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INTEGRALS AND PHASE PORTRAITS OF PLANAR QUADRATIC DIFFERENTIAL SYSTEMS WITH INVARIANT LINES OF AT LEAST FIVE TOTAL MULTIPLICITY

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ABSTRACT. In this article we prove that all real quadratic differential systems $dx/dt = p(x,y)$, $dy/dt = q(x,y)$, with $\gcd(p,q) = 1$, having invariant lines of total multiplicity at least five and a finite set of singularities at infinity, are Darboux integrable having integrating factors whose inverses are polynomials over $\mathbf{R}$. We also classify these systems under two equivalence relations: 1) topological equivalence and 2) equivalence of their associated cubic projective differential equations when the cubic projective differential equations are acted upon by the group $\mathrm{PGL}\,(3,\mathbf{R})$. For each one of the 28 topological classes obtained, we give necessary and sufficient conditions for a quadratic system to belong to this class, in terms of its coefficients in $\mathbf{R}^{12}$.

1. Introduction. We consider here real planar differential systems of the form

$$(S) \qquad \frac{dx}{dt} = p(x,y), \qquad \frac{dy}{dt} = q(x,y), \tag{1.1}$$

where p, $q \in \mathbf{R}[x,y]$, i.e., p, q are polynomials in x, y over $\mathbf{R}$, their associated vector fields

$$\widetilde{D} = p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y} \tag{1.2}$$

and differential equations

$$q(x,y)\,dx - p(x,y)\,dy = 0. \tag{1.3}$$

We call the *degree* of a system (1.1) (or of a vector field (1.2) or of a differential equation (1.3)) the integer $n = \deg(S) = \max(\deg p, \deg q)$. In particular we call *quadratic* a differential system (1.1) with $n = 2$.

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There are several hard open problems on the class of all quadratic differential systems (1.1). Among them the most famous one is the second part of Hilbert's 16th problem which asks for the determination of the so-called Hilbert number $H(n)$ for this class, i.e., for $n = 2$ where

$$H(n) = \max\{\mathrm{LC}\,(S) \,\big|\, \deg\,(S) = n\}$$

and $\mathrm{LC}\,(S)$ is the number of limit cycles of (S).

In [**3**] the authors studied the class of all quadratic systems possessing a second order weak focus. In the bifurcation diagram drawn in [**3**] the maximum number of limit cycles (two limit cycles) which one sees for this class occurs in perturbations of an integrable quadratic system with a rational first integral which foliates the plane into conic curves. This shows the deep connection existing between perturbations of integrable quadratic systems (even with a rational first integral) and the second part of Hilbert's 16th problem and adds to the motivation for studying the class of integrable quadratic systems having invariant algebraic curves, see Definition 1.4. This study is interesting for its own sake as it is at the crossroads of differential equations and algebraic geometry.

Each differential system (1.1) generates a complex differential system when the variables range over $\mathbf{C}$. In [**11**] Darboux gave a beautiful geometric method of integration of planar complex differential equations (1.3) using algebraic curves which are invariant for the equations, see Definition 1.4.

Poincaré was enthusiastic about the work of Darboux [**11**], which he called "admirable" in [**17**]. This method of integration was applied to give unified proofs of integrability for several families of systems (1.1). For example, in [**22**] it was applied to show in a unified way (unlike previous proofs which used ad hoc methods) the integrability of planar quadratic systems possessing a center.

A brief and easily accessible exposition of the method of Darboux can be found in the survey article [**21**].

The topic of Darboux's paper [**11**] is best treated using the language of differential algebra, subject which started to be developed in the work of Ritt (1893–1951), long after Darboux wrote his paper [**11**]. The term *Differential Algebra* was introduced by Ellis Kolchin, who was a student of Ritt and who, as Buium and Cassidy said in [**5**], "deepened and modernized differential algebra and developed differential algebraic geometry and differential algebraic groups."

Differential algebra began to be developed in the 1930s, e.g., [**19**], as a result of the influence of Emmy Noether's work of the 1920s in algebra. In his book [**20**], Ritt paid this tribute to Noether: "the form in which the results of differential algebra are presented has been deeply influenced by her teachings."

As in this paper we are concerned with questions regarding integrability in the sense of Darboux, we shall use here (in a minimal way) the language of differential algebra. In the present article we are concerned with questions regarding Darboux integrability for a specific class of quadratic systems (1.1), namely, the class of systems (1.1) possessing invariant straight lines of total multiplicity at least five. We work with the notion of multiplicity of an invariant line introduced by us in [**23**].

The goal of this article is to present a full study of this class by:

• proving that all systems in this class are integrable via the method of Darboux yielding integrating factors whose inverses are polynomials in x, y over $\mathbf{R}$ and elementary first integrals of Darboux type, see Definition 1.7 below;

• constructing all the topologically distinct phase portraits of the systems in this class (we have 28 such phase portraits);

• giving necessary and sufficient conditions, invariant under the action of the affine group and time rescaling, in terms of the twelve coefficients of the systems, for which a specific phase portrait is realized;

• determining the representatives of the orbits of their associated projective differential equations under the action of the real projective group $\mathrm{PGL}\,(3, \mathbf{R})$.

To do this, we first need to recall some basic notions. Whenever a definition below is given for a system (1.1) or equivalently for a vector field (1.2), this definition could also be given for an equation (1.3) and vice versa. For brevity we sometimes state only one of the possibilities.

As we are here concerned with real differential systems (S) we first recall below the notion of first integral, integrability and integrating factor for such systems.

Definition 1.1. Let $F : U \to \mathbf{R}$, $U \subseteq \mathbf{R}^2$ a C^1 function on an open set U. If F is constant on all solution curves $(x(t), y(t))$ in U of a system (S), we say that F is a first integral on U of (S). If there exists such an F which is nonconstant on any open subset of U we say that this system is integrable on U.

Remark 1.1. We note that such a C^1 function $F : U \to \mathbf{R}$ is a first integral on U of (1.1) if and only if for all solutions $(x(t), y(t))$ with values in U of (1.1) defined when t is in an open interval of $\mathbf{R}$, we have $(dF(x(t), y(t)))/dt = 0$ for all t in this interval, or equivalently,

$$\widetilde{D}F \equiv p(x,y)\frac{\partial F}{\partial x} + q(x,y)\frac{\partial F}{\partial y} = 0 \tag{1.4}$$

on U.

Definition 1.2. An integrating factor of an equation (1.3) on an open subset U of $\mathbf{R}^2$ is a C^1 function $R(x,y) \not\equiv 0$ such that the 1-form

$$\omega = R\,q(x,y)\,dx - R\,p(x,y)\,dy$$

is exact, i.e., there exists a C^1 function $F : U \to \mathbf{K}$ on U such that

$$\omega = dF. \tag{1.5}$$

Remark 1.2. We observe that if R is an integrating factor on U of (1.3) then the function F such that $\omega = Rq\,dx - Rp\,dy = dF$ is a first integral of the equation $w = 0$ (or a system (1.1)). In this case we necessarily have on U:

$$\frac{\partial(Rq)}{\partial y} = -\frac{\partial(Rp)}{\partial x} \tag{1.6}$$

and, developing the above equality, we obtain $(\partial R/\partial x)p + (\partial R/\partial y)q = -R((\partial p/\partial x) + (\partial q/\partial y))$ or equivalently,

$$\widetilde{D}R = -R\operatorname{div}\widetilde{D}. \tag{1.7}$$

In view of Poincaré's lemma, see for example [**26**], if $R(x,y)$ is a C^1 function on a star-shaped open set U of $\mathbf{R}^2$, then $R(x,y)$ is an integrating factor of (1.3) if and only if (1.6), or equivalently (1.7), holds on U.

In this work we shall apply to our real quadratic system (1.10) the method of integration of Darboux which was developed for complex differential equations (1.3). This method uses multiple-valued complex functions of the form:

$$\begin{gathered} F = e^{G(x,y)} f_1(x,y)^{\lambda_1} \cdots f_s(x,y)^{\lambda_s}, \\ G \in \mathbf{C}(x,y), \quad f_i \in \mathbf{C}[x,y], \quad \lambda_i \in \mathbf{C}, \end{gathered} \tag{1.8}$$

$G = G_1/G_2$, $G_i \in \mathbf{C}[x,y]$, f_i irreducible over $\mathbf{C}$. It is clear that in general an expression (1.8) makes sense only for $G_2 \neq 0$ and for points $(x,y) \in \mathbf{C}^2 \setminus (\{G_2(x,y)=0\} \cup \{f_1(x,y)=0\} \cup \cdots \cup \{f_s(x,y)=0\})$.

The above expression (1.8) yields a multiple-valued function on

$$\mathcal{U} = \mathbf{C}^2 \setminus (\{G_2(x,y)=0\} \cup \{f_1(x,y)=0\} \cup \cdots \cup \{f_s(x,y)=0\}).$$

To continue our discussion, we introduce at this point a bit of differential algebra.

The function F in (1.8) belongs to a differential field extension of $(\mathbf{C}(x,y), (\partial/\partial x), (\partial/\partial y))$ obtained by adjoining to $\mathbf{C}(x,y)$ a finite number of algebraic and of transcendental elements over $\mathbf{C}(x,y)$. For example $f(x,y)^{1/2}$ is an expression of the form (1.8), when $f \in \mathbf{C}[x,y] \setminus \{0\}$. This function belongs to the algebraic differential field extension $(\mathbf{C}(x,y)[u], (\partial/\partial x), (\partial/\partial y))$ of $(\mathbf{C}(x,y), (\partial/\partial x), (\partial/\partial y))$ obtained by adjoining to $\mathbf{C}(x,y)$ a root of the equation $u^2 - f(x,y) = 0$. In general, the expression (1.8) belongs to a differential field extension which is not necessarily algebraic. Indeed, for example, this occurs if $G(x,y)$ is not a constant.

Definition 1.3. A function F in a differential field extension K of $(\mathbf{C}(x,y), (\partial/\partial x), (\partial/\partial y))$ which is finite over $\mathbf{C}(x,y)$, is a first integral (integrating factor, respectively inverse integrating factor) of a complex differential system (1.1) or a vector field (1.2) or a differential equation (1.3) if $\widetilde{D}F = 0$ ($\widetilde{D}F = -F \operatorname{div} \widetilde{D}$, respectively $\widetilde{D}F = F \operatorname{div} \widetilde{D}$).

In 1878 Darboux introduced the notion of the invariant algebraic curve for differential equations on the complex projective plane. This notion can be adapted for equations (1.3) on $\mathbf{C}^2$ or equivalently for systems (1.1) or vector fields (1.2).

Definition 1.4 (Darboux [**11**]). An affine algebraic curve $f(x,y) = 0$, $f \in \mathbf{C}[x,y]$, $\deg f \geq 1$ is invariant for an equation (1.3) or for a system (1.1) if and only if $f \mid \widetilde{D}f$ in $\mathbf{C}[x,y]$, i.e., $k = (\widetilde{D}f/f) \in \mathbf{C}[x,y]$. In this case k is called the cofactor of f.

Definition 1.5 [**11**]. An algebraic solution of an equation (1.3) (respectively (1.1), (1.2)) is an invariant algebraic curve $f(x,y) = 0$, $f \in \mathbf{C}[x,y]$ ($\deg f \geq 1$) with f an irreducible polynomial over $\mathbf{C}$.

Darboux showed that if an equation (1.3) or (1.1) or (1.2) possesses a sufficient number of such invariant algebraic solutions $f_i(x,y) = 0$, $f_i \in \mathbf{C}[x,y]$, $i = 1,2,\dots,s$, then the equation has a first integral of the form (1.8).

Definition 1.6. An expression of the form $F = e^{G(x,y)}$, $G(x,y) \in \mathbf{C}(x,y)$, i.e., G is rational over $\mathbf{C}$, is an exponential factor[1] (see Endnotes) for a system (1.1) or an equation (1.3) if and only if $k = (\widetilde{D}F/F) \in \mathbf{C}[x,y]$. In this case k is called the cofactor of the exponential factor F.

Proposition 1.1 (Christopher [**9**]). *If an equation* (1.3) *admits an exponential factor* $e^{G(x,y)}$ *where* $G(x,y) = (G_1(x,y)/G_2(x,y))$, $G_1, G_2 \in \mathbf{C}[x,y]$ *then* $G_2(x,y) = 0$ *is an invariant algebraic curve of* (1.3).

Definition 1.7. We say that a system (1.1) or an equation (1.3) has a Darboux first integral, respectively Darboux integrating factor, if it admits a first integral, respectively integrating factor, of the form $e^{G(x,y)} \prod_{i=1}^{s} f_i(x,y)^{\lambda_i}$, where $G(x,y) \in \mathbf{C}(x,y)$ and $f_i \in [x,y]$, $\deg f_i \geq 1$, $i = 1,2,\dots,s$, f_i irreducible over $\mathbf{C}$ and $\lambda_i \in \mathbf{C}$.

Proposition 1.2 [**11**]. *If an equation* (1.3) (*or* (1.1), *or* (1.2)) *has an integrating factor, or first integral, of the form* $F = \prod_{i=1}^{s} f_i^{\lambda_i}$, *then for all* $i \in \{1, \dots, s\}$, $f_i = 0$ *is an algebraic invariant curve of* (1.3) ((1.1), (1.2)).

In [**11**] Darboux proved the following remarkable theorem of integrability using invariant algebraic solutions of differential equation (1.3):

Theorem 1.1 [**11**]. *Consider a differential equation* (1.3) *with* $p, q \in \mathbf{C}[x, y]$. *Let us assume that* $m = \max(\deg p, \deg q)$ *and that the equation admits* s *algebraic solutions* $f_i(x, y) = 0$, $i = 1, 2, \dots, s$ $(\deg f_i \geq 1)$. *Then we have*:

I. *If* $s = m(m+1)/2$, *then there exists* $\lambda = (\lambda_1, \dots, \lambda_s) \in \mathbf{C}^s \setminus \{0\}$ *such that* $R = \prod_{i=1}^{s} f_i(x, y)^{\lambda_i}$ *is an integrating factor of* (1.3).

II. *If* $s \geq m(m+1)/2+1$, *then there exists* $\lambda = (\lambda_1, \dots, \lambda_s) \in \mathbf{C}^s \setminus \{0\}$ *such that* $F = \prod_{i=1}^{s} f_i(x, y)^{\lambda_i}$ *is a first integral of* (1.3).

Remark 1.3. We stated the theorem for the equation (1.3) but clearly we could have stated it for the vector field $\widetilde{D}$ (1.2) or for the polynomial differential system (1.1). We point out that Darboux's work was done for differential equations in the complex projective plane. The above formulation is an adaptation of his theorem for the complex affine plane.

In [**14**] Jouanolou proved the following theorem which improves part II of Darboux's theorem.

Theorem 1.2 [**14**]. *Consider a polynomial differential equation* (1.3) *over* $\mathbf{C}$ *and assume that it has* s *algebraic solutions* $f_i(x, y) = 0$, $i = 1, 2, \dots, s$ $(\deg f_i \geq 1)$. *Suppose that* $s \geq m(m+1)/2 + 2$. *Then there exists* $(n_1, \dots, n_s) \in \mathbf{Z}^s \setminus \{0\}$ *such that* $F = \prod_{i=1}^{s} f_i(x, y)^{n_i}$ *is a first integral of* (1.3). *In this case* $F \in \mathbf{C}(x, y)$, *i.e.*, F *is a rational function over* $\mathbf{C}$.

The above-mentioned theorem of Darboux gives us sufficient conditions for integrability via the method of Darboux using algebraic solutions of systems (1.1). However, these conditions are not necessary as can be seen from the following example. The system

$$\frac{dx}{dt} = -y - x^2 - y^2, \quad \frac{dy}{dt} = x + xy$$

has two invariant algebraic curves: the invariant line $1 + y = 0$ and a conic invariant curve $f = 6x^2 + 3y^2 + 2y - 1 = 0$. This system is integrable having as first integral $F = (1+y)^2 f$ but here $s = 2 < 3 = m(m+1)/2$.

Sufficient conditions for Darboux integrability were obtained by Christopher and Kooij in [**15**] and Zoladek in [**27**]. Their theorems say that if a system has s invariant algebraic solutions in "generic position" (with "generic" as defined in the work) such that $\sum_{i=1}^{s} \deg f_i = m + 1$ then the system has as an inverse integrating factor of the form $\prod_{i=1}^{s} f_i$. But their theorem does not cover the above case as the two curves are not in "generic position." Indeed, the line $1 + y = 0$ is tangent to the curve $f = 0$ at $(0, -1)$. For similar reasons the above example is not covered by the more general result: Theorem 7.1 in [**10**]. Other sufficient conditions for integrability covering the example above were given in [**8**]. However, to this day, we do not have necessary and sufficient conditions for Darboux integrability and the search is on for finding such conditions.

Problem resulting from the work [11] of Darboux. Give necessary and sufficient conditions for a polynomial system (1.1) to have: (i) a polynomial inverse integrating factor; (ii) an integrating factor of the form $\prod_{i=1}^{s} f_i(x,y)^{\lambda_i}$; (iii) a Darboux integrating factor (or a Darboux first integral); (iv) a rational first integral.

The last problem (iv), above, appeared as the problem of algebraic integrability in 1891 in the articles [**17, 18**] of Poincaré. In recent years there has been much activity in this area of research, e.g., [**6, 7**].

The goal of this work is to provide us with specific data to be used along with similar material for higher degree curves, for the purpose of dealing with questions regarding Darboux and algebraic integrability. We collect here in a systematic way information starting with quadratic systems having invariant lines of total multiplicity at least five. This material may also be used in studying quadratic systems which are small perturbations of integrable ones. As Arnold said in [**1**, page 405], "...these integrable cases allow us to collect a large amount of information about the motion in more important systems. . . .” In fact, as we have already indicated at the beginning of this introduction, the maximum number of limit cycles of some subclasses of the quadratic class can be obtained by perturbing integrable systems even having a rational first integral.

Definition 1.8. We call configuration of invariant lines of a system (1.1) the set of all its (complex) invariant lines (which could have real coefficients), each endowed with its own multiplicity [**23**] and together with all the real singular points of this system located on these lines, each one endowed with its own multiplicity.

One of the results in this article is that all quadratic differential systems which have invariant lines of at least five total multiplicity are integrable via the method of Darboux, having polynomial inverse integrating factors.

This article is organized as follows:

In Section 2, we associate to each real quadratic system (1.1) possessing invariant lines with corresponding multiplicities, a divisor on the complex projective plane which encodes this information. We also define several integer-valued affine invariants of such systems using divisors on the line at infinity or zero-cycles on $\mathbf{P}_2(\mathbf{C})$ defined in [**23**] and [**24**], which encode the multiplicities of the singularities of the systems.

In Section 3, respectively Section 4, we classify in Theorem 3.1, respectively Theorem 4.1, all quadratic systems having invariant lines of total multiplicity six, respectively five, modulo the action of the group $Aff\,(2,\mathbf{R})\times\mathbf{R}^*$ of real affine transformation and time rescaling ($\mathbf{R}^*=\mathbf{R}\backslash\{0\}$). In Table 1, respectively Table 2, we give representatives of the orbits under this action. All these systems have Darboux first integrals listed in Tables 1 and 2 and polynomial inverse integrating factors.

Using the integer-valued invariants constructed in Section 2, we define a multi integer-valued invariant which distinguishes the configurations of all quadratic systems (1.1) having invariant lines of total multiplicity at least five. We also construct the corresponding phase portraits of such systems. The classifications of the systems are stated in Theorems 3.2 and 4.2 using Diagrams 1 and 3 which sum up the information.

In Theorem 3.3, respectively Theorem 4.3, we give the classifications of all quadratic systems with invariant lines of total multiplicity six, respectively five, modulo the action of the group $\mathrm{PGL}\,(3,\mathbf{R})$ of projective transformations of their associated differential equations on the real projective plane. This classification yields only 6, respectively 16, classes, while the classification modulo $\mathit{Aff}\,(2,\mathbf{R})\times\mathbf{R}^*$ in [**23**] yields 11, respectively 30, classes.

In Theorem 5.2 we give necessary and sufficient conditions for quadratic systems with invariant lines of total multiplicity at least five to topologically distinguish the 28 possible phase portraits for this class. These conditions are formulated only in terms of algebraic invariants and comitants, see [**25**], depending upon the coefficients of the systems: $\mathbf{a}\in\mathbf{R}^{12}$.

2. Divisors associated to invariant lines configurations. Consider real differential systems of the form:

$$(2.1)\qquad (\mathrm{S})\qquad \begin{cases} dx/dt = p_0 + p_1(x,y) + p_2(x,y) \equiv p(x,y),\\ dy/dt = q_0 + q_1(x,y) + q_2(x,y) \equiv q(x,y) \end{cases}$$

with

$$\begin{aligned} p_0 &= a_{00}, \qquad p_1(x,y) = a_{10}x + a_{01}y,\\ p_2(x,y) &= a_{20}x^2 + 2a_{11}xy + a_{02}y^2,\\ q_0 &= b_{00}, \qquad q_1(x,y) = b_{10}x + b_{01}y,\\ q_2(x,y) &= b_{20}x^2 + 2b_{11}xy + b_{02}y^2. \end{aligned}$$

Let $a=(a_{00},a_{10},a_{01},a_{20},a_{11},a_{02},b_{00},b_{10},b_{01},b_{20},b_{11},b_{02})$ be the 12-tuple of the coefficients of system (2.1), and denote

$$\mathbf{R}[a,x,y]=\mathbf{R}[a_{00},a_{10},a_{01},a_{20},a_{11},a_{02},b_{00},b_{10},b_{01},b_{20},b_{11},b_{02},x,y].$$

Notation 2.1. Whenever we consider a concrete point in $\mathbf{R}^{12}$ we shall denote it in boldface: $\mathbf{a}=(\mathbf{a}_{00},\mathbf{a}_{10}\ldots,\mathbf{b}_{02})\in\mathbf{R}^{12}$.

Notation 2.2. Let

$$P(X,Y,Z)=p_0(\mathbf{a})Z^2+p_1(\mathbf{a},X,Y)Z+p_2(\mathbf{a},X,Y)=0,$$

$$Q(X,Y,Z) = q_0(\mathbf{a})Z^2 + q_1(\mathbf{a},X,Y)Z + q_2(\mathbf{a},X,Y) = 0.$$

We denote $\sigma(P,Q) = \{w \in \mathbf{P}_2(\mathbf{C}) \mid P(w) = Q(w) = 0\}$.

Definition 2.1. We consider formal expressions $\mathbf{D} = \sum n(w)w$ where $n(w)$ is an integer and only a finite number of $n(w)$ are nonzero. Such an expression is called: i) a zero-cycle of $\mathbf{P}_2(\mathbf{C})$ if all w appearing in $\mathbf{D}$ are points of $\mathbf{P}_2(\mathbf{C})$; ii) a divisor of $\mathbf{P}_2(\mathbf{C})$ if all w appearing in $\mathbf{D}$ are irreducible algebraic curves of $\mathbf{P}_2(\mathbf{C})$; iii) a divisor of an irreducible algebraic curve $\mathfrak{C}$ in $\mathbf{P}_2(\mathbf{C})$ if all w in $\mathbf{D}$ belong to the curve $\mathfrak{C}$. We call degree of the expression $\mathbf{D}$ the integer $\deg(\mathbf{D}) = \sum n(w)$. We call support of $\mathbf{D}$ the set $\mathrm{Supp}(\mathbf{D})$ of all w appearing in $\mathbf{D}$ such that $n(w) \neq 0$.

Definition 2.2. We say that an invariant affine straight line $\mathcal{L}(x,y) = ux + vy + w = 0$, respectively the line at infinity $Z = 0$, for a quadratic vector field $\widetilde{D}$ has multiplicity m if there exists a sequence of real quadratic vector fields $\widetilde{D}_k$ converging to $\widetilde{D}$, such that each $\widetilde{D}_k$ has m, respectively $m-1$, distinct invariant affine straight lines $\mathcal{L}_i^j = u_i^j x + v_i^j y + w_i^j = 0$, $(u_i^j, v_i^j) \neq (0,0)$, $(u_i^j, v_i^j, w_i^j) \in \mathbf{C}^3$, converging to $\mathcal{L} = 0$ as $k \to \infty$ (with the topology of their coefficients), and this does not occur for $m+1$, respectively m.

Notation 2.3. Let us denote by

$$\mathbf{QS} = \left\{ (S) \;\middle|\; \begin{array}{c} (S) \text{ is a real system (1.1) such that} \\ \gcd(p(x,y), q(x,y)) = 1 \quad \text{and} \\ \max\big(\deg(p(x,y)), \deg(q(x,y))\big) = 2 \end{array} \right\};$$

$$\mathbf{QSL} = \left\{ (S) \in \mathbf{QS} \;\middle|\; \begin{array}{l} (S) \text{ possesses at least one invariant affine} \\ \text{line or the line at infinity has multiplicity} \\ \text{at least two} \end{array} \right\}.$$

In this section we shall assume that systems (2.1) belong to **QS**.

We define below the geometrical objects (divisors or zero-cycles) which play an important role in constructing the invariants of the systems.

Definition 2.3.

$$\begin{aligned}
\mathbf{D}_S(P,Q) &= \sum_{w\in\sigma(P,Q)} I_w(P,Q)w;\\
\mathbf{D}_S(P,Q;Z) &= \sum_{w\in\{Z=0\}} I_w(P,Q)w;\\
\widehat{\mathbf{D}}_S(P,Q,Z) &= \sum_{w\in\{Z=0\}} \Big(I_w(C,Z),\ I_w(P,Q)\Big)w;\\
\mathbf{D}_S(C,Z) &= \sum_{w\in\{Z=0\}} I_w(C,Z)w \quad \text{if } Z\nmid C(X,Y,Z),
\end{aligned}$$

where $C(X,Y,Z)=YP(X,Y,Z)-XQ(X,Y,Z)$, $I_w(F,G)$ is the intersection number (see [**12**]) of the curves defined by homogeneous polynomials $F,\ G\in\mathbf{C}[X,Y,Z]$, $\deg(F),\deg(G)\geq 1$ and $\{Z=0\}=\{[X:Y:0]\mid(X,Y)\in\mathbf{C}^2\setminus(0,0)\}$.

Notation 2.4.

$$n_{\mathbf{R}}^{\infty}=\#\{w\in \operatorname{Supp}\mathbf{D}_S(C,Z)\mid w\in\mathbf{P}_2(\mathbf{R})\}. \tag{2.2}$$

A complex projective line $uX+vY+wZ=0$ is invariant for the system (S) if either it coincides with $Z=0$ or it is the projective completion of an invariant affine line $ux+vy+w=0$.

Notation 2.5. Let $S\in\mathbf{QSL}$. Let us denote

$$\mathbf{IL}\,(S)=\left\{\,l\;\middle|\;\begin{array}{c} l \text{ is a line in } \mathbf{P}_2(\mathbf{C}) \text{ such}\\ \text{that } l \text{ is invariant for (S)}\end{array}\right\};$$

$$M(l)=\text{the multiplicity of the invariant line } l \text{ of (S)}.$$

Remark 2.6. We note that the line $l_\infty:Z=0$ is included in $\mathbf{IL}\,(S)$ for any $(S)\in\mathbf{QS}$.

Let $l_i:f_i(x,y)=0,\ i=1,\dots,k$, be all the distinct invariant affine lines (real or complex) of a system $(S)\in\mathbf{QSL}$. Let $l_i':\mathcal{F}_i(X,Y,Z)=0$ be the complex projective completion of l_i.

Notation 2.7. We denote

$$\begin{aligned}
&\mathcal{G} : \prod_i \mathcal{F}_i(X,Y,Z)\, Z = 0;\\
&\operatorname{Sing}\mathcal{G} = \{w \in \mathcal{G} \mid w \text{ is a singular point of } \mathcal{G}\};\\
&\nu(w) = \text{the multiplicity of the point } w, \text{ as a point of } \mathcal{G}.
\end{aligned}$$

Definition 2.4.

$$\begin{aligned}
\mathbf{D}_{\mathbf{IL}}(S) &= \sum_{l \in \mathbf{IL}\,(S)} M(l)l, \quad (S) \in \mathbf{QSL};\\
\operatorname{Supp}\mathbf{D}_{\mathbf{IL}}(S) &= \{l \mid l \in \mathbf{IL}\,(S)\}.
\end{aligned}$$

Notation 2.8.

$$\begin{aligned}
M_{\mathbf{IL}} &= \deg \mathbf{D}_{\mathbf{IL}}(S);\\
N_{\mathbf{C}} &= \#\operatorname{Supp}\mathbf{D}_{\mathbf{IL}};\\
N_{\mathbf{R}} &= \#\{l \in \operatorname{Supp}\mathbf{D}_{\mathbf{IL}} \mid l \in \mathbf{P}_2(\mathbf{R})\};\\
n^{\mathbf{R}}_{\mathcal{G},\,\sigma} &= \#\{\omega \in \operatorname{Supp}\mathbf{D}_S(P,Q) | \omega \in \mathcal{G}|_{\mathbf{R}^2}\};\\
d^{\mathbf{R}}_{\mathcal{G},\,\sigma} &= \sum_{\omega \in \mathcal{G}_{|\mathbf{R}^2}} I_\omega(P,Q);\\
m_{\mathcal{G}} &= \max\{\nu(\omega) \mid \omega \in \operatorname{Sing}\mathcal{G}_{|\mathbf{C}^2}\};\\
m^{\mathbf{R}}_{\mathcal{G}} &= \max\{\nu(\omega) \mid \omega \in \operatorname{Sing}\mathcal{G}_{|\mathbf{R}^2}\}.
\end{aligned} \tag{2.3}$$

3. The class of quadratic systems with $M_{\mathbf{IL}} = 6$.

Definition 3.1 (Poincaré [**18**]). Let $F = F_1/F_2$, where $F_1, F_2 \in \mathbf{C}[x,y]$, be a fist integral of a system (1.1). We call remarkable values in $\mathbf{C}$ for F, constants $K \in \mathbf{C}$ for which $F_1 - KF_2$ are reducible. If K is such a remarkable value for F and $F_1 - KF_2 = u_1^{\alpha_1} u_2^{\alpha_2} \cdots u_k^{\alpha_k}$ where $u_i \in \mathbf{C}[x,y]$ are irreducible over $\mathbf{C}$ and not all integers α_i's are 1, then such a K is called a critical value for F.

An easy corollary from [**2**] is the following proposition:

TABLE 1 ($M_{\mathbf{IL}} = 6$).

Orbit representative	Invariant lines and their multiplicities / Respective cofactors	Inverse integrating factor $\mathcal{R}_i$ / First integral $\mathcal{F}_i$
1) $\begin{cases} \dot{x} = -1 + x^2, \\ \dot{y} = -1 + y^2 \end{cases}$	$x \pm 1(1),\ y \pm 1(1),\ x - y(1)$	$\mathcal{R}_1 = (x-1)(x-y)(y+1)$
	$x \mp 1,\ y \mp 1,\ x + y$	$\mathcal{F}_1 = \dfrac{(x+1)(y-1)}{(x-1)(y+1)}$
2) $\begin{cases} \dot{x} = 1 + x^2, \\ \dot{y} = 1 + y^2 \end{cases}$	$x \pm i(1),\ y \pm i(1),\ x - y(1)$	$\mathcal{R}_2 = (x^2+1)(y^2+1)$
	$x \mp i,\ y \mp i,\ x + y$	$\mathcal{F}_2 = \dfrac{(x^2+1)(y^2+1)}{(x-y)^2}$ or $(xy+1)/(x-y)$
3) $\begin{cases} \dot{x} = 2xy, \\ \dot{y} = y^2 - x^2 - 1 \end{cases}$	$x \pm i(y-1)(1),$ $x \pm i(y+1)(1),\ x(1)$	$\mathcal{R}_3 = \left[x^2 + (y-1)^2\right]\left[x^2 + (y+1)^2\right]$
	$y + 1 \mp ix,\ y - 1 \mp ix,\ y$	$\mathcal{F}_3 = x^{-2}\left[2x^2(y^2+1) + x^4 + (y^2-1)^2\right]$ or $(x^2 + y^2 - 1)/x$
4) $\begin{cases} \dot{x} = 2xy, \\ \dot{y} = 1 - x^2 + y^2 \end{cases}$	$x - 1 \pm iy(1),$ $x + 1 \pm iy(1),\ x(1)$	$\mathcal{R}_4 = \left[y^2 + (x-1)^2\right]\left[y^2 + (x+1)^2\right]$
	$y \mp i(x+1),\ y \mp i(x-1),\ y$	$\mathcal{F}_4 = \dfrac{(x-1)^2 + y^2}{x}$
5) $\dot{x} = x^2, \dot{y} = y^2$	$x(2),\ y(2),\ x - y(1)$	$\mathcal{R}_5 = x^2y^2$ or $xy(x-y)$
	$x, y,\ x + y$	$\mathcal{F}_5 = (x-y)x^{-1}y^{-1}$
6) $\begin{cases} \dot{x} = 2xy, \\ \dot{y} = -x^2 + y^2 \end{cases}$	$x \pm iy(2), x(1)$	$\mathcal{R}_6 = (x^2+y^2)^2$ or $x(x^2+y^2)$
	$y \mp ix, x - y$	$\mathcal{F}_6 = x^{-1}(x^2+y^2)$
7) $\begin{cases} \dot{x} = -1 + x^2, \\ \dot{y} = 2y \end{cases}$	$x + 1(1),\ y(1),\ x - 1(2)$	$\mathcal{R}_7 = (x^2-1)y$
	$x - 1, 2,\ x + 1$	$\mathcal{F}_7 = \dfrac{(x+1)y}{x-1}$
8) $\begin{cases} \dot{x} = x^2 - 1, \\ \dot{y} = 2xy \end{cases}$	$x \pm 1(2), y(1)$	$\mathcal{R}_7 = (x^2-1)y$ or $(x^2-1)^2$
	$\pm x - 1, -2x$	$\mathcal{F}_7 = (x^2-1)y^{-1}$
9) $\begin{cases} \dot{x} = 1 + x^2, \\ \dot{y} = 2xy \end{cases}$	$x \pm i(2), y(1)$	$\mathcal{R}_9 = (x^2+1)y$ or $(x^2+1)^2$
	$-x \pm i, -2x$	$\mathcal{F}_9 = (x^2+1)y^{-1}$
10) $\dot{x} = x^2, \dot{y} = 1$	$x(3)$	$\mathcal{R}_{10} = x^2$
	x	$\mathcal{F}_{10} = (xy+1)x^{-1}$
11) $\begin{cases} \dot{x} = x, \\ \dot{y} = y - x^2 \end{cases}$	$x(3)$	$\mathcal{R}_{11} = x^2$
	1	$\mathcal{F}_{11} = (x^2+y)x^{-1}$

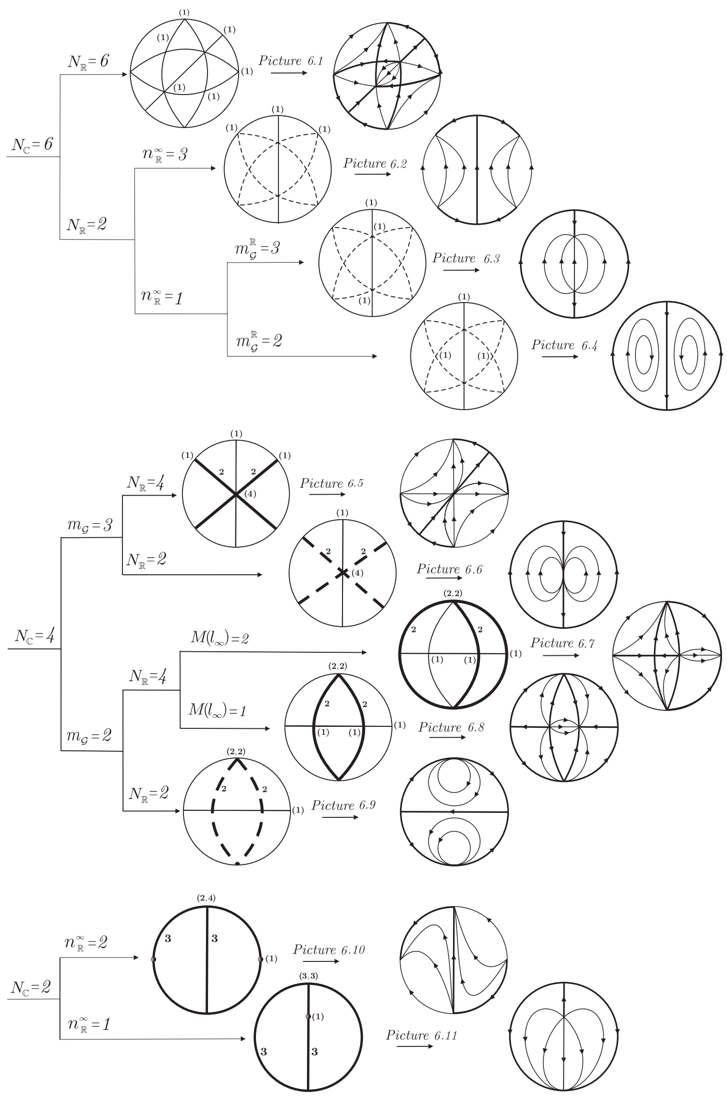


DIAGRAM 1. ($M_{\mathbf{IL}}=6$).

Proposition 3.1. *The maximum number of invariant lines, including the line at infinity and including multiplicities, which a quadratic system could have is six.*

Notation 3.1. We denote by $\mathbf{QSL_6}$ the class of all real quadratic differential systems (2.1) with p and q relatively prime, i.e., $\gcd_{\mathbf{R}}(p, q) = 1$, $Z \nmid C$, and possessing a configuration of invariant straight lines of total multiplicity $M_{\mathbf{IL}} = 6$ including the line at infinity and including possible multiplicities of the lines.

3.1 Darboux integrating factors and first integrals.

Theorem 3.1. *Consider a quadratic system* (2.1) *in* $\mathbf{QSL_6}$. *Then this system has a polynomial inverse integrating factor which splits into linear factors over* $\mathbf{C}$ *and it has a rational first integral, foliating the plane into conic curves. Furthermore, under the action of the affine group and time rescaling, a system* (2.1) *in* $\mathbf{QSL_6}$ *is equivalent to one of the eleven systems indicated in Table* 1 *which form a system of representatives of the orbits. This table also lists the corresponding cofactors of the lines as well as the inverse integrating factors and first integrals of the systems.*

Proof. Orbit representatives and invariant affine lines with their multiplicities which are listed here in the second columns of Table 1 were determined in [**23**]. The corresponding co-factors, Darboux integrating factors and first integrals from Table 1 are obtained via straightforward computations. We observe that the first integrals computed via the method of Darboux and which are listed in the last column of Table 1 yield foliations by conics of the plane in all cases. We note that in the cases 2) and 3) we first obtain first integrals with higher degree polynomials. To obtain the second first integral listed for the cases 2) and 3) in the last column of Table 1 we observe that $K = 1$ is a critical value for the first integral $F = (x^2 + 1)(y^2 + 1)/(x - y)^2$. Indeed, we have $(x^2 + 1)(y^2 + 1) - (x - y)^2 = (xy + 1)^2 = 0$. Hence, $xy + 1 = 0$ is an invariant conic of the system and by Darboux's theory we obtain the first integral $G = (xy + 1)/(x - y)$.

In case 3) we obtain that $K = 4$ is a critical value for the first integral

$$F = \frac{x^4 + 2x^2(y^2+1) + (y^2-1)^2}{x^2}.$$

Indeed, we have $x^4 + 2x^2(y^2+1) + (y^2-1)^2 - 4x^2 = (x^2+y^2-1)^2 = 0$. Then by the theory of Darboux we obtain the first integral $G = (x^2+y^2-1)/x$ and hence in case 3) we also have a foliation of the plane by conics.

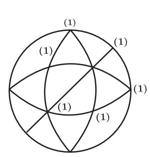
3.2 Phase portraits. In order to construct the phase portraits corresponding to quadratic systems given by Tables 1 and 2 we use the configurations of invariant straight lines already established in [**23**].

Theorem 3.2. *Consider the action of the group of real affine transformations and time rescaling on the class* $\mathbf{QSL_6}$. *The orbits of* $\mathbf{QSL_6}$ *under this action are classified by the multi integer-valued invariant* $(N_{\mathbf{C}}, m_{\mathcal{G}}, N_{\mathbf{R}}, n_{\mathbf{R}}^{\infty}, M(l_\infty), m_{\mathcal{G}}^{\mathbf{R}})$. *The full classification of these orbits is given in Diagram* 1 *according to the possible values of this invariant. Diagram* 1 *also contains all the types of the configurations of the lines and all the corresponding phase portraits of such systems.*

Proof. We shall examine step by step each orbit representative given by Table 1. For each representative, we place below its configuration of invariant lines and next to it, its phase portrait and the types of finite singularities. We have drawn the phase curves so as to suggest the type of conics they are part of (hyperbolas, parabolas, ellipses or reducible conics).

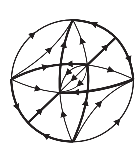
1) $\dot{x} = -1 + x^2$, $\dot{y} = -1 + y^2$; $M_{1,2}(\mp 1, \pm 1)$ are saddles; $M_{3,4}(\mp 1, \mp 1)$ are nodes;

[**23**]⇒



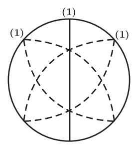
Config. 6.1

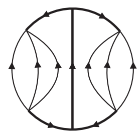
→



Picture 6.1

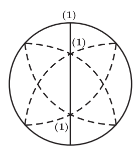
2) $\dot{x} = 1 + x^2$, $\dot{y} = 1 + y^2$; there are no real singular points

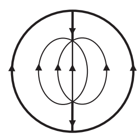
[**23**]⇒ 

→ 

Config. 6.2 *Picture 6.2*

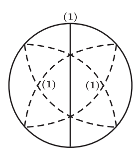
3) $\dot{x} = 2xy$, $\dot{y} = -1 - x^2 + y^2$; $M_{1,2}(0, \pm 1)$ are nodes

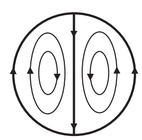
[**23**]⇒ 

→ 

Config. 6.3 *Picture 6.3*

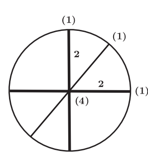
4) $\dot{x} = 2xy$, $\dot{y} = 1 - x^2 + y^2$; $M_{1,2}(\pm 1, 0)$ are centers

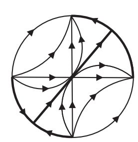
[**23**]⇒ 

→ 

Config. 6.4 *Picture 6.4*

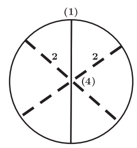
5) $\dot{x} = x^2$, $\dot{y} = y^2$; $M_0(0, 0)$ of multiplicity four

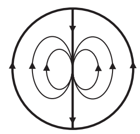
[**23**]⇒ 

→ 

Config. 6.5 *Picture 6.5*

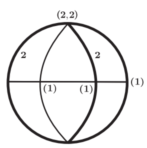
6) $\dot{x} = 2xy$, $\dot{y} = -x^2 + y^2$; $M_0(0,0)$ of multiplicity four

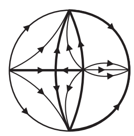
[23]$\Rightarrow$ 

$\longrightarrow$ 

Config. 6.6 *Picture 6.6*

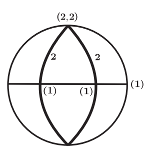
7) $\dot{x} = -1 + x^2$, $\dot{y} = 2y$; $M_1(-1,0)$ is a saddle and $M_1(1,0)$ is a node

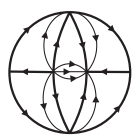
[23]$\Rightarrow$ 

$\longrightarrow$ 

Config. 6.7 *Picture 6.7*

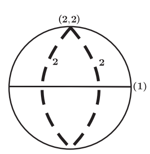
8) $\dot{x} = x^2 - 1$, $\dot{y} = 2xy$; $M_{1,2}(\pm 1, 0)$ are nodes

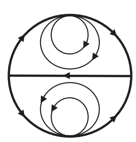
[23]$\Rightarrow$ 

$\longrightarrow$ 

Config. 6.8 *Picture 6.8*

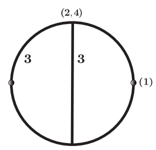
9) $\dot{x} = x^2 + 1$, $\dot{y} = 2xy$; there are no real singular points

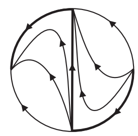
[23]$\Rightarrow$ 

$\longrightarrow$ 

Config. 6.9 *Picture 6.9*

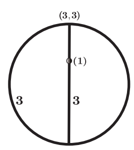
10) $\dot{x} = x^2$, $\dot{y} = 1$; there are no real singular points

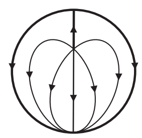
[**23**]$\Rightarrow$ 

$\longrightarrow$ 

Config. 6.10 *Picture 6.10*

11) $\dot{x} = x$, $\dot{y} = y - x^2$; $M_1(0,0)$ is a node

[**23**]$\Rightarrow$ 

$\longrightarrow$ 

Config. 6.11 *Picture 6.11*

As all the cases from Table 1 have been examined, Theorem 3.2 is proved. $\square$

As a byproduct of Diagram 1, we also obtain the following

Corollary 3.2. a) *The total multiplicity six yields configurations corresponding to partitions of the number* 6 *into individual multiplicities as follows:* $(1,1,1,1,1,1)$, $(2,2,1,1)$, $(3,3)$.

b) *The remaining partitions of the number* 6

$$(2,1,1,1,1),\ (3,1,1,1),\ (2,2,2),\ (3,2,1),\ (4,1,1),\ (4,2),\ (5,1),\ (6)$$

cannot be realized in configurations of distinct invariant straight line of quadratic systems possessing invariant lines whose total multiplicity is six.

Comment. It would be nice to have a direct geometrical reason explaining the above corollary.

3.3 The projective classification in $\mathbf{P}_2(\mathbf{C})$ of the invariant lines configurations of systems in $\mathbf{QSL_6}$. To a system (1.1) we can associate the equation (1.3) defined by the 1-form $\omega_1 = q(x,y)dx - p(x,y)\,dy$. We consider the map $j : \mathbf{C}^3 \setminus \{Z=0\} \to \mathbf{C}^2$, given by $j(X,Y,Z) = (X/Z, Y/Z) = (x,y)$ and suppose that $\max(\deg(p), \deg(q)) = m > 0$. Since $x = X/Z$ and $y = Y/Z$ we have: $dx = (ZdX - XdZ)/Z^2$, $dy = (ZdY - YdZ)/Z^2$, the pull-back form $j^*(\omega_1)$ has poles at $Z=0$ and its associated equation $j^*(\omega_1) = 0$ can be written as

$$\begin{aligned} j^*(\omega_1) = q(X/Z, Y/Z)(ZdX - XdZ)/Z^2 \\ p(X/Z, Y/Z)(ZdY - YdZ)/Z^2 = 0. \end{aligned}$$

Then the 1-form $\omega = Z^{m+2} j^*(\omega_1)$ in $\mathbf{C}^3 \setminus \{Z=0\}$ has homogeneous polynomial coefficients of degree $m+1$, and for $Z \neq 0$ the equations $\omega = 0$ and $j^*(\omega_1) = 0$ have the same solutions. Therefore, the differential equation $\omega = 0$ can be written as

$$AdX + BdY + CdZ = 0 \tag{3.1}$$

where

$$\begin{aligned} A(X,Y,Z) &= ZQ(X,Y,Z) = Z^{m+1} q(X/Z, Y/Z), \\ B(X,Y,Z) &= -ZP(X,Y,Z) = -Z^{m+1} p(X/Z, Y/Z), \\ C(X,Y,Z) &= YP(X,Y,Z) - XQ(X,Y,Z) \end{aligned}$$

and $P(X,Y,Z) = Z^m p(X/Z, Y/Z)$, $Q(X,Y,Z) = Z^m q(X/Z, Y/Z)$. Clearly A, B and C are homogeneous polynomials of degree $m+1$ satisfying $AX + BY + CZ = 0$.

The equation (3.1) becomes in this case

$$(E_0) \qquad QZdX - PZdY + (YP - XQ)\,dZ = 0. \tag{3.2}$$

Notation 3.3. We consider the set **EQ** of all real differential equations

$$(E) \qquad AdX + BdY + CdZ = 0$$

where A, B and C are cubic homogeneous polynomials in X, Y and Z over $\mathbf{R}$ subject to the identity

$$AX + BY + CZ = 0.$$

On **EQ** acts the group $PGL(3, \mathbf{R})$ of projective transformations of $\mathbf{P}_2(\mathbf{R})$.

We consider the set **Eq** of all equations (E) in **EQ** of the form (E_0) obtained from equations (1.3).

Notation 3.4. We denote by $\mathbf{Eql_6}$ the class of all equations in **Eq** of the form (E_0) obtained from equations (1.3) and possessing invariant lines of total multiplicity six.

On **Eq** (and $\mathbf{Eql_6}$) acts $Aff\,(2, \mathbf{R})$. As $Aff\,(2, \mathbf{R})$ is a subgroup of $PGL\,(3, \mathbf{R})$, we may have two distinct orbits under the action of $Aff\,(2, \mathbf{R})$ of systems in **Eq** contained in a single orbit of **EQ** under the action of the bigger group $PGL\,(3, \mathbf{R})$. As we show in Theorem 3.3 below this indeed occurs.

Definition 3.2. We define a projective invariant for equation in $\mathbf{Eql_6}$ as follows:

$$m^{\mathbf{R}}_{\text{Sing}} = \max_{i\in\{1,\dots,N_{\mathbf{R}}\}} \Big(\#\big\{w \in l_i \mid w \in \operatorname{Sing}\mathcal{G} \cap \mathbf{P}_2(\mathbf{R})\big\}\Big).$$

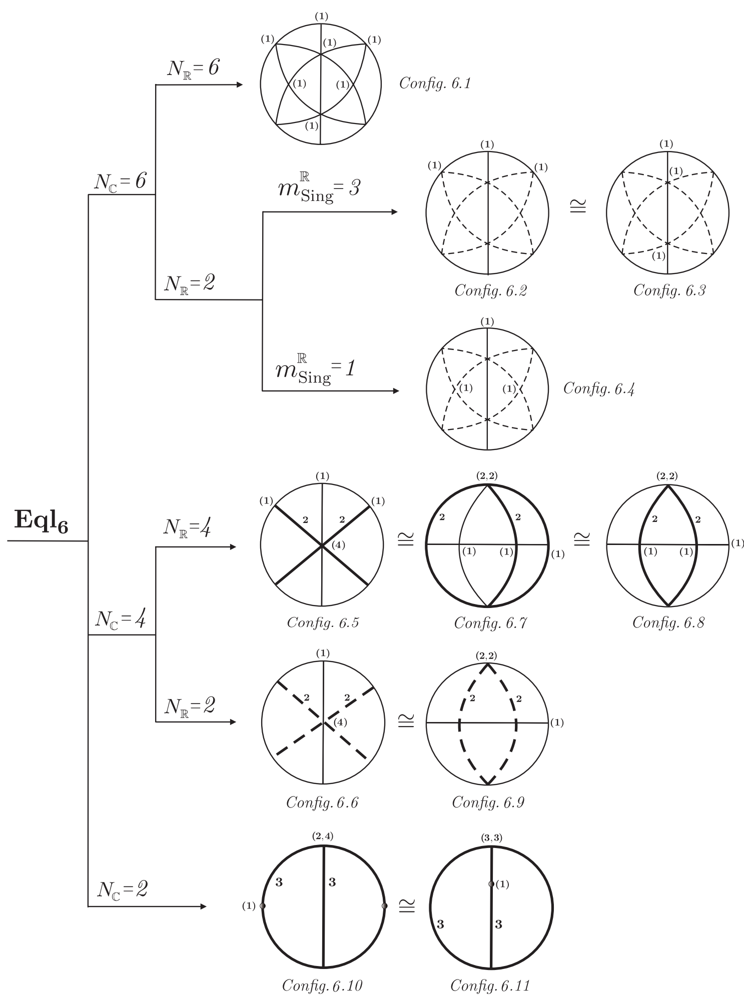
Theorem 3.3. *We consider the systems in $\mathbf{QSL_6}$ and their associated real equations in $\mathbf{Eql_6}$. We consider the action of the group $PGL(3, \mathbf{R})$ of real projective transformations of the plane on the class* **EQ**.

i) *Two systems (S_1) and (S_2) in $\mathbf{QSL_6}$ located on distinct orbits under the action of the real affine group and time rescaling could yield equations (E_1) and (E_2) located on the same orbit under the action of $PGL\,(3, \mathbf{R})$ on* **EQ**.

ii) *The classification of the orbits of equations (E_0) in* **EQ** *associated to systems (S) in $\mathbf{QSL_6}$, under the action of $PGL\,(3, \mathbf{R})$ on* **EQ** *is given in Diagram* 2.

Proof. i) Consider the systems

$$(S_1)\qquad \begin{cases} \dot{x} = 1 + x^2, \\ \dot{y} = 1 + y^2 \end{cases} \qquad \text{and} \qquad (S_2)\qquad \begin{cases} \dot{x} = 2xy, \\ \dot{y} = y^2 - x^2 - 1. \end{cases}$$



$\mathbf{Eql_6}$
$N_{\mathbb{C}} = 6$
$N_{\mathbb{R}} = 6$
Config. 6.1
$N_{\mathbb{R}} = 2$
$m^{\mathbb{R}}_{\mathrm{Sing}} = 3$
Config. 6.2
$\cong$
Config. 6.3
$m^{\mathbb{R}}_{\mathrm{Sing}} = 1$
Config. 6.4
$N_{\mathbb{C}} = 4$
$N_{\mathbb{R}} = 4$
Config. 6.5
Config. 6.7
Config. 6.8
$N_{\mathbb{R}} = 2$
Config. 6.6
Config. 6.9
$N_{\mathbb{C}} = 2$
Config. 6.10
Config. 6.11
(1)
(4)
(2,2)
(2,4)
(3,3)

DIAGRAM 2.

Their corresponding invariant line configurations are: Configuration 6.2 and Configuration 6.3 which are not equivalent under the group of real affine transformations and time rescaling.

Consider now their associated differential equations (3.2) in $\mathbf{P}_2(\mathbf{R})$:

$$(E_1)\quad \begin{bmatrix} Z(Y^2+Z^2)dX - Z(X^2+Z^2)\,dY+ \\ (X-Y)(XY-Z^2)\,dZ = 0 \end{bmatrix} \quad \text{and}$$

$$(E_2)\quad \begin{bmatrix} Z(Y^2-X^2-Z^2)dX - 2XYZ\,dY+ \\ X(X^2+Y^2+Z^2)\,dZ = 0. \end{bmatrix}$$

It can easily be checked that the real projective transformation of $\mathbf{P}_2(\mathbf{R})$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix}$$

transform (E_1) into (E_2).

ii) We consider the 11 representatives of the orbits of $\mathbf{QSL_6}$ under the action of the affine group and time rescaling and their respective configurations in Diagram 2. In i) we showed that the equations associated to systems with the distinct configurations Configuration 6.2 and Configuration 6.3 in Diagram 2, lie on the same orbit under the action of $PGL\,(3,\mathbf{R})$ on $\mathbf{EQ}$. We also show below that the equations associated to systems with Configuration 6.i, $i=5,7,8$ (respectively Configuration 6.j, $j=6,9$, or Configuration 6.j, $j=10,11$) are also located on the same orbit. The proofs are obtained in analogous way to the cases in i) and we only list below the corresponding transformations (the respective equations can be easily computed directly having canonical systems and the equation (3.2).

$$1)\quad \begin{bmatrix} \text{Config. } 6.5: \\ \dot{x}=x^2,\ \dot{y}=y^2 \end{bmatrix} : \begin{pmatrix} 0 & 0 & 2 \\ -1 & 0 & 1 \\ 0 & 4 & 0 \end{pmatrix} \Longrightarrow \begin{bmatrix} \text{Config. } 6.7: \\ \dot{x}=-1+x^2, \\ \dot{y}=2y \end{bmatrix};$$

$$2)\quad \begin{bmatrix} \text{Config. } 6.5: \\ \dot{x}=x^2,\ \dot{y}=y^2 \end{bmatrix} : \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \Longrightarrow \begin{bmatrix} \text{Config. } 6.8: \\ \dot{x}=x^2-1, \\ \dot{y}=2xy \end{bmatrix};$$

$$3)\quad \begin{bmatrix} \text{Config. } 6.6: \\ \dot{x}=2xy, \\ \dot{y}=-x^2+y^2 \end{bmatrix} : \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \Longrightarrow \begin{bmatrix} \text{Config. } 6.9: \\ \dot{x}=1+x^2, \\ \dot{y}=2xy \end{bmatrix};$$

$$4)\quad \begin{bmatrix} \text{Config. } 6.10: \\ \dot{x}=x^2,\ \dot{y}=1 \end{bmatrix} : \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \Longrightarrow \begin{bmatrix} \text{Config. } 6.11: \\ \dot{x}=x, \\ \dot{y}=y-x^2 \end{bmatrix};$$

4. The class of quadratic systems with $M_{\mathbf{IL}} = 5$.

Notation 4.1. We denote by $\mathbf{QSL_5}$ the class of all real quadratic differential systems (2.1) with p and q relatively prime ($(p,q) = 1$), $Z \nmid C$, and possessing a configuration of invariant straight lines of total multiplicity $M_{\mathbf{IL}} = 5$ including the line at infinity and including possible multiplicities of the lines.

4.1 Darboux integrating factors and first integrals.

Theorem 4.1. *Consider a quadratic system* (2.1) *in* $\mathbf{QSL_5}$. *Then this system has a polynomial inverse integrating factor which splits into linear factors over* $\mathbf{C}$, *and it has a Darboux first integral. Furthermore, the quotient set under the action of the affine group and time rescaling on* $\mathbf{QSL_5}$ *is formed by*:

(i) *a set of* 19 *orbits*;

(ii) *a set of* 11 *one-parameter families of orbits. A system of representatives of the quotient is given in Table* 2. *This table also lists the corresponding cofactors of the lines as well as the integrating factors and first integrals of the systems.*

Corollary 4.2. *All the systems in* $\mathbf{QSL_5}$ *have elementary real first integrals. We only list in Table* 3 *all real first integrals which correspond to those in the last column of Table* 2 *which are given there in complex form.*

4.2 Phase portraits. In order to construct the phase portraits corresponding to quadratic systems given by Table 2 we use the configurations of invariant straight lines already established in [**23**]. In order to determine the phase portraits in the vicinity of infinity, we shall also use the following CT-comitants constructed in [**24**] (for detailed definitions of the notions involved see [**24**]):

TABLE 2 ($M_{\mathbf{IL}}=5$).

Orbit representative	Invariant lines and their multiplicities / Respective cofactors	Inverse integrating factor $\mathcal{R}_i$ / First integral $\mathcal{F}_i$
1) $\begin{cases}\dot x=(x+1)(gx+1),\\ \dot y=(g-1)xy+y^2,\\ g(g^2-1)\neq 0\end{cases}$	$x+1(1),\ gx+1(1),$ $x-y+1(1),\ y(1)$	$\mathcal{R}_1=y(gx+1)(x-y+1)$
	$gx+1,\ g(x+1),\ gx+y+1,$ $(g-1)x+y$	$\mathcal{F}_1=y^g(gx+1)\times$ $(x-y+1)^{-g}$
2) $\begin{cases}\dot x=g(x^2-4), g\neq 0\\ \dot y=(g^2-4)-x^2-y^2\\ +(g^2+4)x+gxy\end{cases}$	$y\pm i(x-2)+g(1),$ $x\pm 2(1)$	$\mathcal{R}_2=(x+2)\times$ $\left[(x-2)^2+(y+g)^2)\right]$
	$(g\pm i)x-y+g\mp 2i,$ $g(x\mp 2)$	$\mathcal{F}_2=(x+2)^2\times$ $\dfrac{[i(x-2)+y+g]^{ig}}{[i(2-x)+y+g]^{ig}}$
3) $\begin{cases}\dot x=-1+x^2,\\ \dot y=g(y^2-1),\\ g(g^2-1)\neq 0\end{cases}$	$x\pm 1(1),\ y\pm 1(1)$	$\mathcal{R}_3=(x^2-1)(y^2-1)$
	$x\mp 1,\ g(y\mp 1)$	$\mathcal{F}_3=\dfrac{(x+1)^g(y-1)}{(x-1)^g(y+1)}$
4) $\begin{cases}\dot x=-1+x^2,\\ \dot y=g(y^2+1),\ g\neq 0\end{cases}$	$x\pm 1(1),\ y\pm i(1)$	$\mathcal{R}_4=(x^2-1)(y^2+1)$
	$x\mp 1,\ g(y\mp i)$	$\mathcal{F}_4=\dfrac{(x+1)^g(y+i)^i}{(x-1)^g(y-i)^i}$
5) $\begin{cases}\dot x=1+x^2, \lvert g\rvert\neq 0,1\\ \dot y=g(y^2+1)\end{cases}$	$x\pm i(1),\ y\pm i(1)$	$\mathcal{R}_5=(x^2+1)(y^2+1)$
	$x\mp i,\ g(y\mp i)$	$\mathcal{F}_5=\dfrac{(x+i)^g(y-i)}{(x-i)^g(y+i)}$
6) $\begin{cases}\dot x=1+2xy,\\ \dot y=g-x^2+y^2,\\ g=c^2-1/(4c^2)\end{cases}$	$I'_{\pm}=x+c\pm i(y-1/(2c))(1),$ $I''_{\pm}=x-c\pm i(y+1/(2c))(1)$	$\mathcal{R}_6=I'_+\times I'_-\times I''_+\times I''_-$
	$\mp i(x-c)+(y+1/(2c)),$ $\mp i(x+c)+(y-1/(2c))$	$\mathcal{F}_6=\dfrac{(I'_+)^{2c^2+i}(I'_-)^{2c^2-i}}{(I''_+)^{2c^2+i}(I''_-)^{2c^2-i}}$
7) $\begin{cases}\dot x=1+x,\\ \dot y=-xy+y^2\end{cases}$	$x+1(1),\ y(1),\ x-y+1(1)$	$\mathcal{R}_7=y(x-y+1)$
	$1,\ y-x,\ y+1$	$\mathcal{F}_7=e^{-x}y^{-1}(x-y+1)$
8) $\begin{cases}\dot x=gx^2, \lvert g\rvert\neq 0,1\\ \dot y=(g-1)xy+y^2\end{cases}$	$x(2),\ x-y(1),\ y(1)$	$\mathcal{R}_8=xy(x-y)$
	$gx,\ gx+y,\ (g-1)x+y$	$\mathcal{F}_8=xy^g(x-y)^{-g}$
9) $\begin{cases}\dot x=2x,\\ \dot y=1-x^2-y^2\end{cases}$	$1+y\pm ix(1),\ x(1)$	$\mathcal{R}_9=x^2+(y+1)^2$
	$1-y\pm ix,\ 2$	$\mathcal{F}_9=\dfrac{e^{ix}(1+y-ix)}{1+y+ix}$

TABLE 2 ($M_{\mathbf{IL}} = 5$) (continued).

Orbit representative	Invariant lines and their multiplicities / Respective cofactors	Inverse integrating factor $\mathcal{R}_i$ / First integral $\mathcal{F}_i$
10) $\begin{cases} \dot x = gx^2, \ g \neq 0 \\ \dot y = -x^2 + gxy - y^2 \end{cases}$	$x \pm iy(1), \ x(2)$	$\mathcal{R}_{10} = x(x^2 + y^2)$
	$(g \mp i)x - y, \ gx$	$\mathcal{F}_{10} = \dfrac{x^2(ix+y)^{ig}}{(ix-y)^{ig}}$
11) $\begin{cases} \dot x = x^2 + xy, \\ \dot y = y + y^2 \end{cases}$	$x(2), \ y+1(1), \ y(1)$	$\mathcal{R}_{11} = x^2 y$
	$x+y, \ y, \ y+1$	$\mathcal{F}_{11} = ye^{(y+1)/x}$
12) $\dot x = -1 + x^2, \ \dot y = y^2$	$x \pm 1(1), \ y(2)$	$\mathcal{R}_{12} = y^2(x^2-1)$
	$x \mp 1, \ y$	$\mathcal{F}_{12} = \dfrac{e^{2/y}(x-1)}{(x+1)}$
13) $\begin{cases} \dot x = g(x^2-1), \\ \dot y = 2y \ \ g(g^2-1) \neq 0 \end{cases}$	$x \pm 1(1), \ y(1)$	$\mathcal{R}_{13} = y(x^2-1)$
	$g(x \mp 1), \ 2$	$\mathcal{F}_{13} = \dfrac{y^g(x+1)}{(x-1)}$
14) $\begin{cases} \dot x = (x+1)(gx+1), \\ \dot y = (g-1)xy, \ g(g^2-1) \neq 0 \end{cases}$	$x+1(2), \ y(1), \ gx+1(1)$	$\mathcal{R}_{14} = y(x+1)(gx+1)$
	$gx+1, \ (g-1)x, \ g(x+1)$	$\mathcal{F}_{14} = \dfrac{(gx+1)y^g}{(x+1)^g}$
15) $\dot x = g(x^2+1), \ \dot y = 2y, \ g \neq 0$	$x \pm i(1), \ y(1)$	$\mathcal{R}_{15} = y(x^2+1)$
	$g(x \mp i), \ 2$	$\mathcal{F}_{15} = \dfrac{y^g(x-i)^i}{(x+i)^i}$
16) $\dot x = 1 + x^2, \ \dot y = y^2$	$x \pm i(1), \ y(2)$	$\mathcal{R}_{16} = y^2(x^2+1)$
	$x \mp i, y$	$\mathcal{F}_{16} = \dfrac{e^{2/y}(x+i)^i}{(x-i)^i}$
17) $\dot x = x^2, \quad \dot y = 2y$	$x(2), \ y(1)$	$\mathcal{R}_{17} = x^2 y$
	$x, \ 2$	$\mathcal{F}_{17} = ye^{2/x}$
18) $\dot x = 1 + x, \ \dot y = -xy$	$x+1(2), y(1)$	$\mathcal{R}_{18} = (x+1)y$
	$1, \ -x$	$\mathcal{F}_{18} = ye^x(x+1)^{-1}$
19) $\dot x = x^2 + xy, \ \dot y = y^2$	$x(2), \ y(2)$	$\mathcal{R}_{19} = x^2 y$
	$x+y, \ y$	$\mathcal{F}_{19} = ye^{y/x}$
20) $\dot x = -1 + x^2, \ \dot y = 1$	$x \pm 1(1)$	$\mathcal{R}_{20} = x^2 - 1$
	$x \mp 1$	$\mathcal{F}_{20} = \dfrac{e^{2/y}(x+1)}{(x-1)}$

TABLE 2 ($M_{\mathbf{IL}}=5$) (continued).

Orbit representative	Invariant lines and their multiplicities / Respective cofactors	Inverse integrating factor $\mathcal{R}_i$ / First integral $\mathcal{F}_i$
21) $\begin{cases}\dot x=-1+x^2,\\ \dot y=x+2y\end{cases}$	$x-1(2),\ x+1(1)$	$\mathcal{R}_{21}=(x-1)^2$
	$x+1,\ x-1$	$\mathcal{F}_{21}=(x-1)^{-1}\exp\left[\dfrac{xy+y+1}{x-1}\right]$
22) $\begin{cases}\dot x=1-x^2,\\ \dot y=1-2xy\end{cases}$	$x\pm 1(2)$	$\mathcal{R}_{22}=(x^2-1)^2$
	$-x\pm 1$	$\mathcal{F}_{22}=\exp\left[\dfrac{2x-4y}{x^2-1}\right](x-1)(x+1)^{-1}$
23) $\begin{cases}\dot x=-1+x^2,\\ \dot y=-3+y-x^2+xy\end{cases}$	$x-1(3),\ x+1(1)$	$\mathcal{R}_{23}=(x+1)(x-1)^2$
	$x+1,\ x-1$	$\mathcal{F}_{23}=(x+1)\exp\left[\dfrac{y-2}{x-1}\right]$
24) $\dot x=1+x^2,\ \dot y=1$	$x\pm i(1)$	$\mathcal{R}_{24}=x^2+1$
	$x\mp i$	$\mathcal{F}_{24}=e^{2y}\left(\dfrac{i-x}{i+x}\right)^i$
25) $\dot x=1+x^2,\ \dot y=1+2xy$	$x\pm i(2)$	$\mathcal{R}_{25}=(x^2+1)^2$
	$x\mp i$	$\mathcal{F}_{25}=\exp\left[\dfrac{4y-2x}{x^2+1}\right](i+x)^i(i-x)^{-i}$
26) $\dot x=-x,\ \dot y=y-x^2$	$x(1)$	$\mathcal{R}_{26}=1$
	-1	$\mathcal{F}_{26}=x(x^2-3y)$
27) $\dot x=1+x,\ \dot y=y-x^2$	$x+1(2)$	$\mathcal{R}_{27}=(x+1)^2$
	1	$\mathcal{F}_{27}=(x+1)^{-2}\exp\left[x+\dfrac{y-1}{x+1}\right]$
28) $\dot x=x^2,\ \dot y=1+x$	$x(3)$	$\mathcal{R}_{28}=x^2$
	x	$\mathcal{F}_{28}=x^{-1}\exp\left[y+\dfrac{1}{x}\right]$
29) $\dot x=x^2,\ \dot y=1+2xy$	$x(4)$	$\mathcal{R}_{29}=x^4$
	x	$\mathcal{F}_{29}=\dfrac{3xy+1}{x^3}$
30) $\dot x=1,\ \dot y=x^2$	–	$\mathcal{R}_{30}=1$
	–	$\mathcal{F}_{30}=x^3-3y$

TABLE 3.

System	First integral	System	First integral
2)	$(x+2)^2\exp\{2g\,\mathrm{arctg}\,[(2-x)/(g+y)]\}$	4)	$\ln\lvert(x+1)/(x-1)\rvert^g$ $+2\,\mathrm{arctg}\,(y)$
5)	$\mathrm{arctg}\,(y)-g\,\mathrm{arctg}\,(x)$	6)	‡ $\mathcal{F}_1^{2c^2}\mathcal{F}_2^{-2c^2}\exp\{2\times$ $\mathrm{arctg}\,[(4cx+8c^3y)/$ $(4c^2x^2+4c^2y^2-4c^4-1)]\}$
9)	$[x\tilde{u}-(1+y)\tilde{v}]/[x\tilde{v}+(1+y)\tilde{u}]$, $\tilde{u}=\cos(x/2),\tilde{v}=\sin(x/2)$	10)	$\ln\lvert x\rvert+g\,\mathrm{arctg}\,(y/x)$
15)	$g\ln\lvert y\rvert-2\,\mathrm{arctg}\,(x)$	16)	$1/y+\mathrm{arctg}\,(x)$
24)	$y-\mathrm{arctg}\,(x)$	25)	$(2y-x)/(x^2+1)$ $-\mathrm{arctg}\,(x)$

‡ $\mathcal{F}_{1,2}=4c^2(x^2+y^2)\pm 8c^3x\mp 4cy+4c^4+1$.

$$
\begin{aligned}
C_i(a,x,y) &= yp_i(x,y)-xq_i(x,y), \quad i=0,1,2;\\
D_i(a,x,y) &= \frac{\partial}{\partial x}p_i(a,x,y)+\frac{\partial}{\partial y}q_i(a,x,y), \quad i=1,2;\\
M(a,x,y) &= 2\,\mathrm{Hess}\,\big(C_2(a,x,y)\big);\\
\eta(a) &= \mathrm{Discriminant}\,\big(C_2(a,x,y)\big);\\
K(a,x,y) &= \mathrm{Jacob}\,\big(p_2(x,y),q_2(x,y)\big);\\
\mu_0(a) &= \mathrm{Res}_x(p_2,q_2)/y^4=\mathrm{Discriminant}\,\big(K(a,x,y)\big)/16;\\
H(a,x,y) &= -\,\mathrm{Discriminant}\,(\alpha p_2(x,y)+\beta q_2(x,y))|_{\{\alpha=y,\,\beta=-x\}};\\
L(a,x,y) &= 4K+8H-M;\\
K_1(a,x,y) &= p_1(x,y)q_2(x,y)-p_2(x,y)q_1(x,y).
\end{aligned} \tag{4.1}
$$

Remark 4.3. We note that by Discriminant (C_2) of the cubic form $C_2(a,x,y)$ we mean the expression given in Maple via the function "discrim $(C_2,x)/y^6$."

In order to construct other necessary invariant polynomials let us consider the differential operator $\mathcal{L}=x\cdot\mathbf{L}_2-y\cdot\mathbf{L}_1$ acting on $\mathbf{R}[a,x,y]$ constructed in [**4**], where

$$\mathbf{L}_1=2a_{00}\frac{\partial}{\partial a_{10}}+a_{10}\frac{\partial}{\partial a_{20}}+\frac{1}{2}a_{01}\frac{\partial}{\partial a_{11}}+2b_{00}\frac{\partial}{\partial b_{10}}+b_{10}\frac{\partial}{\partial b_{20}}+\frac{1}{2}b_{01}\frac{\partial}{\partial b_{11}},$$

$$\mathbf{L}_2 = 2a_{00}\frac{\partial}{\partial a_{01}} + a_{01}\frac{\partial}{\partial a_{02}} + \frac{1}{2}a_{10}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{01}} + b_{01}\frac{\partial}{\partial b_{02}} + \frac{1}{2}b_{10}\frac{\partial}{\partial b_{11}},$$

as well as the classical differential operator $(f,\varphi)^{(k)}$ acting on $\mathbf{R}[x,y]^2$ which is called *transvectant* of index k (see, for example, [**13, 16**]):

$$(f,g)^{(k)} = \sum_{h=0}^{k}(-1)^h\binom{k}{h}\frac{\partial^k f}{\partial x^{k-h}\partial y^h}\ \frac{\partial^k g}{\partial x^h \partial y^{k-h}}. \tag{4.2}$$

So, by using these operators and the GL-comitants $\mu_0(a)$, $M(a,x,y)$, $K(a,x,y)$, $D_i(a,x,y)$ and $C_i(a,x,y)$ we construct the following polynomials:

$$\mu_i(a,x,y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0), \quad i = 1,\dots,4,$$

$$\kappa(a) = (M,K)^{(2)}, \quad \kappa_1(a) = (M,C_1)^{(2)},$$

(4.3)

$$K_2 = 4\,\mathrm{Jacob}\,(J_2,\xi) + 3\,\mathrm{Jacob}\,(C_1,\xi)D_1 - \xi(16J_1 + 3J_3 + 3D_1^2),$$

$$K_3 = 2C_2^2(2J_1 - 3J_3) + C_2(3C_0K - 2C_1J_4) + 2K_1(3K_1 - C_1D_2),$$

where $\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$ and

$$J_1 = \mathrm{Jacob}\,(C_0,D_2), \qquad J_2 = \mathrm{Jacob}\,(C_0,C_2),$$

$$J_3 = \mathrm{Discrim}\,(C_1), \qquad J_4 = \mathrm{Jacob}\,(C_1,D_2),$$

$$\xi = M - 2K.$$

Notation 4.4. $\mathcal{J}_f(S) = \prod_{w\in\sigma(p,q)} i_w$ where i_w is the Poincaré index of w.

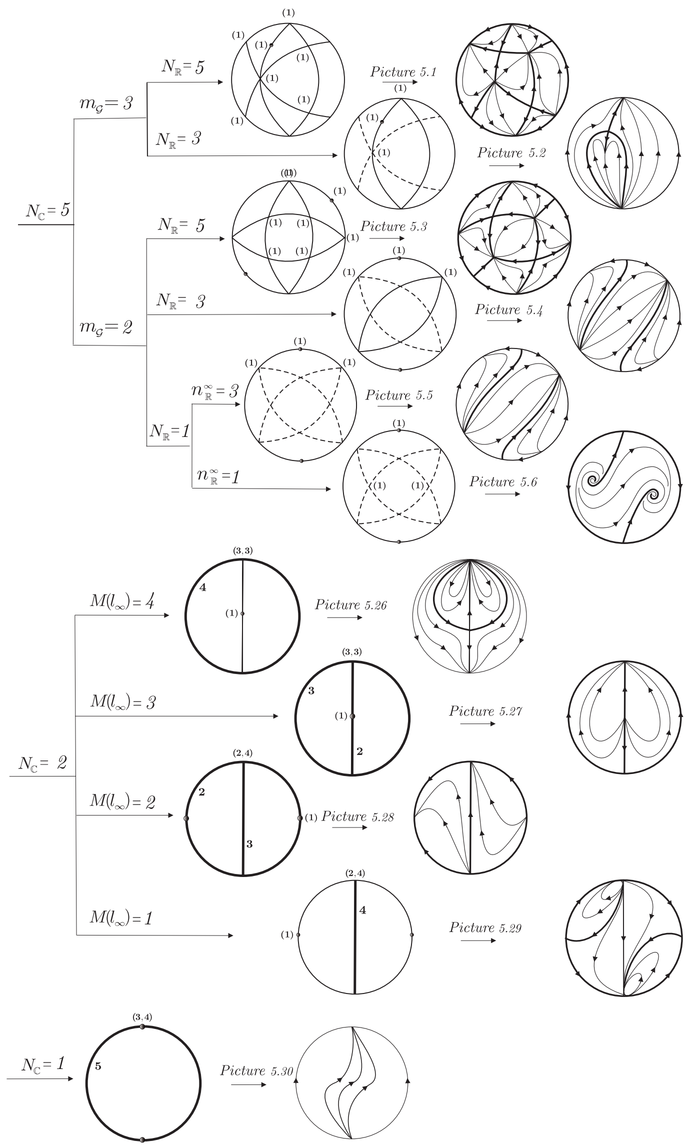


DIAGRAM 3. ($M_{\mathbf{IL}}=5$).

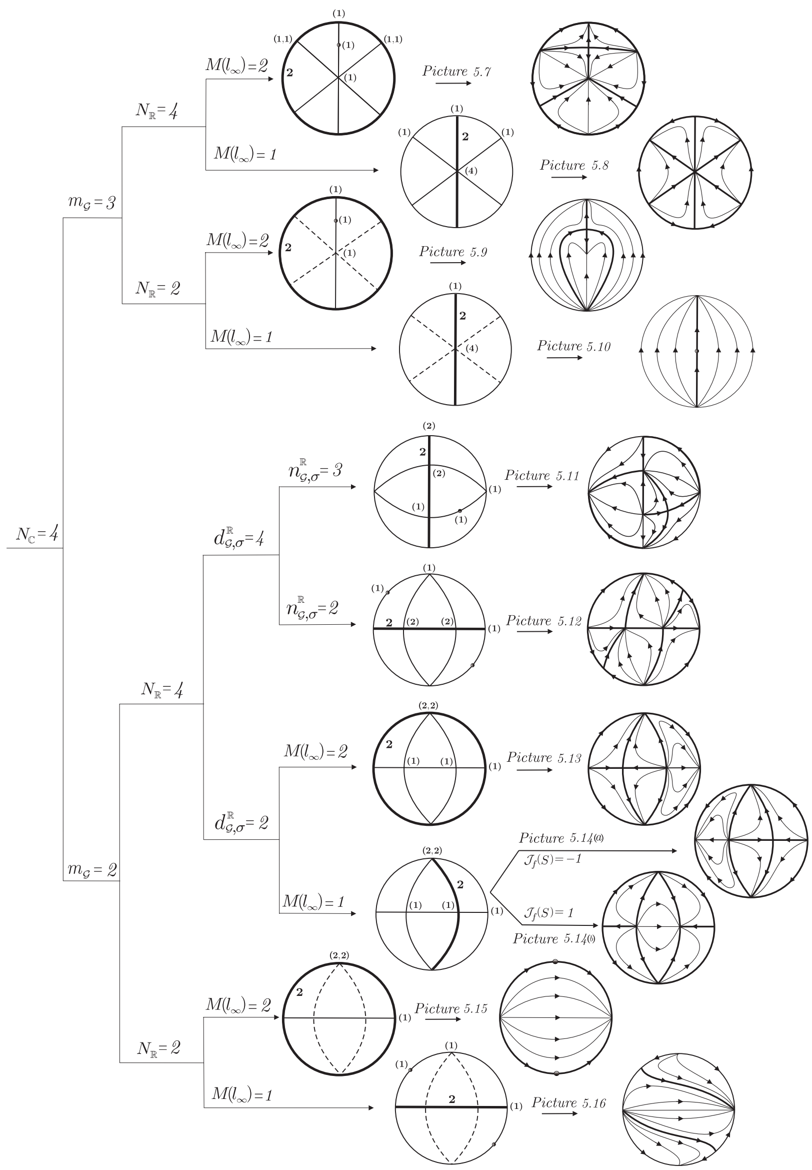


DIAGRAM 3. ($M_{\mathbf{IL}}=5$) (Continued).

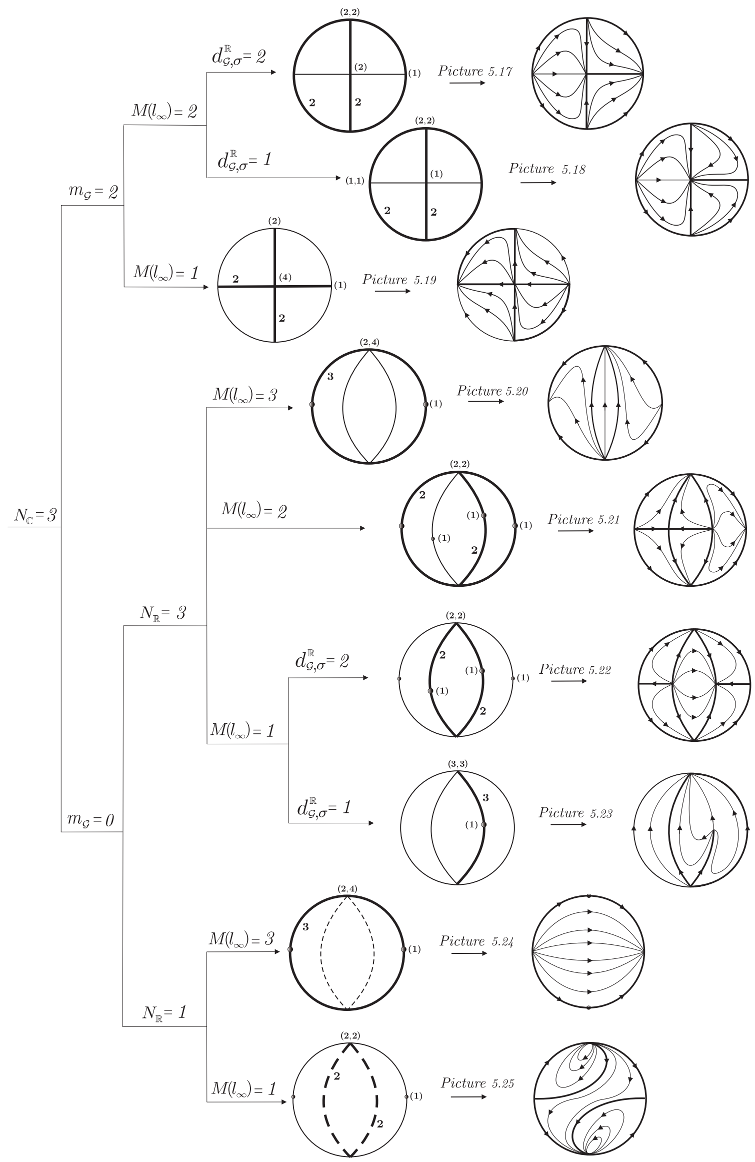


DIAGRAM 3. ($M_{\mathbf{IL}} = 5$) (Continued).

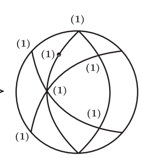
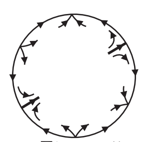
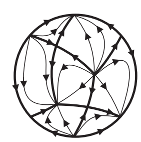
Theorem 4.2. *Consider the systems of the form* (2.1) *in* $\mathbf{QSL_5}$. *Then the action of the affine group and time rescaling yields a classification which corresponds to the distinct values of the invariant* $(N_{\mathbf{C}}, m_{\mathcal{G}}, N_{\mathbf{R}}, \overset{\infty}{n_{\mathbf{R}}}, M(l_\infty), n^{\mathbf{R}}_{\mathcal{G},\sigma}, d^{\mathbf{R}}_{\mathcal{G},\sigma}, \mathcal{J}_f(S))$. *This classification appears in Diagram* 3 *where all the types of configurations of the lines and all the corresponding phase portraits of such systems are listed.*

Proof. Using the results of [**23, 24**] we shall examine step by step each orbit representative given by Table 2. From [**23**] we get the respective configurations Configuration 5.i of invariant lines and from [**24**] we get the phase portraits Figures j in the neighborhood of infinity.

1) *Configuration* 5.1. $\dot{x} = (x+1)(gx+1)$, $\dot{y} = (g-1)xy + y^2$, $g(g^2-1) \neq 0$;

1.1. *Finite singular points*: $M_1(-1,0)$ is a node, $M_2(g-1,-1)$ is a saddle; for $g > 0$ $M_3(-1/g, 0)$ is a saddle, $M_4((g-1)/g, -1/g)$ is a node; for $g < 0$ M_3 is a node, M_4 is a saddle;

1.2. *Infinite singular points*: $\eta = 1 > 0$, $\mu_0 = g^2 > 0$

By [**23, 24**] $\Rightarrow$  and  then $\Longrightarrow$ 

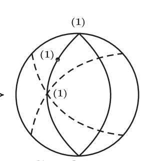
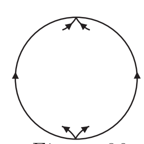
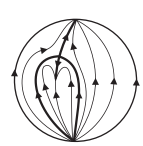
Config. 5.1 *Figure 5* *Picture 5.1*

2) *Configuration* 5.2.

$$\begin{cases} \dot{x} = g(x^2-4), \quad g \neq 0, \\ \dot{y} = (g^2-4) + (g^2+4)x - x^2 + gxy - y^2; \end{cases}$$

2.1. *Finite singular points*: $M_1(2,-g)$ is a node, $M_2(2,3g)$ is a saddle;

2.2. *Infinite singular points*: $\eta = -4 < 0$, $\mu_0 = g^2 > 0$

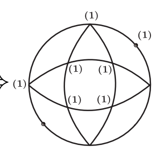
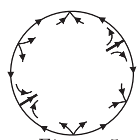
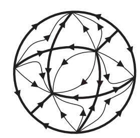
By [**23, 24**] $\Rightarrow$  and  then $\Longrightarrow$ 

Config. 5.2 *Figure 30* *Picture 5.2*

3) *Configuration* 5.3. $\dot{x} = -1 + x^2$, $\dot{y} = g(y^2 - 1)$, $g \neq 0$;

3.1. *Finite singular points*: $M_{1,2}(\pm 1, \pm 1)$ are saddles for $g < 0$ and nodes for $g > 0$; $M_{3,4}(\pm 1, \mp 1)$ are nodes for $g < 0$ and saddles for $g > 0$;

3.2. *Infinite singular points*: $\eta = g^2 > 0$, $\mu_0 = g^2 > 0$;

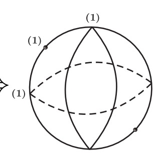
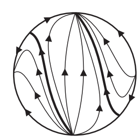
By [**23, 24**] $\Rightarrow$  and  then $\Longrightarrow$ 

Config. 5.3 *Figure 5* *Picture 5.3*

4) *Configuration* 5.4. $\dot{x} = -1 + x^2$, $\dot{y} = g(y^2 + 1)$, $g \neq 0$;

4.1. *Finite singular points*: there are no real singular points;

4.2. *Infinite singular points*: $\eta = g^2 > 0$, $\mu_0 = g^2 > 0$;

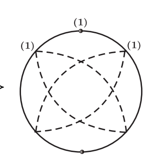
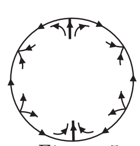
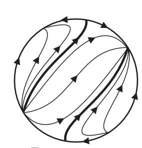
By [**23, 24**] $\Rightarrow$  and  then $\Longrightarrow$ 

Config. 5.4 *Figure 5* *Picture 5.4*

5) *Configuration* 5.5. $\dot{x} = 1 + x^2$, $\dot{y} = g(y^2 + 1)$, $g(g^2 - 1) \neq 0$;

5.1. *Finite singular points*: there are no real singular points;

5.2. *Infinite singular points*: $\eta = g^2 > 0$, $\mu_0 = g^2 > 0$;

By [**23, 24**] $\Rightarrow$  and  then $\Longrightarrow$ 

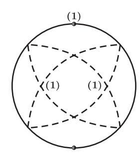
Config. 5.5 *Figure 5* *Picture 5.5*

6) *Configuration* 5.6. $\dot{x} = 1 + 2xy$, $\dot{y} = g - x^2 + y^2$, $g \in \mathbf{R}$;

6.1. *Finite singular points*: $M_{1,2}(\pm(-2g + 2\sqrt{g^2+1})^{-1/2}, \mp(-g/2 + \sqrt{g^2+1}/2)^{1/2})$ are foci;

6.2. *Infinite singular points*: $\eta = -4 < 0$, $\mu_0 = -4 < 0$;

By [**23, 24**] $\Rightarrow$



Config. 5.6

and

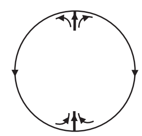
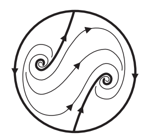


Figure 34

then $\Longrightarrow$



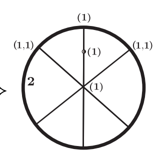
Picture 5.6

7) *Configuration* 5.7. $\dot{x} = 1 + x$, $\dot{y} = -xy + y^2$;

7.1. *Finite singular points*: $M_1(-1,-1)$ is a saddle and $M_2(-1,0)$ is a node;

7.2. *Infinite singular points*: $\eta = 1 > 0$, $\mu_0 = \mu_1 = \kappa = 0$, $\mu_2 = y(y-x) = L/8$, $\mu_2 L > 0$;

By [**23, 24**] $\Rightarrow$



Config. 5.7

and

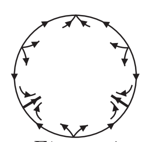
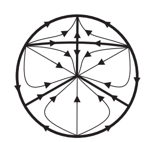


Figure 4

then $\Longrightarrow$



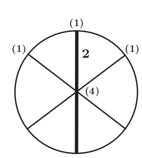
Picture 5.7

8) *Configuration* 5.8. $\dot{x} = gx^2$, $\dot{y} = (g-1)xy + y^2$, $g(g^2-1) \neq 0$;

8.1. *Finite singular points*: the systems are homogeneous and the singular point $M_1(0,0)$ of multiplicity four has 2 parabolic and 2 hyperbolic sectors;

8.2. *Infinite singular points*: $\eta = 1 > 0$, $\mu_0 = g^2 > 0$;

By [**23, 24**] $\Rightarrow$



Config. 5.8

and

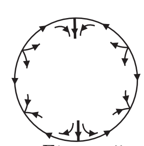
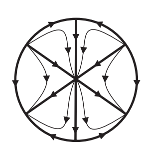


Figure 5

then $\Longrightarrow$

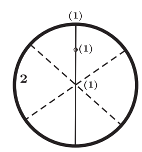


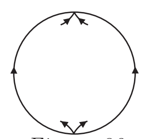
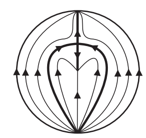
Picture 5.8

9) *Configuration* 5.9. $\dot{x} = 2x$, $\dot{y} = 1 - x^2 - y^2$;

9.1. *Finite singular points*: $M_1(0,1)$ is a saddle and $M_2(0,-1)$ is a (dicritical) node;

9.2. *Infinite singular points*: $\eta = -4 < 0$, $\mu_0 = \mu_1 = \kappa = 0$, $\mu_2 = 4(x^2 + y^2) \neq 0$;

By [**23, 24**] $\Rightarrow$ 

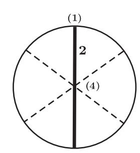
and  then $\Longrightarrow$ 

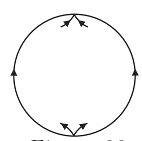
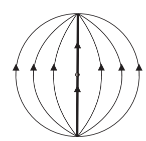
Config. 5.9 *Figure 30* *Picture 5.9*

10) *Configuration* 5.10. $\dot{x} = gx^2$, $\dot{y} = -x^2 + gxy - y^2$, $g \neq 0$;

10.1. *Finite singular points*: the systems are homogeneous and the singular point $M_1(0,0)$ of multiplicity four has 2 hyperbolic sectors;

10.2. *Infinite singular points*: $\eta = -4 < 0$, $\mu_0 = g^2 > 0$;

By [**23, 24**] $\Rightarrow$ 

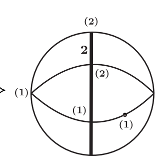
and  then $\Longrightarrow$ 

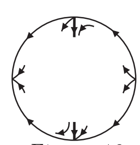
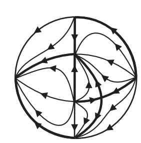
Config. 5.10 *Figure 30* *Picture 5.10*

11) *Configuration* 5.11. $\dot{x} = x^2 + xy$, $\dot{y} = y + y^2$;

11.1. *Finite singular points*: $M_1(0,-1)$ is a node, $M_2(1,-1)$ is a saddle, $M_3(0,0)$ is a saddle-node;

11.2. *Infinite singular points*: $\eta = 0$, $M = -8x^2 \neq 0$, $\mu_0 = 1 > 0$;

By [**23, 24**] $\Rightarrow$ 

and  then $\Longrightarrow$ 

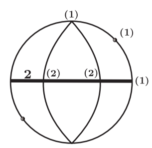
Config. 5.11 *Figure 18* *Picture 5.11*

12) *Configuration* 5.12. $\dot{x} = -1 + x^2$, $\dot{y} = y^2$;

12.1. *Finite singular points*: $M_1(-1, 0)$ and $M_2(1, 0)$ are both saddle-nodes;

12.2. *Infinite singular points*: $\eta = 1 > 0$, $\mu_0 = 1 > 0$;

By [**23, 24**] $\Rightarrow$



Config. 5.12

and

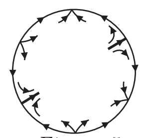
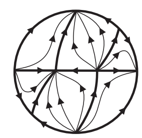


Figure 5

then $\Longrightarrow$



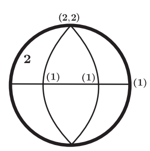
Picture 5.12

13) *Configuration* 5.13. $\dot{x} = g(x^2 - 1)$, $\dot{y} = 2y$, $g(g^2 - 1) \neq 0$;

13.1. *Finite singular points*: for $g > 0$ $M_1(-1, 0)$ is a saddle and $M_2(1, 0)$ is a node; for $g < 0$ $M_1(-1, 0)$ is a node and $M_2(1, 0)$ is a saddle;

13.2. *Infinite singular points*: $\eta = 0$, $M = -8g^2x^2 \neq 0$, $\mu_0 = \mu_1 = \kappa = \kappa_1 = 0$, $L = 8g^2x^2 > 0$, $\mu_2 = 4g^2x^2 > 0$, $K_2 = 384g^4x^2 > 0$;

By [**23, 24**] $\Rightarrow$



Config. 5.13

and

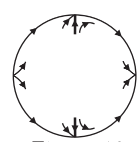
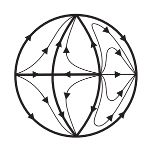


Figure 19

then $\Longrightarrow$



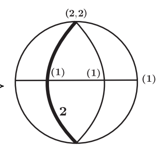
Picture 5.13

14) *Configuration* 5.14. $\dot{x} = (x + 1)(gx + 1)$, $\dot{y} = (g - 1)xy$, $g(g^2 - 1) \neq 0$;

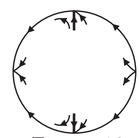
14.1. *Finite singular points*: $M_1(-1, 0)$ is a node; $M_2(-1/g, 0)$ is a saddle for $g > 0$ and it is a node for $g < 0$;

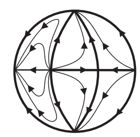
14.2. *Infinite singular points*: $\eta = 0$, $M = -8x^2$, $\mu_0 = \mu_1 = \kappa = \kappa_1 = 0$, $\mu_2 = g(g - 1)^2x^2$, $L = 8gx^2$, $K_2 = 48(g - 1)^2(g^2 - g + 2)x^2$. We shall consider two subcases: $g > 0$ and $g < 0$.

a) $g > 0$: $\eta = \mu_0 = \mu_1 = \kappa = \kappa_1 = 0$, $M \neq 0$, $\mu_2 > 0$, $L > 0$, $K_2 > 0$;

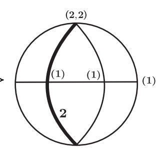
By [**23, 24**] $\Rightarrow$ 

Config. 5.14

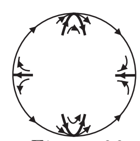
and 
Figure 19

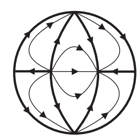
then $\Longrightarrow$ 
Picture 5.14(a)

b) $g < 0$: $\eta = \mu_0 = \mu_1 = \kappa = \kappa_1 = 0$, $M \neq 0$, $\mu_2 < 0$, $L < 0$;

By [**23, 24**] $\Rightarrow$ 

Config. 5.14

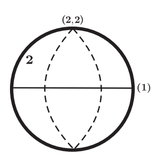
and 
Figure 29

then $\Longrightarrow$ 
Picture 5.14(b)

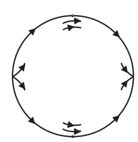
15) *Configuration* 5.15. $\dot{x} = g(x^2 + 1)$, $\dot{y} = 2y$, $g \neq 0$;

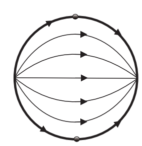
15.1. *Finite singular points*: there are no real singular points;

15.2. *Infinite singular points*: $\eta = 0$, $M = -8g^2x^2 \neq 0$, $\mu_0 = \mu_1 = \kappa = \kappa_1 = 0$, $L = 8g^2x^2 > 0$, $\mu_2 = 4g^2x^2 > 0$, $K_2 = -384g^4x^2 < 0$;

By [**23, 24**] $\Rightarrow$ 

Config. 5.15

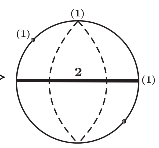
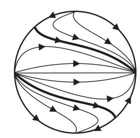
and 
Figure 8

then $\Longrightarrow$ 
Picture 5.15

16) *Configuration* 5.16. $\dot{x} = x^2 + 1$, $\dot{y} = y^2$;

16.1. *Finite singular points*: there are no real singular points;

16.2. *Infinite singular points*: $\eta = 1 > 0$, $\mu_0 = 1 > 0$;

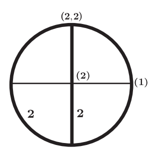
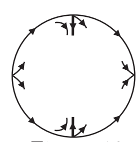
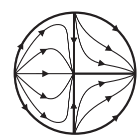
By [**23, 24**] $\Rightarrow$  and  then $\Longrightarrow$ 

Config. 5.16 *Figure 5* *Picture 5.16*

17) *Configuration* 5.17. $\dot{x} = x^2$, $\dot{y} = 2y$;

17.1. *Finite singular points*: $M_1(0,0)$ is a saddle-node;

17.2. *Infinite singular points*: $\eta = 0$, $M = -8x^2 \neq 0$, $\mu_0 = \mu_1 = \kappa = \kappa_1 = 0$, $L = 8x^2 > 0$, $\mu_2 = 4x^2 > 0$, $K_2 = 0$;

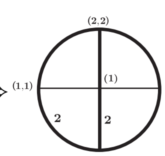
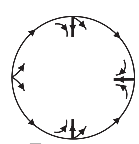
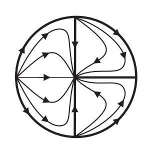
By [**23, 24**] $\Rightarrow$  and  then $\Longrightarrow$ 

Config. 5.17 *Figure 19* *Picture 5.17*

18) *Configuration* 5.18. $\dot{x} = 1 + x$, $\dot{y} = -xy$;

18.1. *Finite singular points*: $M_1(-1,0)$ is a (dicritical) node;

18.2. *Infinite singular points*: $\eta = 0$, $M = -8x^2 \neq 0$, $\mu_0 = \mu_1 = \mu_2 = \kappa = \kappa_1 = 0$, $L = 0$, $\mu_3 = -x^2y$, $K_1 = -x^2y$, $\mu_3 K_1 > 0$;

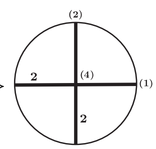
By [**23, 24**] $\Rightarrow$  and  then $\Longrightarrow$ 

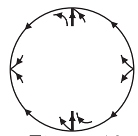
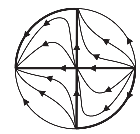
Config. 5.18 *Figure 21* *Picture 5.18*

19) *Configuration* 5.19. $\dot{x} = x^2 + xy$, $\dot{y} = y^2$;

19.1. *Finite singular points*: the systems are homogeneous and the singular point $M_1(0,0)$ of multiplicity four has 2 parabolic and 2 hyperbolic sectors;

19.2. *Infinite singular points*: $\eta = 0$, $M = -8x^2 \neq 0$, $\mu_0 = 1 > 0$;

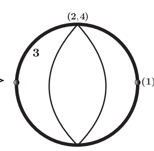
By [**23, 24**] $\Rightarrow$ 

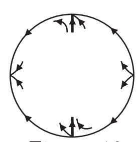
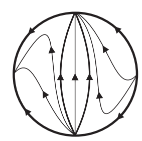
Config. 5.19
and 
Figure 18
then $\Longrightarrow$ 
Picture 5.19

20) *Configuration* 5.20. $\dot{x} = -1 + x^2$, $\dot{y} = 1$;

20.1. *Finite singular points*: there are no singular points;

20.2. *Infinite singular points*: $\eta = 0$, $M = -8x^2 \neq 0$, $L = 8x^2 > 0$, $\mu_0 = \mu_1 = \mu_2 = \mu_3 = \kappa = \kappa_1 = 0$, $\mu_4 = x^4 > 0$, $K = 0$, $K_2 = 384x^2 > 0$;

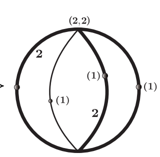
By [**23, 24**] $\Rightarrow$ 

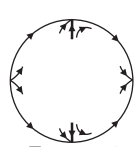
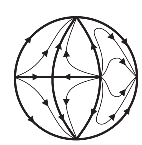
Config. 5.20
and 
Figure 18
then $\Longrightarrow$ 
Picture 5.20

21) *Configuration* 5.21. $\dot{x} = -1 + x^2$, $\dot{y} = x + 2y$;

21.1. *Finite singular points*: $M_1(-1, 1/2)$ is a saddle and $M_2(1, -1/2)$ is a node;

21.2. *Infinite singular points*: $\eta = 0$, $M = -8x^2 \neq 0$, $\mu_0 = \mu_1 = \kappa = \kappa_1 = 0$, $\mu_2 = 4x^2 > 0$, $L = 8x^2 > 0$, $K_2 = 384x^2 > 0$;

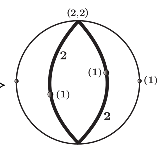
By [**23, 24**] $\Rightarrow$ 

Config. 5.21
and 
Figure 19
then $\Longrightarrow$ 
Picture 5.21

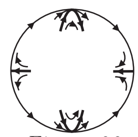
22) *Configuration* 5.22. $\dot{x} = 1 - x^2$, $\dot{y} = 1 - 2xy$;

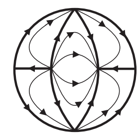
22.1. *Finite singular points*: $M_1(1, 1/2)$ and $M_2(-1, -1/2)$ are nodes;

22.2. *Infinite singular points*: $\eta = 0$, $M = -8x^2 \neq 0$, $\mu_0 = \mu_1 = \kappa = \kappa_1 = 0$, $\mu_2 = -4x^2 < 0$, $L = -8x^2 < 0$;

By [**23, 24**] $\Rightarrow$ 

Config. 5.22

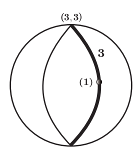
and 
Figure 29

then $\Longrightarrow$ 
Picture 5.22

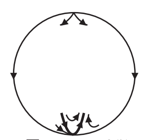
23) *Configuration* 5.23. $\dot{x} = -1 + x^2$, $\dot{y} = -3 + y - x^2 + xy$;

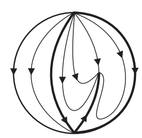
23.1. *Finite singular points*: $M_1(1, 2)$ is a (dicritical) node;

23.2. *Infinite singular points*: $\eta = 0$, $M = 0$, $\mu_0 = \mu_1 = \mu_2 = 0$, $\mu_3 = -8x^3$, $K = 2x^2$, $K_3 = 24x^6 > 0$;

By [**23, 24**] $\Rightarrow$ 

Config. 5.23

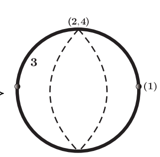
and 
Figure 37

then $\Longrightarrow$ 
Picture 5.23

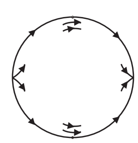
24) *Configuration* 5.24. $\dot{x} = 1 + x^2$, $\dot{y} = 1$;

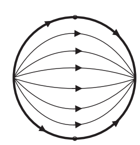
24.1. *Finite singular points*: there are no singular points;

24.2. *Infinite singular points*: $\eta = 0$, $M = -8x^2 \neq 0$, $\mu_4 = x^4 > 0$, $\mu_0 = \mu_1 = \mu_2 = \mu_3 = \kappa = \kappa_1 = 0$, $K = 0$, $L = 8x^2 > 0$, $K_2 = -384x^2 < 0$;

By [**23, 24**] $\Rightarrow$ 

Config. 5.24

and 
Figure 8

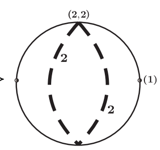
then $\Longrightarrow$ 
Picture 5.24

25) *Configuration* 5.25. $\dot{x} = 1 + x^2$, $\dot{y} = 1 + 2xy$;

25.1. *Finite singular points*: there are no singular points;

25.2. *Infinite singular points*: $\eta = 0$, $M = -8x^2 \neq 0$, $\mu_0 = \mu_1 = \kappa = \kappa_1 = 0$, $\mu_2 = 4x^2 > 0$, $L = -8x^2 < 0$;

By [**23, 24**] $\Rightarrow$



Config. 5.25

and

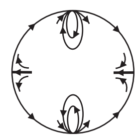
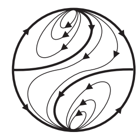


Figure 17

then $\Longrightarrow$



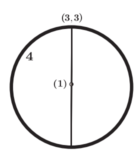
Picture 5.25

26) *Configuration* 5.26. $\dot{x} = -x$, $\dot{y} = y - x^2$;

26.1. *Finite singular points*: $M_1(0,0)$ is a saddle;

26.2. *Infinite singular points*: $\eta = 0$, $M = 0$, $C_2 = x^3 \neq 0$, $\mu_0 = \mu_1 = \mu_2 = 0$, $\mu_3 = -x^3 \neq 0$, $K = 0$, $K_1 = x^3$, $\mu_3 K_1 < 0$;

By [**23, 24**] $\Rightarrow$



Config. 5.26

and

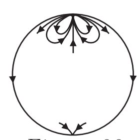
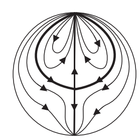


Figure 33

then $\Longrightarrow$



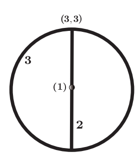
Picture 5.26

27) *Configuration* 5.27. $\dot{x} = 1 + x$, $\dot{y} = y - x^2$;

27.1. *Finite singular points*: $M_1(-1,1)$ is a node;

27.2. *Infinite singular points*: $\eta = 0$, $M = 0$, $C_2 = x^3 \neq 0$, $\mu_0 = \mu_1 = \mu_2 = 0$, $K = 0$, $\mu_3 K_1 = x^6 > 0$, $K_3 = 6x^6 > 0$;

By [**23, 24**] $\Rightarrow$



Config. 5.27

and

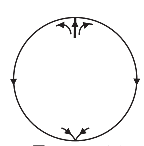
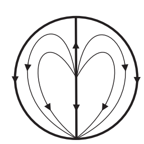


Figure 31

then $\Longrightarrow$



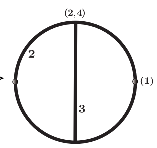
Picture 5.27

28) *Configuration* 5.28. $\dot{x} = x^2$, $\dot{y} = 1 + x$;

28.1. *Finite singular points*: there are no singular points;

28.2. *Infinite singular points*: $\eta = 0$, $M = -8x^2 \neq 0$, $L = 8x^2 > 0$, $\mu_0 = \mu_1 = \mu_2 = \mu_3 = \kappa = \kappa_1 = 0$, $\mu_4 = x^4 > 0$, $K = 0 = K_2$;

By [**23, 24**] $\Rightarrow$



Config. 5.28

and

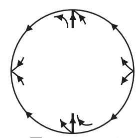
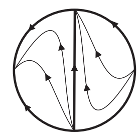


Figure 18

then $\Longrightarrow$

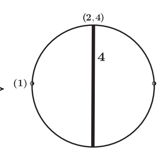

Picture 5.28

29) *Configuration* 5.29. $\dot{x} = x^2$, $\dot{y} = 1 + 2xy$;

29.1. *Finite singular points*: there are no singular points;

29.2. *Infinite singular points*: $\eta = 0$, $M = -8x^2 \neq 0$, $\mu_0 = \mu_1 = \mu_2 = \mu_3 = \kappa = \kappa_1 = 0$, $\mu_4 = x^4 > 0$, $L = -8x^2 < 0$;

By [**23, 24**] $\Rightarrow$



Config. 5.29

and

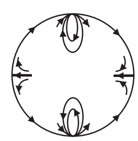
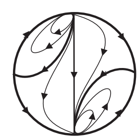


Figure 17

then $\Longrightarrow$

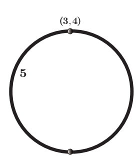

Picture 5.29

30) *Configuration* 5.30. $\dot{x} = 1$, $\dot{y} = x^2$;

30.1. *Finite singular points*: there are no singular points;

30.2. *Infinite singular points*: $\eta = 0 = M$, $C_2 = -x^3 \neq 0$, $\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0$, $\mu_4 = x^4 > 0$, $K_3 = 0$;

By [**23, 24**] $\Rightarrow$



Config. 5.30

and

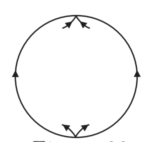
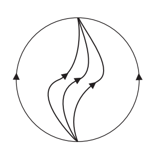


Figure 30

then $\Longrightarrow$


Picture 5.30

As all the classes from Table 2 are examined, Theorem 4.2 is proved. □

4.3 The projective classification in $\mathbf{P}_2(\mathbf{C})$ of the invariant lines configurations of systems in $\mathbf{QSL_5}$. Using the same notations introduced at the beginning of subsection 3.3 we add here the following notation.

Notation 4.5. We denote by $\mathbf{Eql_5}$ the class of all equations $\mathbf{EQ}$ of the form (E_0) obtained from an equation (1.3) possessing invariant lines of total multiplicity five.

We also need an integer-valued projective invariant.

Notation 4.6. Let us denote $\mathcal{N}_{m^{\mathbf{R}}_{\mathrm{Sing}}}$ is the number of the real invariant lines on which lie exactly $m^{\mathbf{R}}_{\mathrm{Sing}}$ real singularities of $\mathcal{G}$.

Notation 4.7. We denote by $N_{\mathbf{C}}$ the number of all distinct invariant lines $l_1, \dots, l_{N_{\mathbf{C}}}$ of a system (2.1). We denote by $\mathfrak{M}_i$ the multiplicity of the line l_i and by $n_{\mathfrak{M}_i}$ the number of the lines l_j, $j \in \{1, \dots, N_{\mathbf{C}}\}$ with multiplicity $\mathfrak{M}_i$. We denote $\mathfrak{M} = \max\{\mathfrak{M}_i \mid i = 1, \dots, N_{\mathbf{C}}\}$.

Then we have:

$$\sum_{i=1}^{N_{\mathbf{C}}} \mathfrak{M}_i \, n_{\mathfrak{M}_i} = M_{IL}.$$

Notation 4.8. Let us denote by $\mathfrak{m}_S$ the maximum of the multiplicities of singularities of the system which are located on any one of the lines and by $n_{\mathfrak{m}_S}$ the number of the singularities of the system with multiplicity $\mathfrak{m}_S$.

Notation 4.9. We denote $m^{\mathbf{P}}_{\mathcal{G}} = \max\{\nu(\omega) \mid \omega \in \mathrm{Sing}\,\mathcal{G}\}$.

Theorem 4.3. *We consider the systems in* $\mathbf{QSL_5}$ *and their associated real equations in* $\mathbf{Eql_5}$. *We consider the action of the group* $PGL(3, \mathbf{R})$ *of real projective transformations of the plane on the class* $\mathbf{EQ}$. *The classification of the orbits of equations* (E_0) *in* $\mathbf{EQ}$ *associ-*

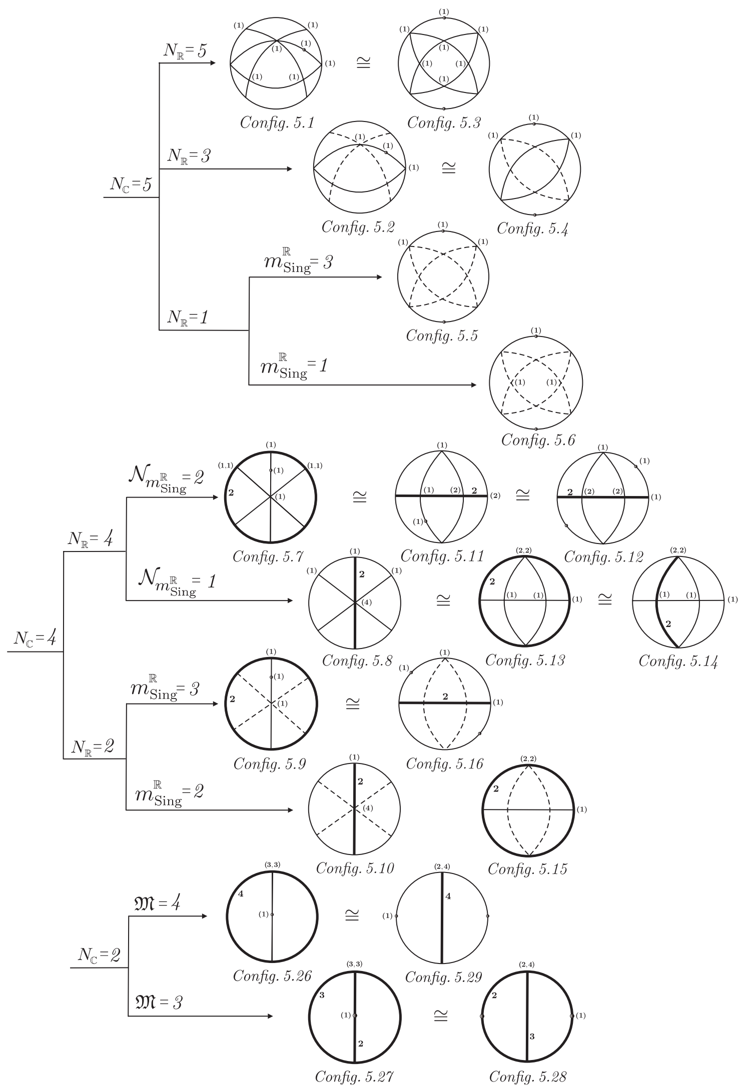


DIAGRAM 4 ($\mathbf{Eql_5}$).

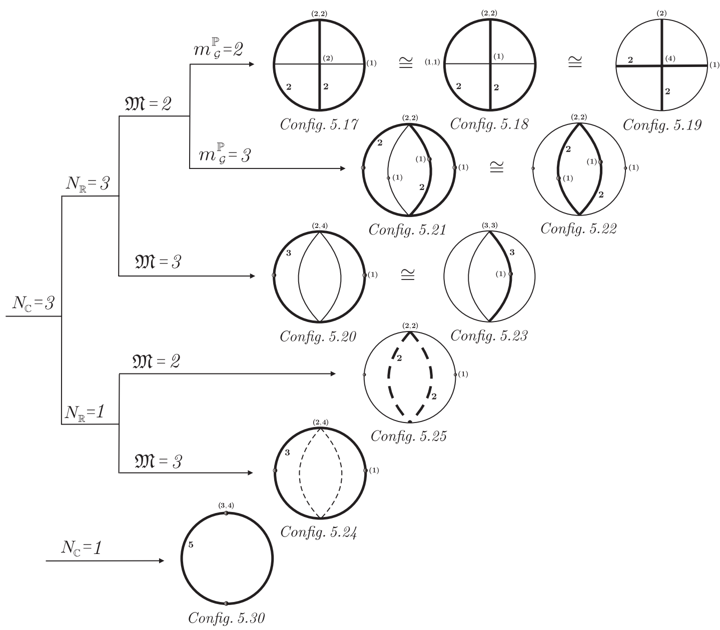


DIAGRAM 4 ($\mathbf{Eql_5}$) (Continued).

ated to systems (S) *in* $\mathbf{QSL_5}$*, under the action of* $PGL(3,\mathbf{R})$ *on* $\mathbf{EQ}$ *is given in Diagram* 4.

Proof. The proofs are obtained in an analogous way to case ii) in Theorem 3.3, and we only list below the corresponding transformations (the respective equations in $\mathbf{Eql_5}$ can be easily computed directly having canonical systems and the equation (3.2)).

$$1)\quad \begin{bmatrix} \text{Config. 5.1}: \\ \dot{x}=(x+1)(gx+1), \\ \dot{y}=(g-1)xy+y^2 \end{bmatrix} : \begin{pmatrix} 0 & -g & g-2 \\ g-1 & 0 & g-1 \\ 0 & g & g \end{pmatrix} \Longrightarrow \begin{bmatrix} \text{Config. 5.3}: \\ \dot{x}=-1+x^2, \\ \dot{y}=y=g'(y^2-1) \end{bmatrix};$$

$$2)\quad \begin{bmatrix} \text{Config. 5.2}: \\ \dot{x}=g(x^2-4), \\ \dot{y}=(g^2-4)-x^2 \\ +(g^2+4)x \\ +gxy-y^2 \end{bmatrix} : \begin{pmatrix} -2 & 0 & 6 \\ g & 8 & g \\ -1 & 0 & -1 \end{pmatrix} \Longrightarrow \begin{bmatrix} \text{Config. 5.4}: \\ \dot{x}=-1+x^2, \\ \dot{y}=g'(y^2+1) \end{bmatrix};$$

$$3)\ \begin{bmatrix}\text{Config. 5.7}:\\ \dot{x}=1+x,\\ \dot{y}=-xy+y^2\end{bmatrix}:\begin{pmatrix}-1&1&1\\0&1&0\\1&0&0\end{pmatrix}\Longrightarrow\begin{bmatrix}\text{Config. 5.11}:\\ \dot{x}=x^2+xy,\\ \dot{y}=y+y^2\end{bmatrix};$$

$$4)\ \begin{bmatrix}\text{Config. 5.7}:\\ \dot{x}=1+x,\\ \dot{y}=-xy+y^2\end{bmatrix}:\begin{pmatrix}0&-1&2\\-1&0&1\\0&1&0\end{pmatrix}\Longrightarrow\begin{bmatrix}\text{Config. 5.12}:\\ \dot{x}=-1+x^2,\\ \dot{y}=y^2\end{bmatrix};$$

$$5)\ \begin{bmatrix}\text{Config. 5.8}:\\ \dot{x}=gx^2,\\ \dot{y}=(g-1)xy+y^2\end{bmatrix}:\begin{pmatrix}0&0&-2\\1&0&-1\\0&1/(2g)&0\end{pmatrix}\Longrightarrow\begin{bmatrix}\text{Config. 5.13}:\\ \dot{x}=g'(x^2-1),\\ \dot{y}=2y\end{bmatrix};$$

$$6)\ \begin{bmatrix}\text{Config. 5.8}:\\ \dot{x}=gx^2,\\ \dot{y}=(g-1)xy+y^2\end{bmatrix}:\begin{pmatrix}1&0&1\\0&0&1-g\\0&\frac{1}{g(g-1)}&0\end{pmatrix}\Longrightarrow\begin{bmatrix}\text{Config. 5.14}:\\ \dot{x}=(x+1)\times\\ (g'x+1),\\ \dot{y}=(g'-1)xy\end{bmatrix};$$

$$7)\ \begin{bmatrix}\text{Config. 5.9}:\\ \dot{x}=2x,\\ \dot{y}=1-x^2-y^2\end{bmatrix}:\begin{pmatrix}0&0&-1\\1&-1&0\\0&1&0\end{pmatrix}\Longrightarrow\begin{bmatrix}\text{Config. 5.16}:\\ \dot{x}=1+x^2,\\ \dot{y}=y^2\end{bmatrix};$$

$$8)\ \begin{bmatrix}\text{Config. 5.10}:\\ \dot{x}=gx^2,\\ \dot{y}=-x^2+gxy-y^2\end{bmatrix}:\begin{pmatrix}0&0&-1\\1&0&0\\0&1&0\end{pmatrix}\Longrightarrow\begin{bmatrix}\text{Config. 5.15}:\\ \dot{x}=g'(x^2+1),\\ \dot{y}=2y\end{bmatrix};$$

$$9)\ \begin{bmatrix}\text{Config. 5.17}:\\ \dot{x}=x^2,\\ \dot{y}=2y\end{bmatrix}:\begin{pmatrix}0&0&2\\0&1/4&0\\1&0&1\end{pmatrix}\Longrightarrow\begin{bmatrix}\text{Config. 5.18}:\\ \dot{x}=1+x,\\ \dot{y}=-xy\end{bmatrix};$$

$$10)\ \begin{bmatrix}\text{Config. 5.17}:\\ \dot{x}=x^2,\\ \dot{y}=2y\end{bmatrix}:\begin{pmatrix}-2&0&0\\0&0&-1/4\\0&1&0\end{pmatrix}\Longrightarrow\begin{bmatrix}\text{Config. 5.19}:\\ \dot{x}=x^2+xy,\\ \dot{y}=y^2\end{bmatrix};$$

$$11)\ \begin{bmatrix}\text{Config. 5.20}:\\ \dot{x}=-1+x^2,\\ \dot{y}=1\end{bmatrix}:\begin{pmatrix}1&0&3\\-1&1/2&0\\1&0&-1\end{pmatrix}\Longrightarrow\begin{bmatrix}\text{Config. 5.23}:\\ \dot{x}=-1+x^2,\\ \dot{y}=-3+y\\ -x^2+xy\end{bmatrix};$$

$$12)\ \begin{bmatrix}\text{Config. 5.21}:\\ \dot{x}=-1+x^2,\\ \dot{y}=x+2y\end{bmatrix}:\begin{pmatrix}-2&0&6\\1&-4&-1\\2&0&2\end{pmatrix}\Longrightarrow\begin{bmatrix}\text{Config. 5.22}:\\ \dot{x}=1-x^2,\\ \dot{y}=1-2xy\end{bmatrix};$$

$$13)\ \begin{bmatrix}\text{Config. 5.26}:\\ \dot{x}=-x,\\ \dot{y}=y-x^2\end{bmatrix}:\begin{pmatrix}0&0&1\\0&1&0\\-1&0&0\end{pmatrix}\Longrightarrow\begin{bmatrix}\text{Config. 5.29}:\\ \dot{x}=x^2,\\ \dot{y}=1+2xy\end{bmatrix};$$

$$14)\ \begin{bmatrix}\text{Config. 5.27}:\\ \dot{x}=1+x,\\ \dot{y}=y-x^2\end{bmatrix}:\begin{pmatrix}-1&0&-2\\1&4&0\\1&0&0\end{pmatrix}\Longrightarrow\begin{bmatrix}\text{Config. 5.28}:\\ \dot{x}=x^2,\\ \dot{y}=1+x\end{bmatrix}.$$

5. Invariant conditions for distinguishing topological phase portraits. We consider two equivalence relations on the classes of all real quadratic differential systems which possess invariant lines of total multiplicity five ($\mathbf{QSL_5}$) and six ($\mathbf{QSL_6}$): the topological equivalence relation of the phase portraits and the equivalence relation induced by the action of the affine group and time rescaling.

We note that the first equivalence relation is coarser than the second. Indeed, we could even have an infinite set of equivalence classes of the second relation, all included in the same equivalence class of the first relation.

In this section we consider the following problem:

Give necessary and sufficient conditions for quadratic systems in $\mathbf{QSL_5}$ *and for* $\mathbf{QSL_6}$*, formulated only in terms of algebraic invariants and comitants depending on the coefficients of the systems*: $\mathbf{a} \in \mathbf{R}^{12}$ *such that two systems have topologically equivalent phase portraits.*

We shall use here the CT-comitants constructed in [**23**] as follows.

Notation 5.1. Consider the polynomial $\Phi_{\alpha,\beta} = \alpha P + \beta Q \in \mathbf{R}[a, X, Y, Z, \alpha, \beta]$ where $P = Z^2 p(X/Z, Y/Z)$, $Q = Z^2 q(X/Z, Y/Z)$, $p, q \in \mathbf{R}[a, x, y]$ and $\max(\deg_{(x,y)} p, \deg_{(x,y)} q) = 2$. Then

$$\begin{aligned}\Phi_{\alpha,\beta} = \ & c_{11}(\alpha,\beta)X^2 + 2c_{12}(\alpha,\beta)XY + c_{22}(\alpha,\beta)Y^2 \\ & + 2c_{13}(\alpha,\beta)XZ + 2c_{23}(\alpha,\beta)YZ + c_{33}(\alpha,\beta)Z^2,\end{aligned}$$

$$\Delta(a,\alpha,\beta) = \det \|c_{ij}(\alpha,\beta)\|_{i,j\in\{1,2,3\}},$$
$$D(a,x,y) = 4\Delta(a,-y,x),$$
$$H(a,x,y) = 4\big[\det \|c_{ij}(-y,x)\|_{i,j\in\{1,2\}}\big].$$

Let us consider the polynomials

$$C_i(a,x,y) = yp_i(a,x,y) - xq_i(a,x,y) \in \mathbf{R}[a,x,y], \quad i = 0,1,2,$$
$$D_i(a,x,y) = \frac{\partial}{\partial x}p_i(a,x,y) + \frac{\partial}{\partial y}q_i(a,x,y) \in \mathbf{R}[a,x,y], \quad i = 1,2.$$

We construct the following T-comitants and CT-comitants, cf. [**23**]:

$$
\begin{aligned}
B_3(a,x,y) &= (C_2,D)^{(1)} = \text{Jacob}\,(C_2,D)\,,\\
B_2(a,x,y) &= (B_3,B_3)^{(2)} - 6B_3(C_2,D)^{(3)},\\
B_1(a) &= \text{Res}_x\,(C_2,D)\,/y^9 = -2^{-9}3^{-8}\,(B_2,B_3)^{(4)}\,,\\
\mu(a) &= \text{Discriminant}\,\big(K(a,x,y)\big) = \mu_0(a),\\
N(a,x,y) &= K(a,x,y) + H(a,x,y),\\
\theta(a) &= \text{Discriminant}\,\big(N(a,x,y)\big).
\end{aligned}
\tag{5.1}
$$

Notation 5.2.

$$
\begin{aligned}
H_1(a) &= -\big((C_2,C_2)^{(2)},C_2)^{(1)},D\big)^{(3)},\\
H_2(a,x,y) &= (C_1,\ 2H-N)^{(1)} - 2D_1N,\\
H_3(a,x,y) &= (C_2,D)^{(2)},\\
H_4(a) &= \big((C_2,D)^{(2)},(C_2,D_2)^{(1)}\big)^{(2)},\\
H_5(a) &= \big((C_2,C_2)^{(2)},(D,D)^{(2)}\big)^{(2)}\\
&\quad + 8\big((C_2,D)^{(2)},(D,D_2)^{(1)}\big)^{(2)},\\
H_6(a,x,y) &= 16N^2(C_2,D)^{(2)} + H_2^2(C_2,C_2)^{(2)},\\
N_1(a,x,y) &= C_1(C_2,C_2)^{(2)} - 2C_2(C_1,C_2)^{(2)},\\
N_2(a,x,y) &= D_1(C_1,C_2)^{(2)} - \Big((C_2,C_2)^{(2)},C_0\Big)^{(1)},\\
N_3(a,x,y) &= (C_2,C_1)^{(1)}\,,\\
N_4(a,x,y) &= 4\,(C_2,C_0)^{(1)} - 3C_1D_1,\\
N_5(a,x,y) &= \big[(D_2,C_1)^{(1)} + D_1D_2\big]^2 - 4\big(C_2,C_2\big)^{(2)}\big(C_0,D_2\big)^{(1)},\\
N_6(a,x,y) &= 8D + C_2\Big[8(C_0,D_2)^{(1)} - 3(C_1,C_1)^{(2)} + 2D_1^2\Big].
\end{aligned}
$$

TABLE 4 ($M_{\mathbf{IL}} = 6$).

Orbit representative	Necessary and sufficient conditions	Configuration
$(VI.1)\begin{cases}\dot{x} = -1 + x^2,\\ \dot{y} = -1 + y^2\end{cases}$	$\eta > 0,\ B_3 = N = 0,\ H_1 > 0$	*Config. 6.1*
$(VI.2)\begin{cases}\dot{x} = 1 + x^2,\\ \dot{y} = 1 + y^2\end{cases}$	$\eta > 0,\ B_3 = N = 0,\ H_1 < 0$	*Config. 6.2*
$(VI.3)\begin{cases}\dot{x} = 2xy,\\ \dot{y} = y^2 - x^2 - 1\end{cases}$	$\eta < 0,\ B_3 = N = 0,\ H_1 < 0$	*Config. 6.3*
$(VI.4)\begin{cases}\dot{x} = 2xy,\\ \dot{y} = 1 - x^2 + y^2\end{cases}$	$\eta < 0,\ B_3 = N = 0,\ H_1 > 0$	*Config. 6.4*
$(VI.5)\ \dot{x} = x^2,\ \dot{y} = y^2$	$\eta > 0,\ B_3 = N = H_1 = 0$	*Config. 6.5*
$(VI.6)\begin{cases}\dot{x} = 2xy,\\ \dot{y} = -x^2 + y^2\end{cases}$	$\eta < 0,\ B_3 = N = H_1 = 0$	*Config. 6.6*
$(VI.7)\begin{cases}\dot{x} = -1 + x^2,\\ \dot{y} = 2y\end{cases}$	$MD \neq 0,\ \eta = B_3 = N = 0,$ $H = N_1 = N_2 = 0$	*Config. 6.7*
$(VI.8)\begin{cases}\dot{x} = x^2 - 1,\\ \dot{y} = 2xy\end{cases}$	$MH \neq 0,\ \eta = B_3 = N = 0,$ $H_2 = 0, H_3 > 0$	*Config. 6.8*
$(VI.9)\begin{cases}\dot{x} = 1 + x^2,\\ \dot{y} = 2xy\end{cases}$	$MH \neq 0,\ \eta = B_3 = N = 0,$ $H_2 = 0, H_3 < 0$	*Config. 6.9*
$(VI.10)\ \dot{x} = x^2, \dot{y} = 1$	$M \neq 0,\ \eta = B_3 = N = 0,$ $H = D = N_1 = N_2 = 0$	*Config. 6.10*
$(VI.11)\begin{cases}\dot{x} = x,\\ \dot{y} = y - x^2\end{cases}$	$\eta = M = B_3 = N = 0,$ $N_3 = N_4 = 0$	*Config. 6.11*

The following theorem, which is proved in [**23**] using these invariant polynomials, will be applied here to construct the conditions mentioned in the above problem.

TABLE 5 ($M_{\mathbf{IL}} = 5$).

Orbit representative	Necessary and sufficient conditions	Configuration
(V.1) $\begin{cases} \dot{x} = (x+1)(gx+1), \\ \dot{y} = (g-1)xy + y^2, \ g(g^2-1) \neq 0 \end{cases}$	$\eta > 0, \ B_3 = \theta = 0,$ $N \neq 0, \ \mu \neq 0, \ H_1 \neq 0$	Config. 5.1
(V.2) $\begin{cases} \dot{x} = g(x^2-4), \ g \neq 0 \\ \dot{y} = (g^2-4) - x^2 - y^2 + (g^2+4)x + gxy \end{cases}$	$\eta < 0, \ B_3 = \theta = 0,$ $N \neq 0, \ \mu \neq 0, \ H_1 \neq 0$	Config. 5.2
(V.3) $\begin{cases} \dot{x} = -1 + x^2, \ \dot{y} = g(y^2-1), \\ g(g^2-1) \neq 0 \end{cases}$	$\eta > 0, \ B_2 = N = 0, \ B_3 \neq 0,$ $H_1 > 0, \ H_4 = 0, \ H_5 > 0$	Config. 5.3
(V.4) $\begin{cases} \dot{x} = -1 + x^2, \ g \neq 0 \\ \dot{y} = g(y^2+1), \end{cases}$	$\eta > 0, \ B_2 = N = 0, \ B_3 \neq 0,$ $H_4 = 0, \ H_5 < 0$	Config. 5.4
(V.5) $\begin{cases} \dot{x} = 1 + x^2, \ \lvert g \rvert \neq 0, 1 \\ \dot{y} = g(y^2+1) \end{cases}$	$\eta > 0, \ B_2 = N = 0, \ B_3 \neq 0,$ $H_1 < 0, \ H_4 = 0, \ H_5 > 0$	Config. 5.5
(V.6) $\dot{x} = 1 + 2xy, \ \dot{y} = g - x^2 + y^2, \ g \in \mathbf{R}$	$\eta < 0, \ B_3 \neq 0, \ B_2 = N = 0$	Config. 5.6
(V.7) $\dot{x} = 1 + x, \ \dot{y} = -xy + y^2$	$\eta > 0, \ B_3 = \theta = 0,$ $N \neq 0, \ \mu = H_6 = 0$	Config. 5.7
(V.8) $\begin{cases} \dot{x} = gx^2, \lvert g \rvert \neq 0, 1 \\ \dot{y} = (g-1)xy + y^2 \end{cases}$	$\eta > 0, \ B_3 = \theta = 0,$ $N \neq 0, \ \mu \neq 0, \ H_1 = 0$	Config. 5.8
(V.9) $\dot{x} = 2x, \ \dot{y} = 1 - x^2 - y^2$	$\eta < 0, \ B_3 = \theta = 0,$ $N \neq 0, \ \mu = H_6 = 0$	Config. 5.9
(V.10) $\begin{cases} \dot{x} = gx^2, \ g \neq 0 \\ \dot{y} = -x^2 + gxy - y^2 \end{cases}$	$\eta < 0, \ B_3 = \theta = 0,$ $N \neq 0, \ \mu \neq 0, \ H_1 = 0$	Config. 5.10
(V.11) $\dot{x} = x^2 + xy, \dot{y} = y + y^2$	$\eta = 0, \ M \neq 0, \ B_3 = \theta = 0,$ $\mu \neq 0, \ N \neq 0, \ D \neq 0$	Config. 5.11
(V.12) $\dot{x} = -1 + x^2, \dot{y} = y^2$	$\eta > 0, \ B_2 = N = 0, \ B_3 \neq 0,$ $H_1 > 0, \ H_4 = H_5 = 0$	Config. 5.12
(V.13) $\begin{cases} \dot{x} = g(x^2-1), \\ \dot{y} = 2y, \lvert g \rvert \neq 0, 1 \end{cases}$	$\eta = 0, \ M \neq 0, \ B_3 = N = 0,$ $H = N_1 = 0, \ N_2 D \neq 0, \ N_5 > 0$	Config. 5.13
(V.14) $\begin{cases} \dot{x} = (x+1)(gx+1), \\ \dot{y} = (g-1)xy, \ g(g^2-1) \neq 0 \end{cases}$	$\eta = 0, \ M \neq 0, \ B_3 = \theta = 0,$ $NK \neq 0, \ \mu = H_6 = 0$	Config. 5.14
(V.15) $\begin{cases} \dot{x} = g(x^2+1), \\ \dot{y} = 2y, g \neq 0 \end{cases}$	$\eta = 0, \ M \neq 0, \ B_3 = N = 0,$ $H = N_1 = 0, \ N_2 D \neq 0, \ N_5 < 0$	Config. 5.15

TABLE 5 ($M_{\mathbf{IL}} = 5$) (continued).

Orbit representative	Necessary and sufficient conditions	Configuration
(V.16) $\dot{x} = 1 + x^2, \dot{y} = y^2$	$\eta > 0,\ B_2 = N = 0,\ B_3 \neq 0,$ $H_1 < 0,\ H_4 = H_5 = 0$	Config. 5.16
(V.17) $\dot{x} = x^2, \dot{y} = 2y$	$\eta = 0,\ M \neq 0,\ B_3 = N = 0,$ $H = N_1 = 0,\ N_2 D \neq 0,\ N_5 = 0$	Config. 5.17
(V.18) $\dot{x} = 1 + x, \dot{y} = -xy$	$\eta = 0,\ M \neq 0,\ B_3 = \theta = 0,$ $N \neq 0,\ \mu = K = H_6 = 0$	Config. 5.18
(V.19) $\dot{x} = x^2 + xy, \dot{y} = y^2$	$\eta = 0,\ M \neq 0,\ B_3 = \theta = 0,$ $\mu \neq 0,\ N \neq 0,\ D = 0$	Config. 5.19
(V.20) $\dot{x} = -1 + x^2, \dot{y} = 1$	$\eta = 0,\ M \neq 0,\ B_3 = N = 0,$ $H = D = N_1 = 0,\ N_2 \neq 0,\ N_5 > 0$	Config. 5.20
(V.21) $\begin{cases} \dot{x} = -1 + x^2, \\ \dot{y} = x + 2y \end{cases}$	$\eta = 0,\ M \neq 0,\ B_3 = N = 0,$ $H = N_2 = 0,\ D \neq 0,\ N_1 \neq 0$	Config. 5.21
(V.22) $\begin{cases} \dot{x} = 1 - x^2, \\ \dot{y} = 1 - 2xy \end{cases}$	$\eta = 0,\ M \neq 0,\ B_2 = N = 0,$ $B_3 \neq 0,\ H_2 = 0,\ H_3 > 0$	Config. 5.22
(V.23) $\begin{cases} \dot{x} = -1 + x^2, \\ \dot{y} = -3 + y - x^2 + xy \end{cases}$	$\eta = M = 0,\ N \neq 0,$ $B_3 = \theta = N_6 = 0$	Config. 5.23
(V.24) $\dot{x} = 1 + x^2,\ \dot{y} = 1$	$\eta = 0,\ M \neq 0,\ B_3 = N = 0,$ $H = D = N_1 = 0,\ N_2 \neq 0,\ N_5 < 0$	Config. 5.24
(V.25) $\begin{cases} \dot{x} = 1 + x^2, \\ \dot{y} = 1 + 2xy \end{cases}$	$\eta = 0,\ M \neq 0,\ B_2 = N = 0,$ $B_3 \neq 0,\ H_2 = 0,\ H_3 < 0$	Config. 5.25
(V.26) $\dot{x} = -x,\ \dot{y} = y - x^2$	$\eta = M = 0,\ N_3 \neq 0,$ $B_3 = N = D_1 = 0$	Config. 5.26
(V.27) $\dot{x} = 1 + x,\ \dot{y} = y - x^2$	$\eta = M = 0,\ N_4 \neq 0,$ $B_3 = N = N_3 = 0,\ D_1 \neq 0$	Config. 5.27
(V.28) $\dot{x} = x^2, \dot{y} = 1 + x$	$\eta = 0,\ M \neq 0,\ B_3 = N = 0,$ $H = D = N_2 = 0,\ N_1 \neq 0$	Config. 5.28
(V.29) $\dot{x} = x^2, \dot{y} = 1 + 2xy$	$\eta = 0,\ M \neq 0,\ B_2 = N = 0,$ $B_3 \neq 0,\ H_2 = H_3 = 0$	Config. 5.29
(V.30) $\dot{x} = 1, \dot{y} = x^2$	$\eta = M = 0,\ N_4 \neq 0,$ $B_3 = N = N_3 = D_1 = 0$	Config. 5.30

Theorem 5.1 [**23**]. *We consider the orbits of the class* $\mathbf{QSL_6}$, *respectively* $\mathbf{QSL_5}$, *under the action of the real affine group and time rescaling. The systems* (VI.1) *up to* (VI.11), *respectively* (V.1) *up to* (V.30), *from Table* 4, *respectively Table* 5, *form a system of representatives of these orbits under this action. A differential system* (S) *in* $\mathbf{QSL_6}$, *respectively* $(S) \in \mathbf{QSL_5}$, *is in the orbit of a system belonging to* (VI.i), *respectively* (V.i), *if and only if the corresponding conditions in the middle column are verified for this system* (S). *The conditions indicated in the middle column, jointly taken, are invariant under the action of this group.*

Theorems 3.2 and 4.2 yield 42 phase portraits not necessarily topologically distinct, each being generated by one of the 41 configurations. We first retain only all topologically distinct phase portraits obtained from the 42 pictures in Diagrams 1 and 3 after topological identification. Furthermore, for each one of these topologically distinct phase portraits, we want to find necessary and sufficient invariant conditions under the action of the group $Aff\,(2, \mathbf{R}) \times \mathbf{R}^*$.

Theorem 5.2. *Assume that a quadratic system* (2.1) *belongs to* $\mathbf{QSL_6} \cup \mathbf{QSL_5}$. *Then its phase portrait is topologically equivalent with one of the* 28 *portraits, listed below, which all appear in Diagrams* 1 *and* 3. *A particular phase portrait occurs if and only if the associated, respectively one of the associated, set of the conditions, listed below, jointly taken are satisfied*:

1) *Picture 6.2* $\Longleftrightarrow \eta > 0,\ B_3 = N = 0,\quad H_1 < 0;$
2) *Picture 6.3* $\Longleftrightarrow \eta < 0,\ B_3 = N = 0,\quad H_1 < 0;$
3) *Picture 6.4* $\Longleftrightarrow \eta < 0,\ B_3 = N = 0,\quad H_1 > 0;$
4) *Picture 6.6* $\Longleftrightarrow \eta < 0,\ B_3 = N = 0,\quad H_1 = 0;$
5) *Picture 6.9* $\Longleftrightarrow \eta = 0,\ M \neq 0,\ B_3 = N = H_2 = 0,\ H \neq 0,\ H_3 < 0;$
6) *Picture 5.2* $\Longleftrightarrow \eta < 0,\ B_3 = \theta = 0,\ N \neq 0,\ \mu H_1 \neq 0;$
7) *Picture 5.6* $\Longleftrightarrow \eta < 0,\ B_2 = N = 0,\ B_3 \neq 0;$
8) *Picture 5.7* $\Longleftrightarrow \eta > 0,\ B_3 = \theta = 0,\ N \neq 0,\ \mu = H_6 = 0;$
9) *Picture 5.9* $\Longleftrightarrow \eta < 0,\ B_3 = \theta = 0,\ N \neq 0,\ \mu = H_6 = 0;$
10) *Picture 5.10* $\Longleftrightarrow \eta < 0,\ B_3 = \theta = 0,\ N \neq 0,\ \mu \neq 0,\ H_1 = 0;$

11) *Picture 5.11* $\Longleftrightarrow \eta=0,\ M\neq 0,\ B_3=\theta=0,\ N\neq 0,\ \mu D\neq 0;$

12) *Picture 5.12* $\Longleftrightarrow \begin{bmatrix}\eta>0,\ B_2=N=0,\ B_3\neq 0,\\ H_4=H_5=0,\ H_1>0;\end{bmatrix}$

13) *Picture 5.17* $\Longleftrightarrow \begin{bmatrix}\eta=0,\ M\neq 0,\ B_3=N=H=0,\\ N_1=N_5=0,\ N_2D\neq 0;\end{bmatrix}$

14) *Picture 5.18* $\Longleftrightarrow \begin{bmatrix}\eta=0,\ M\neq 0,\ B_3=\theta=0,\ N\neq 0,\\ \mu=K=H_6=0;\end{bmatrix}$

15) *Picture 5.19* $\Longleftrightarrow \begin{bmatrix}\eta=0,\ M\neq 0,\ B_3=\theta=0,\ N\neq 0,\\ \mu\neq 0,\ D=0;\end{bmatrix}$

16) *Picture 5.20* $\Longleftrightarrow \begin{bmatrix}\eta=0,\ M\neq 0,\ B_3=N=H=D=0,\\ N_1=0,\ N_2\neq 0,\ N_5>0;\end{bmatrix}$

17) *Picture 5.23* $\Longleftrightarrow \eta=M=0,\ C_2\neq 0,\ N\neq 0,\ B_3=\theta=N_6=0;$

18) *Picture 5.26* $\Longleftrightarrow \begin{bmatrix}\eta=M=0,\ C_2\neq 0,\ B_3=N=0,\\ D_1=0,\ N_3\neq 0;\end{bmatrix}$

19) *Picture 5.30* $\Longleftrightarrow \begin{bmatrix}\eta=M=0,\ C_2\neq 0,\ B_3=N=N_3=0,\\ D_1=0,\ N_4\neq 0;\end{bmatrix}$

20) *Picture 6.1* $\cong \begin{bmatrix}5.1,\\ 5.3\end{bmatrix} \Longleftrightarrow \begin{bmatrix}\eta>0,\ B_2=\theta=B_3N=H_4=0,\\ \mu\neq 0,\ H_5>0,\ H_1>0;\end{bmatrix}$

21) *Picture 6.5* $\cong \begin{bmatrix}5.8\end{bmatrix} \Longleftrightarrow \eta>0,\ B_3=\theta=0,\ \mu\neq 0,\ H_1=0;$

22) *Picture 6.7* $\cong \begin{bmatrix}5.13,\\ 5.21,\\ 5.14(a)\end{bmatrix} \Longleftrightarrow \left\{\begin{array}{l}\begin{bmatrix}\eta=0, M\neq 0, B_3=\theta=\mu=0,\\ H_6=0, NK\neq 0, L>0 \quad \text{or}\end{bmatrix}\\ \begin{bmatrix}\eta=0, M\neq 0, B_3=N=H=0,\\ N_1N_2=0, N_5>0, D\neq 0;\end{bmatrix}\end{array}\right.$

23) *Picture 6.8* $\cong \begin{bmatrix}5.22,\\ 5.14(b)\end{bmatrix} \Longleftrightarrow \left\{\begin{array}{l}\begin{bmatrix}\eta=0, M\neq 0, B_2=N=0\\ H_2=0, H_3>0 \quad \text{or}\end{bmatrix}\\ \begin{bmatrix}\eta=0, M\neq 0, B_3=\theta=\mu=0,\\ H_6=0, NK\neq 0,\ L<0;\end{bmatrix}\end{array}\right.$

24) *Picture 6.10* $\cong \begin{bmatrix}5.28\end{bmatrix} \Longleftrightarrow \begin{bmatrix}\eta=0,\ M\neq 0,\ B_3=N=0,\\ H=D=N_2=0;\end{bmatrix}$

25) *Picture 6.11* $\cong \begin{bmatrix}5.27\end{bmatrix} \Longleftrightarrow \eta=M=0,\ B_3=N=N_3=0;$

26) *Picture 5.4* $\cong \begin{bmatrix}5.5,\\ 5.16\end{bmatrix} \Longleftrightarrow \left\{\begin{array}{l}\begin{bmatrix}\eta>0,\ B_2=N=H_4=0,\\ B_3\neq 0,\ H_5<0 \quad \text{or}\end{bmatrix}\\ \begin{bmatrix}\eta>0, B_2=N=H_4=0,\\ B_3\neq 0,\ H_5\geq 0,\ H_1<0;\end{bmatrix}\end{array}\right.$

27) *Picture 5.15* $\cong \begin{bmatrix}5.24\end{bmatrix} \Longleftrightarrow \begin{bmatrix}\eta=0,\ M\neq 0,\ B_3=N=H=0,\\ N_1=0,\ N_2\neq 0,\ N_5<0;\end{bmatrix}$

28) *Picture 5.25* $\cong \begin{bmatrix}5.29\end{bmatrix} \Longleftrightarrow \begin{bmatrix}\eta=0,\ M\neq 0,\ B_2=N=0,\\ B_3\neq 0,\ H_2=0,\ H_3\leq 0;\end{bmatrix}$

Proof. From Diagrams 1 and 3 it can easily be seen that we have the topological equivalences indicated in points 20) $\rightarrow$ 28) above.

On the other hand, it can easily be seen that the pictures listed in points 1) $\rightarrow$ 19) which correspond to the distinct configurations, are topologically distinct among themselves and topologically distinct from any one of the pictures listed in points 20) $\rightarrow$ 28). Therefore, the corresponding conditions, invariant under the action of the group $\mathit{Aff}\,(2,\mathbf{R})\times\mathbf{R}^*$, for each one of the pictures listed in points 1) $\rightarrow$ 19) are those which have already appeared in Table 4 (for Configuration 6.i, $i\in\{2,3,4,6,9\}$) and in Table 5 (for Configuration 5.j, $j\in\{2,6,7,9,\dots,12,17,\dots,20,23,26,30\}$). It only remains to find invariant conditions for the cases 20) $\rightarrow$ 28). A common feature of all these last cases is that for each one of them a single topological phase portrait is associated to at least two distinct configurations.

We first observe that according to (5.1) we have the following:

Remark 5.3. The condition $B_3=0$ yields $B_2=0=B_1$ and the condition $N=0$ yields $\theta=0$.

Case 20) According to Theorem 5.1 and Remark 5.3, the conditions $\eta>0$, $B_2=0$ and $\theta=0$ are satisfied for each one of the canonical systems (VI.1), (V.1) and (V.3).

However, there are more conditions which should be satisfied and we list the remaining ones:

$$\begin{aligned}
&\text{(VI.1)}: B_3=0,\ N=0,\ H_1>0;\\
&\text{(V.1)}: B_3=0,\ N\neq0,\ \mu\neq0,\ H_1\neq0;\\
&\text{(V.3)}: B_3\neq0,\ N=0,\ H_1>0,\ H_4=0,\ H_5>0.
\end{aligned}\tag{5.2}$$

To integrate in one sequence of conditions, covering all these cases, we calculate: for (VI.1), μ, H_4 and H_5; for (V.1), H_1, H_4 and H_5; for (V.3), μ, and we obtain

$$\begin{aligned}
&\text{(VI.1)}: \quad \mu=16\neq0,\ H_4=0,\ H_5=6144>0;\\
&\text{(V.1)}: H_1=576(g-1)^2>0,\ H_4=0,\ H_5=384(g-1)^4>0;\\
&\text{(V.3)}: \quad \mu=16g^2\neq0.
\end{aligned}\tag{5.3}$$

Looking at (5.2) and (5.3) we observe that the three sequences of conditions for the three canonical forms (VI.1), (V.1) and (V.3) can be integrated as one sequence, which is exactly the one indicated in point 20) of Theorem 5.2.

Case 21) Taking into account Remark 5.3 and Tables 4 and 5 we have that besides the conditions $\eta > 0$, $B_3 = 0$, $\theta = 0$ and $H_1 = 0$ which are common to the canonical systems (VI.5) and (V.8), we have additional conditions:

$$(\text{VI}.5) : N = 0,\ \mu = 16 \neq 0; \qquad (\text{V}.8) : N \neq 0,\ \mu \neq 0.$$

Analogously to the previous case, for the realization of the phase portrait corresponding to both Configurations 6.5 and 5.8 we obtain here exactly the sequence of conditions (jointly taken), indicated in point 21) of Theorem 5.2.

Case 22) It is easy to establish that the phase portraits given by Pictures 6.7, 5.13, 5.14 (a) and 5.21 (see Diagrams 1 and 3) are topologically equivalent.

Remark 5.4. As shown earlier, Configuration 5.14 yields the phase portrait given by Picture 5.14 (a) for $g > 0$ and given by Picture 5.14 (b) for $g < 0$. Since for the systems (V.14) we have $L = 8gx^2$, we obtain Picture 5.14 (a) for $L > 0$ and Picture 5.14 (b) for $L < 0$.

According to Lemma 6.3 from [**23**] all invariant polynomials in Tables 4 and 5 which distinguish Configurations 6.7, 5.13, 5.14 and 5.21 are T-comitants except for N_1, N_2 and N_5 which are CT-comitants (for detailed definitions, see [**23**]). More precisely, the polynomial N_1 is a CT-comitant modulo $\langle \eta, H \rangle$, whereas N_2 and N_5 are CT-comitants modulo $\langle \eta, H, B_3 \rangle$, see [**23**, Lemma 62]. Since for systems (V.14) we have $H = -(g-1)^2x^2 \neq 0$ the polynomials N_i, $i = 1, 2, 5$, cannot be applied to these systems. Hence, there cannot exist one common set of conditions in terms of these polynomials for the phase portrait given by all these four configurations and we search the common conditions only for Configurations 6.7, 5.13 and 5.21.

In a similar way to Case 20), according to Theorem 5.1, the canonical systems (VI.7), (V.13) and (V.21) have the conditions $\eta = 0$, $M \neq 0$, $B_3 = N = H = 0$ in common. Furthermore, in addition to the common conditions for these systems, we have more conditions as follows:

$$\begin{aligned}
&(\mathrm{VI}.7): N_1 = 0,\ N_2 = 0,\ N_5 = 64x^2 > 0,\ D = -4x^2y \neq 0;\\
&(\mathrm{V}.13): N_1 = 0,\ N_2 \neq 0,\ N_5 > 0,\ D \neq 0;\\
&(\mathrm{V}.21): N_1 \neq 0,\ N_2 = 0,\ N_5 = 64x^2 > 0,\ D \neq 0.
\end{aligned}$$

Therefore, for the realization of the phase portrait corresponding to Configurations 6.7, 5.13 and 5.21 the set of conditions, jointly taken, are exactly those indicated in the second line of point 22) in Theorem 5.2.

Regarding the remaining Picture 5.14 (a), we note that the conditions from the first set in Case 22) are exactly the conditions for the realization of the Configuration 5.14 plus the condition $L > 0$, see Remark 5.4.

Case 23) According to Remark 5.4 the Pictures 5.14 (b) can be realized if in addition to the conditions for Configuration 5.14, see Table 5, the condition $L < 0$ is verified. According to Remark 5.3 from Tables 4 and 5 we see that for the systems (VI.8) and (V.22) the only conditions which differ, involve the polynomial B_3 differ: $B_3 = 0$ for system (VI.8) and $B_3 \neq 0$ for system (V.22). This obviously leads to the first sequence of conditions for Case 23) in Theorem 5.2. We note that the second sequence of conditions for the Case 23) are exactly those for the realization of Configuration 5.14 plus the condition $L < 0$, see Remark 5.4.

Case 24) From Tables 4 and 5 we observe that all the conditions for systems (VI.10) and (V.28) coincide with the exception of the conditions for the polynomial N_1: for the system (VI.10) we have $N_1 = 0$ whereas for the system (V.28) we have $N_1 \neq 0$. This obviously leads to the sequence of conditions from Theorem 5.2 for the Case 24).

Case 25) According to Tables 4 and 5 the conditions $\eta = M = B_3 = N = N_3 = 0$ appear for both canonical systems (VI.11) and (V.27). Furthermore, for the system (V.27) we have $N_4 D_1 \neq 0$, see Table 5, whereas for the system (VI.11) calculations yield $N_4 = 0$ and $D_1 = 2 \neq 0$. This leads to the sequence of conditions from Theorem 5.2 for case 25).

Case 26) According to Theorem 5.1, besides the common conditions $\eta > 0$, $B_3 \neq 0$, $B_2 = N = H_4 = 0$, see Table 5, for the canonical systems (V.4), (V.5) and (V.16) we have respectively additional conditions:

$$\begin{aligned}(V.4) &: H_1 = -1152g^2(g^2-1),\ H_5<0;\\ (V.5) &: H_1<0,\ H_5>0;\\ (V.16) &: H_1<0,\ H_5=0.\end{aligned}$$

Hence, we have either $H_5 < 0$ or $H_5 \geq 0$ and in this case $H_1 < 0$. This leads to the conditions from Theorem 5.2 for the Case 26).

Case 27) From Table 5 we observe that all the conditions for the systems (V.15) and (V.24) coincide except for the conditions for the polynomial D: for the system (V.15) we have $D \neq 0$ whereas for the system (V.24) $D = 0$ occurs. This obviously leads to the common conditions from Theorem 5.2 for Case 27).

Case 28) According to Table 5 all the conditions for the systems (V.25) and (V.29) coincide with the exception of the conditions for the polynomial H_3: for the system (V.25) we have $H_3 < 0$ whereas for (V.29) we have $H_3 = 0$. These conditions obviously can be amalgamated as $H_3 \leq 0$, and this leads to the conditions from Theorem 5.2 for Case 28). $\square$

Comments. In the proof of Theorem 4.2 we used the global topological classification of phase portraits in the neighborhood of infinity of all quadratic systems, obtained by us in [**24**]. This serves as an example for the usefulness of this topological classification for studying subclasses of the quadratic class, in cutting short what otherwise would be repetitive and much longer calculations. This also shows the importance of algebraic invariants and comitants in applying to whatever normal form we choose, characterizations in terms of these invariant polynomials.

ENDNOTES

1. Under the name *degenerate invariant algebraic curve* this notion was introduced by Christopher in [**9**].

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