac{\sigma(\lambda)}{m} + \rho(m) \frac{q_n}{n} + \eta(\lambda) + 4^{-n} \right\}, \end{aligned}$$

with C independent of n, m, λ, ρ and σ. The functions σ and ρ obey the conventions listed at the beginning of this section, and are independent of f, n, m and p, as is the constant C. For a given large enough $n \geq 1$, we choose $m = m(n)$ to be the largest integer $\leq n/2$ such that

$$\rho(m)\frac{q_n}{n} \leq \left(\frac{q_n}{n}\right)^{1/2}.$$

Since (by Theorem 2.1) $q_n/n \to 0$ as $n \to \infty$, while ρ is increasing and finite-valued, necessarily $m = m(n)$ approaches ∞ as $n \to \infty$. Next, for the given $m = m(n)$, we choose the largest $\lambda = \lambda(n) \leq m$ such that

$$\sigma(\lambda) \leq \sqrt{m}$$

As σ is finite-valued, necessarily $\lambda(n) \to \infty$, so $\eta(\lambda(n)) \to 0$, $n \to \infty$. Then, for some sequence $\{\eta_n\}_{n=1}^{\infty}$ with limit 0, and which is independent of f,

$$\inf_{\deg(P)\leq n} \|(f-P)W\|_{L_p(\mathbf{R})} \leq \eta_n \|f'W\|_{L_p(\mathbf{R})}.$$

For the remaining finitely many n, we can set $\eta_n = \eta(0)$ and use

$$\inf_{\deg(P)\leq n} \|(f-P)W\|_{L_p(\mathbf{R})} \leq \|fW\|_{L_p(\mathbf{R})} \leq \eta(0)\|f'W\|_{L_p(\mathbf{R})}. \qquad \square$$

Proof of the necessity part of Theorem 1.2. We assume that (5) is true for every absolutely continuous f with $\|f'W\|_{L_p(\mathbf{R})}$ finite. In particular, if we choose f to be 0 outside $[-1,1]$, and not almost everywhere a polynomial in $[-1,1]$, we obtain for some sequence $\{P_n\}_{n=1}^{\infty}$ of polynomials with degrees tending to ∞,

$$\|P_nW\|_{L_p(|x|\geq 1)} \longrightarrow 0, \quad n \to \infty.$$

As P_n behaves for large $|x|$ like its leading term, this forces

$$\|x^n W(x)\|_{L_p(\mathbf{R})} < \infty,$$

for each $n \geq 0$. Then the hypothesis (21) of Lemma 2.4 is fulfilled, and consequently there exist $\{\xi_n\}_{n=1}^{\infty}$ such that (22) holds for all polynomials P_n of degree $\leq n$. Let us consider an absolutely continuous f with $f(0) = 0$ and $\|f'W\|_{L_p(\mathbf{R})}$ finite. Our hypothesis asserts that there are for large n polynomials $\{P_n\}_{n=1}^{\infty}$ of degree $\leq n$ with

$$\begin{aligned}&\|\,(f-P_n)\,W\|_{L_p(\mathbf{R})} \leq \eta_n \|f'W\|_{L_p(\mathbf{R})}\\ &\qquad \Longrightarrow \|fW\|_{L_p(|x|\geq \xi_n)} \leq \eta_n \|f'W\|_{L_p(\mathbf{R})} + \|P_n W\|_{L_p(|x|\geq \xi_n)}.\end{aligned}$$

By Lemma 2.4, and then our hypothesis on $\{P_n\}_{n=1}^{\infty}$,

$$\begin{aligned}\|P_n W\|_{L_p(|x|\geq \xi_n)} &\leq C2^{-n} \|P_n W\|_{L_p[-1,1]}\\ &\leq C2^{-n} (\|fW\|_{L_p[-1,1]} + \eta_n \|f'W\|_{L_p(\mathbf{R})}).\end{aligned}$$

Here,

$$\begin{aligned}\|fW\|_{L_p[0,1]} &\leq \|W\|_{L_\infty[0,1]} \left\| \int_0^x f'(t)\,dt \right\|_{L_p[0,1]}\\ &\leq \|W\|_{L_\infty[0,1]} \, \|f'\|_{L_p[0,1]}\\ &\leq \|W\|_{L_\infty[0,1]} \|W^{-1}\|_{L_\infty[0,1]} \, \|f'W\|_{L_p[0,1]}\,.\end{aligned}$$

A similar inequality holds over $[-1,0]$, and hence

$$\|fW\|_{L_p[-1,1]} \leq 2\|W\|_{L_\infty[-1,1]} \|W^{-1}\|_{L_\infty[-1,1]} \|f'W\|_{L_p[-1,1]}.$$

Combining all of the above inequalities gives

$$\|fW\|_{L_p(|x|\geq \xi_n)} \leq \eta_n^* \|f'W\|_{L_p(\mathbf{R})},$$

where $\{\eta_n^*\}_{n=1}^{\infty}$ has limit 0 and is independent of f. The same inequality then holds for the L_p norm of fW over $|x| \geq \lambda$, where $\lambda \in [\xi_n, \xi_{n+1}]$. It follows that there is a positive decreasing function η with limit 0 at ∞ such that (23) holds for absolutely continuous f with $f(0) = 0$ and $\|f'W\|_{L_p(\mathbf{R})}$ finite. Then Theorem 3.1 gives the limit (6). $\square$

5. Proof of Theorem 1.3. In this section, we let

$$W_2(x) = \exp(-x^2), \quad x \in \mathbf{R},$$

denote the Hermite weight. Moreover, we determine q, s by the equations

$$\frac{1}{r} + \frac{1}{s} = 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1.$$

The construction is more complicated than that in [**10**], but the general idea is the same. We choose intervals

$$[j-\alpha_j, j+\alpha_j], \quad j \geq 3,$$

where $\alpha_j \leq 1/(2j)$, $j \geq 3$. We set

$$W(x) = W_2(x), \quad x \in \mathbf{R} \setminus \bigcup_{j=3}^{\infty} (j-\alpha_j, j+\alpha_j). \tag{36}$$

(I) For the case where $p < r$, we set

$$W(j) = \frac{W_2(j)}{[j \log j]}, \quad j \geq 3, \tag{37}$$

choose

$$\beta \in (s, q) \tag{38}$$

and

$$\alpha_j = \frac{1}{2j(\log j)^{\beta}}, \quad j \geq 3. \tag{39}$$

(II) For the case where $p > r$, we set

$$W(j) = W_2(j)[j \log j], \quad j \geq 3, \tag{40}$$

and choose

$$\beta \in (r, p) \tag{41}$$

and

$$\alpha_j = \frac{1}{2j(\log j)^{\beta}}, \quad j \geq 3. \tag{42}$$

In both cases we then define W so that W/W_2 is linear in $[j-\alpha_j, j]$ and in $[j, j+\alpha_j]$. This ensures that W is continuous in $\mathbf{R}$. (Of course, we could ensure it is C^∞ by smoothing at j and $j \pm \alpha_j$). It also implies under (38) that

$$1 \geq W(x)/W_2(x) \geq \frac{1}{1+x^2}, \quad x \in \mathbf{R}, \tag{43}$$

and under (40),

$$1 \leq W(x)/W_2(x) \leq 1 + x^2, \quad x \in \mathbf{R}. \tag{44}$$

(Since $\log x = o(x)$, these inequalities are clear for large $|x|$. However they are even true for "small" $|x|$, as shown by some simple calculations.) We shall make repeated use of the fact that, uniformly in j and x,

$$W_2(x) \sim W_2(j), \quad x \in [j - \alpha_j, j + \alpha_j],$$

as follows since $\alpha_j \leq 1/(2j)$. We now show that W fulfills the asymptotic behavior required for Theorem 1.3.

Lemma 4.2. (a) *Let $p < r$ and W satisfy* (37), (38) *and* (39). *Then*

$$\limsup_{x\to\infty} \left\|W^{-1}\right\|_{L_q[0,x]} \|W\|_{L_p[x,\infty)} = \infty, \tag{45}$$

but

$$\lim_{x\to\infty} \left\|W^{-1}\right\|_{L_s[0,x]} \|W\|_{L_r[x,\infty)} = 0. \tag{46}$$

(b) *Let $p > r$ and W satisfy* (40), (41) *and* (42). *Then* (45) *and* (46) *are valid.*

Proof. (a) Note that, as $1 \leq p < r$, so $p, s < \infty$. Let $c > 0$. Some simple calculations show that, for $1 \leq a \leq b$,

$$\int_a^b W_2^{-c} \sim W_2^{-c}(b) \min\left\{\frac{1}{b}, b - a\right\}, \tag{47}$$

and if also $b \leq 2a$,

$$\int_a^b W_2^{c} \sim W_2^{c}(a) \min\left\{\frac{1}{b}, b - a\right\}. \tag{48}$$

Since $\alpha_j = O(1/j)$, we see that $W_2(j+\alpha_j) \sim W_2(j)$, and hence applying (48),

$$\int_j^\infty W^p \geq \int_{j+\alpha_j}^{j+1-\alpha_{j+1}} W_2^p \geq \frac{C}{j} W_2(j)^p. \tag{49}$$

Moreover, by (47), if $q < \infty$,

$$\int_0^j W^{-q} \geq C(j \log j)^q \int_{j-(\alpha_j/2)}^j W_2^{-q} \geq C(j \log j)^q \alpha_j W_2(j)^{-q}.$$

Then

$$\begin{aligned} \left\| W^{-1} \right\|_{L_q[0,j]} \| W \|_{L_p[j,\infty)} &\geq C[j \log j] \alpha_j^{1/q} j^{-1/p} \\ &= C(\log j)^{1-\beta/q} \to \infty, \end{aligned}$$

$j \to \infty$, by (38). We then have (45) for the case $1 < p$, $q < \infty$. If $q = \infty$, it is easy to see that (45) persists, by minor modifications of the above arguments.

The proof of (46) is a little more difficult because it involves a full limit. Let $x \geq 2$ and j_0 denote the least integer $\geq x$. We see that, as $\alpha_j = O(1/j)$,

$$\begin{aligned} \int_0^x W^{-s} &\leq \int_{(0,x)\setminus \cup_{j=3}^{j_0}(j-\alpha_j, j+\alpha_j)} W_2^{-s} + \sum_{j=3}^{j_0-1} \int_{j-\alpha_j}^{j+\alpha_j} W^{-s} \\ &\quad + \int_{[j_0-\alpha_{j_0}, x]} W^{-s} \\ &\leq \int_0^x W_2^{-s} + C \sum_{j=3}^{j_0-1} \alpha_j W_2^{-s}(j)(j \log j)^s \\ &\quad + C\alpha_{j_0} W_2^{-s}(x)(j_0 \log j_0)^s \\ &\leq C W_2(x)^{-s}/x + C W_2^{-s}(x) x^{s-1} (\log x)^{s-\beta}, \end{aligned}$$

as for large enough j, and some $\theta < 1$ independent of j,

$$\frac{\alpha_j W_2^{-s}(j)(j \log j)^s}{\alpha_{j-1} W_2^{-s}(j-1)((j-1)\log(j-1))^s} < \theta.$$

We also used (47). Then this and (43) give

$$\begin{aligned}\left\|W^{-1}\right\|_{L_s[0,x]} \|W\|_{L_r[x,\infty)} \\ &\le CW_2^{-1}(x)x^{1-1/s}(\log x)^{1-\beta/s} \|W_2\|_{L_r[x,\infty)} \\ &\le CW_2^{-1}(x)x^{1-1/s}(\log x)^{1-\beta/s} W_2(x)x^{-1/r} \\ &= C(\log x)^{1-\beta/s} \longrightarrow 0,\end{aligned}$$

$x \to \infty$ as $\beta > s$, recall (38).

(b) This is similar to (a). Note that, as $p > r \ge 1$, so $r, q < \infty$. By (40), if $p < \infty$,

$$\int_j^\infty W^p \ge C \int_j^{j+\alpha_j/2} (j \log j)^p W_2^p \ge C\alpha_j j^p (\log j)^p W_2(j)^p.$$

Moreover,

$$\int_0^j W^{-q} \ge \int_{j-1+\alpha_{j-1}}^{j-\alpha_j} W_2^{-q} \ge Cj^{-1} W_2(j)^{-q},$$

by (47). Then

$$\begin{aligned}\left\|W^{-1}\right\|_{L_q[0,j]} \|W\|_{L_p[j,\infty)} &\ge Cj^{-1/q}\alpha_j^{1/p} j \log j \\ &= C(\log j)^{1-\beta/p} \to \infty,\end{aligned}$$

as $\beta < p$ (recall (41)). If $p = \infty$, this argument requires minor modifications. So we have (45). Next, if j_1 is the largest integer $\le x$,

$$\begin{aligned}\int_x^\infty W^r &\le \int_{(x,\infty)\setminus \cup_{j=j_1}^\infty (j-\alpha_j, j+\alpha_j)} W_2^r \\ &\quad + \sum_{j=j_1}^\infty \int_{j-\alpha_j}^{j+\alpha_j} W_2^r (j\log j)^r + \int_{[x, j_1+\alpha_{j_1}]} W_2^r (j_1 \log j_1)^r \\ &\le \int_x^\infty W_2^r + C \sum_{j=j_1+1}^\infty \alpha_j (j \log j)^r W_2^r(j) \\ &\quad + CW_2^r(x)\alpha_{j_1}(j_1 \log j_1)^r \\ &\le CW_2(x)^r/x + j_1^{r-1}(\log j_1)^{r-\beta} W_2^r(x) \\ &\le Cx^{r-1}(\log x)^{r-\beta} W_2^r(x),\end{aligned}$$

by (48) and as again for large j and some $\theta < 1$,

$$\frac{\alpha_j (j\log j)^r W_2^r(j)}{\alpha_{j-1}((j-1)\log(j-1))^r W_2^r(j-1)} < \theta.$$

Then (46) and (47) give

$$\begin{aligned}\left\|W^{-1}\right\|_{L_s[0,x]} \|W\|_{L_r[x,\infty)} & \\ &\leq C\left\|W_2^{-1}\right\|_{L_s[0,x]} W_2(x)x^{1-1/r}(\log x)^{1-\beta/r} \\ &\leq CW_2^{-1}(x)x^{-1/s}W_2(x)x^{1-1/r}(\log x)^{1-\beta/r} \\ &= C(\log x)^{1-\beta/r} \longrightarrow 0,\end{aligned}$$

$x \to \infty$, as $\beta > r$ (recall (41)). □

Proof of Theorem 1.3. This follows directly from the limit conditions in Lemma 4.2 and from Theorem 1.2. □

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