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ON THE DIOPHANTINE EQUATION $(x^2+k)(y^2+k)=(z^2+k)^2$

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ABSTRACT. In this paper we show that if $k \in \mathbf{Z}$ can be represented in the form $k = \pm(a^2 - 2b^2)$, then there exists an infinite family of three-term geometric progressions of numbers of the form x^2+k. Furthermore, we prove that the set of $k \in \mathbf{Z}$, such that there exists a four-term geometric progression of numbers of the form x^2+k is infinite.

1. Introduction. Schinzel and Sierpiński in [**1**] showed that all solutions of the equation

$$(x^2-1)(y^2-1)=\left(\left(\frac{x-y}{2}\right)^2-1\right)^2 \tag{1}$$

in positive integers x, y, $x<y$, are of the form $x=x_n$, $y=x_{n+1}$, $n=0,1,\dots$, where $x_0=1$, $x_1=3$ and generally $x_n=6x_{n-1}-x_{n-2}$.

Szymiczek in [**2**] generalized the above by showing that all solutions t, x, y, $x<y$ of the equation

$$(x^2-t^2)(y^2-t^2)=\left(\left(\frac{x-y}{2}\right)^2-t^2\right)^2 \tag{2}$$

in distinct positive integers are of the form

$$t=|m^2-2n^2|s, \quad x=(m^2+2n^2)s, \quad y=(3m^2+8mn+6n^2)s,$$

where m, n, s are integers.

Let us point out that (1) and (2) are particular cases of the equation

$$(x^2+k)(y^2+k)=(z^2+k)^2. \tag{3}$$

The question about integer solutions of the equation (3) with fixed $k \in \mathbf{Z}$ is equivalent to the question whether there exists a geometric

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progression of the length three created with values of the polynomial x^2+k.

The fundamental question which arises here is how big can a set $K \subset \mathbf{Z}$ be, such that for $k \in K$ the equation (3) has infinitely many solutions in integers? By examining the particular Pell-type equation we show that if $k \in \mathbf{Z}$ can be represented in the form $k = \pm(a^2-2b^2)$, then the diophantine equation (3) has infinitely many solutions (Theorem 2.1). Next, we prove that the set of $k \in \mathbf{Z}$, such that there exists a four-term geometric progression of numbers of the form x^2+k is infinite (Theorem 3.2). The solutions are obtained from the integer points on a particular quartic surface.

2. The equation $(x^2+k)(y^2+k)=(z^2+k)^2$. Let $f_k(x) := x^2+k$, $k(a,\ b) := a^2-2b^2$ and $K = \{\pm k(a,b) :\ a,\ b \in \mathbf{Z}\}$.

Then we have

Theorem 2.1. *For each $k \in K$ the equation* (3) *has infinitely many solutions in integers.*

Proof. Since k is representable by $-p^2+2q^2$ if and only if k is representable by p^2-2q^2, it is sufficient to prove the theorem when $k = a^2-2b^2$. Put $z=(y-x)/2$ and then, as in [**1, 2**], we have

$$f_k(x)f_k(y) - f_k\left(\frac{y-x}{2}\right)^2 = \frac{1}{16}(x+y)^2(8k-x^2+6xy-y^2).$$

Therefore it is enough to show that for $h(x,\ y) := x^2-6xy+y^2$ the equation $h(x,\ y)=8k$ has infinitely many solutions in integers $x,\ y$. But this equation represents the Pell equation $(x-3y)^2-8y^2=8k$, where we know an initial solution $(x,\ \ y) = (a+2b,\ \ -a+2b)$; consequently, there are infinitely many solutions in integers. $\square$

Now taking $a=-3$, $b=2$, we know that for $k=1$ the equation (3) has infinitely many solutions and in this case the sequence of solutions is the same as the one mentioned in the reference to the work by Schinzel and Sierpiński. Next, the solution proposed by Szymiczek is obtained by solving the equation $a^2-2b^2=t^2$. This equation has integer solution

$a = 1$, $b = 0$, $t = 1$, which can be used for the parametrization of solutions of (2).

3. Geometric progressions of length 4. The aim of this paragraph is to show that the set of $k \in \mathbf{Z}$ such that there exists a geometric progression of length four containing numbers of the form $x^2 + k$ is infinite. We are interested in solving in integers x, y, z and t the following system of equations

$$\frac{f_k(y)}{f_k(x)} = \frac{f_k(z)}{f_k(y)} = \frac{f_k(t)}{f_k(z)}. \tag{4}$$

It is clear that, with a fixed k, the above system has a solution in integers if and only if the system of equations

$$\begin{cases} x^2 + k = A, \\ y^2 + k = Aq, \\ z^2 + k = Aq^2, \\ t^2 + k = Aq^3, \end{cases} \tag{5}$$

has a solution in integers x, y, z, t, A and q. Solving the first three equations of the system (5) with respect to k, A and q, we have

$$\begin{cases} k = (y^4 - x^2\,z^2)/(x^2 - 2\,y^2 + z^2), \\ A = (x^4 - 2\,x^2\,y^2 + y^4)/(x^2 - 2\,y^2 + z^2), \\ q = (y^2 - z^2)/(x^2 - y^2). \end{cases}$$

Substituting the calculated values into the last equation in (5) we obtain the equation

$$t^2(x^2 - y^2) + y^2(y^2 - z^2) + z^2(z^2 - x^2) = 0. \tag{6}$$

We can see that every solution of the equation (6) in integers x, y, z and t corresponds to a certain solution of the system (5) with $k = (y^4 - x^2\,z^2)/(x^2 - 2\,y^2 + z^2)$. A solution in integers x, y, z and t of (6) will be called trivial if $y^4 - x^2\,z^2 = 0$ or $x^2 - 2\,y^2 + z^2 = 0$. It is clear that the equation (6) has infinitely many trivial solutions. Indeed, for p, $q \in \mathbf{Z}$ the solution $x = z = p$, $y = t = q$ is trivial. However, we are interested in nontrivial integer solutions of (6). Now we will show the following

Theorem 3.1. *The equation* (6) *has infinitely many nontrivial solutions in integers.*

Proof. Let E be the surface in $\mathbf{P}^3(\mathbf{Q})$ given by the equation (6). It is worth noting that the set S of singular points of E consists of points of the form $(\pm 1,\ \pm 1,\ \pm 1,\ \pm 1)$ and points $(1,\ 0,\ 0,\ 0)$, $(0,\ 0,\ 0,\ 1)$. Let $P = (p,\ q,\ p,\ q)$, and let us consider a line passing through P. The parametric equation of this line is

$$x = a\,T + p, \quad y = b\,T + q, \quad z = c\,T + p, \quad t = d\,T + q. \tag{7}$$

Substituting x, y, z and t from (7) into the equation of E, we obtain the equation

$$T\left(\sum_{i=0}^{3} A_i(a,b,c,d)\,T^i\right) = 0, \tag{8}$$

where

$$\begin{aligned}
A_0(a,b,c,d) &= -2(p-q)(p+q)(ap-cp+bq-dq),\\
A_1(a,b,c,d) &= -a^2p^2 - b^2p^2 - 4acp^2 + 5c^2p^2 + d^2p^2 - 4bcpq + 4adpq\\
&\quad + a^2q^2 + 5b^2q^2 - c^2q^2 - 4bdq^2 - d^2q^2,\\
A_2(a,b,c,d) &= -2(a^2cp + b^2cp + ac^2p - 2c^3p - ad^2p - 2b^3q + bc^2q)\\
&\quad - 2(-a^2dq + b^2dq + bd^2q),\\
A_3(a,b,c,d) &= b^4 - a^2c^2 - b^2c^2 + c^4 + a^2d^2 - b^2d^2.
\end{aligned}$$

From the system of equations

$$\begin{cases} A_0(a,b,c,d) = 0,\\ A_1(a,b,c,d) = 0,\end{cases}$$

we obtain

$$\begin{cases} c = (2bpq(p^2-q^2) + a(p^2+q^2)^2)/(p^4+4p^2q^2-q^4),\\ d = (2apq(p^2-q^2) - b(p^4-6p^2q^2+q^4))/(p^4+4p^2q^2-q^4).\end{cases} \tag{9}$$

Putting (9) into A_2 and A_3, we see that the equation (8) has a triple root at zero and rational root

$$T = -\frac{4pq(p^2+q^2)(p^4+4p^2q^2-q^4)}{aq\left(3p^6+11p^4q^2+p^2q^4+q^6\right) + bp\left(p^6+9p^4q^2+11p^2q^4-5q^6\right)}.$$

Putting the calculated quantities c, d and T into (7) we obtain a parametric family of rational points on E. Reducing and simplifying the denominators, we get a polynomial family of solutions

$$\begin{cases} x(p,\ q) = p\,(p^6 + 9p^4q^2 + 11p^2q^4 - 5q^6), \\ y(p,\ q) = q\,(3p^6 + 11p^4q^2 + p^2q^4 + q^6), \\ z(p,\ q) = p\,(p^6 + p^4q^2 + 11p^2q^4 + 3q^6), \\ t(p,\ q) = q\,(-5p^6 + 11p^4q^2 + 9p^2q^4 + q^6). \end{cases}$$

Direct examination shows that if $p,\ q \in \mathbf{Z} \setminus \{0\}$ and $p \pm q \neq 0$, then the corresponding point $P(p,\ q) = (x(p,\ q),\ y(p,\ q),\ z(p,\ q),\ t(p,\ q))$ on E is nontrivial. $\square$

The value of k corresponding to the parametrization from Theorem 2 is a rational function of variables p, q. In order to obtain $k \in \mathbf{Q}[p, q]$, we take the parametrization

$$\begin{cases} X(p,\ q) = p\,(p^2 - q^2)\,(p^6 + 9p^4q^2 + 11p^2q^4 - 5q^6), \\ Y(p,\ q) = q\,(p^2 - q^2)\,(3p^6 + 11p^4q^2 + p^2q^4 + q^6), \\ Z(p,\ q) = p\,(p^2 - q^2)\,(p^6 + p^4q^2 + 11p^2q^4 + 3q^6), \\ T(p,\ q) = q\,(p^2 - q^2)\,(-5p^6 + 11p^4q^2 + 9p^2q^4 + q^6). \end{cases}$$

For such X, Y, Z and T we obtain

$$\begin{aligned} k(p,\ q) = -\frac{1}{2}(p^2 + q^2)(p^4 + 6p^2q^2 + q^4)\times \\ \times\,(p^6 - 3p^4q^2 - 13p^2q^4 - q^6)(p^6 + 13p^4q^2 + 3p^2q^4 - q^6). \end{aligned}$$

Therefore, we have shown

Theorem 3.2. *Let $p,\ q \in \mathbf{Z} \setminus \{0\}$ and $p \pm q \neq 0$. For $k = k(p,\ q)$, the system of equations*

$$\frac{f_k(y)}{f_k(x)} = \frac{f_k(z)}{f_k(y)} = \frac{f_k(t)}{f_k(z)}$$

has a nontrivial solution in integers.

Unfortunately the number $k(p,\ q)$ rises very quickly with the growth of $p,\ q$. One can therefore ask what is the smallest possible k such that the system (4) has a nontrivial solution? We have carried out a search to find solutions of (4) with the use of a computer. For $\max\{x, y, z\} < 1000$, $|k| < 100$ the only square-free integers k allowing nontrivial solution of the system (4) are $k = -91,\ -89,\ -34,\ 11,\ 39,\ 95$. Certainly, the list may not be complete. In the table below we list solutions corresponding to those numbers.

k	x	y	z	t
-91	9	11	1	19
-89	13	43	197	923
-34	10	32	122	472
11	3	7	13	23
39	3	21	69	219
39	5	11	19	31
95	16	29	49	81

It is worth noting that, as P. Fermat proved, no four squares exist forming an arithmetic progression, and hence there is no polynomial of degree two whose four values form an arithmetic progression. As we can see, in the case of geometric progression the situation is different. A question arises to determine the maximal number n, for which there exists a nontrivial solution of the system of equations

$$(10)\qquad \frac{f_k(x_2)}{f_k(x_1)} = \frac{f_k(x_3)}{f_k(x_2)} = \cdots = \frac{f_k(x_{n-1})}{f_k(x_{n-2})} = \frac{f_k(x_n)}{f_k(x_{n-1})}$$

in integers. The search done in the case $n = 5$ suggests that for this (and also bigger) values of n our problem does not have any nontrivial solutions. This leads us to the following

Conjecture 3.3. *Let $k \in \mathbf{Z}\setminus\{0\}$. For $n > 4$ the system of equations* (10) *does not have any nontrivial solutions in integer numbers.*

Professor A. Schinzel informed me that the problem concerning the solvability in integers of the diophantine equation $(x^2 + k)(y^2 + k) =$

$(z^2 + k)^2$, where $k = p^2 - 2q^2$ was proposed by J.A.H. Hunter as an advanced problem in American Mathematical Monthly (Problem 5020, April 1962, page 316). Essentially the same result as our Theorem 2.1 was obtained by Hunter and Venkatchalam Iyer (American Mathematical Monthly, May 1963, page 574).

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