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GLOBAL SOLUTIONS FOR A TRITROPHIC FOOD CHAIN MODEL WITH DIFFUSION

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ABSTRACT. In this paper, a tritrophic food chain model with Holling II type functional response is studied. Very few mathematical results are known for this model with diffusion. We first consider the asymptotical stability of equilibrium points for the model of ODE type. Then, the existence and uniform boundedness of global solutions and stability of the equilibrium points for the model of weakly coupled reaction-diffusion type are discussed. Finally, the global existence of solutions for the model of cross-diffusion type is investigated when the space dimension is less than six.

1. Introduction. For several decades, after the pioneering work of Lotka and Volterra in the 1920s, one of the topics of major concern in mathematical ecology has been the study of tritrophic food chains [**19, 20**]. A spatially homogeneous food chain model is given by the ODE system

$$\begin{aligned}
\frac{dX}{dt} &= X\left[R\left(1-\frac{X}{k}\right)-\frac{A_1Y}{B_1+X}\right],\\
\frac{dY}{dt} &= Y\left[E_1\frac{A_1X}{B_1+X}-\frac{A_2Z}{B_2+Y}-D_1\right],\\
\frac{dZ}{dt} &= Z\left[E_2\frac{A_2Y}{B_2+Y}-D_2\right].
\end{aligned}\tag{1.1}$$

The model describes a tritrophic food chain composed of a logistic prey (X), a Holling type II predator (Y), and a Holling type II top-predator (Z). In this model, t is time, R and k are the prey intrinsic

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growth rate and the carrying capacity, the A_i's are maximum predation rates, the B_i's are half saturation constants, the D_i's are death rates, and the E_i's are efficiencies of predator ($i = 1$) and top-predator ($i = 2$) (see [**20, 23**] for more details). In order to preserve the biological meaning of the model, the parameters are assumed to be strictly positive. It is not hard to verify that Y is extinct if $E_1A_1 \leq D_1$ and Z is extinct if $E_2A_2 \leq D_2$, so we always assume that $E_iA_i > D_i$, $i = 1, 2$.

By rescaling the variables [**20**],

$$u = X, \qquad v = \frac{Y}{E_1}, \qquad w = \frac{Z}{E_1E_2},$$

one obtains

$$\begin{aligned} \frac{du}{dt} &= u\left[r\left(1 - \frac{u}{k}\right) - \frac{a_1v}{1 + b_1u}\right], \\ \frac{dv}{dt} &= v\left[\frac{a_1u}{1 + b_1u} - \frac{a_2w}{1 + b_2v} - D_1\right], \\ \frac{dw}{dt} &= w\left[\frac{a_2v}{1 + b_2v} - D_2\right], \end{aligned} \tag{1.2}$$

where

$$r = R, \quad a_1 = \frac{A_1E_1}{B_1}, \quad a_2 = \frac{A_2E_1E_2}{B_2}, \quad b_1 = \frac{1}{B_1}, \quad b_2 = \frac{E_1}{B_1}.$$

For a food chain model, with almost no exceptions, the first contributions dealt with the problem of persistence [**8, 9, 13**]. Complex dynamics of the model (1.2) are investigated in [**20**] by combining numerical continuation techniques with theoretical arguments. In [**20**], it is first shown that (1.2) admits a sequence of pairs of Belyakov bifurcations (codimension-two homoclinic orbits to a critical node), then fold- and period-doubling cycle bifurcation curves associated to each pair of Belyakov points are computed and analyzed. The overall bifurcation scenario explains why stable limit cycles and strange attractors with different geometries can coexist.

To take into account the inhomogeneous distribution of the predators and prey in different spatial locations within a fixed bounded domain

$\Omega \subset \mathbf{R}^N$ at any given time, and the natural tendency of each species to diffuse to areas of smaller population concentration, we are led to the following PDE system of reaction-diffusion type [**20**]:

$$
\begin{aligned}
&u_t = d_1 \Delta u + u\left[r\left(1 - \frac{u}{k}\right) - \frac{a_1 v}{1 + b_1 u}\right], \quad x \in \Omega,\ t > 0,\\
&v_t = d_2 \Delta v + v\left[\frac{a_1 u}{1 + b_1 u} - \frac{a_2 w}{1 + b_2 v} - D_1\right], \quad x \in \Omega,\ t > 0,\\
&w_t = d_3 \Delta w + w\left[\frac{a_2 v}{1 + b_2 v} - D_2\right], \quad x \in \Omega,\ t > 0,\\
&\partial_\eta u(x,t) = \partial_\eta v(x,t) = \partial_\eta w(x,t) = 0, \quad x \in \partial\Omega,\ t > 0,\\
&u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), \quad x \in \Omega,
\end{aligned}
\tag{1.3}
$$

where η is the unit outward normal vector of the boundary $\partial\Omega$ which we will assume is smooth. The homogeneous Neumann boundary condition indicates that the above system is self-contained with zero population flux across the boundary. The constants d_1, d_2 and d_3, called diffusion coefficients, are positive, and the initial data $u_0(x)$, $v_0(x)$, $w_0(x)$ are nonnegative smooth functions.

As far as the authors are aware, the knowledge of system (1.3) is limited. Observe that, if $w = 0$, then the system of equations (1.3) reduces to a Kolmogorov type prey-predator model with diffusion and Michaelis-Menten functional response [**14, 18**]. In this special case, the existence and asymptotic behavior of the solutions have been studied extensively, for example, the existence of traveling front solutions was established in [**14**] using a modification of the Conley index and in [**18**] using the shooting argument and the Hopf bifurcation theorem.

In recent years there has been considerable interest in investigating the global behavior of a system of interacting populations by taking into account the effect of self- as well as cross-diffusion [**5–7, 10–12, 23, 24, 26**]. We are led to the following cross-diffusion system:

$$
u_t = \Delta(d_1 u + \alpha_{11} u^2) + u\left[r\left(1 - \frac{u}{k}\right) - \frac{a_1 v}{1 + b_1 u}\right], \quad x \in \Omega,\ t > 0,
\tag{1.4}
$$

$$
\begin{aligned}
v_t &= \Delta(d_2 v + \alpha_{22} v^2) + v\left[\frac{a_1 u}{1+b_1 u} - \frac{a_2 w}{1+b_2 v} - D_1\right], \\
&\qquad x \in \Omega,\ t > 0, \\
w_t &= \Delta(d_3 w + \alpha_{31} uw + \alpha_{33} w^2) + w\left[\frac{a_2 v}{1+b_2 v} - D_2\right], \\
&\qquad x \in \Omega,\ t > 0, \\
\partial_\eta u(x,t) &= \partial_\eta v(x,t) = \partial_\eta w(x,t) = 0, \\
&\qquad x \in \partial\Omega,\ t > 0, \\
u(x,0) &= u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), \quad x \in \Omega,
\end{aligned}
$$

where d_i $(i,j = 1,2,3)$, a_i, b_i, D_i $(i = 1,2)$, r, k and α_{31} are positive constants andα_{ii} $(i = 1,2,3)$ is a nonnegative constant. d_1, d_2 and d_3 are the diffusion rates of the three species, respectively. α_{ii} $(i = 1,2,3)$ are referred to as self-diffusion pressures and α_{31} as the cross-diffusion pressure. The term self-diffusion implies the movement of individuals from a higher to a lower region of concentration. Cross-diffusion expresses the population fluxes of one species due to the presence of the other species. The value of the cross-diffusion coefficient may be positive, negative or zero. The term *positive cross-diffusion coefficient* denotes the movement of the species in the direction of lower concentration of another species and *negative cross-diffusion coefficient* denotes that one species tends to diffuse in the direction of higher concentration of another species [**7**]. For $\alpha_{ij} \neq 0$, the problem becomes strongly coupled with a full diffusion matrix. Nonlinear problems of this kind are quite difficult to deal with since the usual idea to apply maximum principle arguments to get a priori estimates cannot be used here [**12**]. As far as the authors are aware, very few mathematical results are known for this system.

The main purpose of this paper is to study the asymptotic behavior of solutions of the reaction-diffusion system (1.3) and the global existence of the solution of the cross-diffusion system (1.4). The paper will be organized as follows. In Section 2 a linear stability analysis of equilibrium points for the ODE system (1.2) is given. In Section 3 the uniform bound of the solution to (1.3) and stability of the equilibrium points are proved. Section 4 deals with the existence of global solutions of (1.4).

2. Stability for the ODE system. In this section, we discuss the stability of nonnegative equilibrium points for system (1.2). The following theorem shows that the solution of system (1.2) is bounded.

Theorem 2.1. *Let* $(u(t), v(t), w(t))$ *be the solution of system* (1.2) *with initial values* $(u(0), v(0), w(0)) > 0$*, and let* $[0, T)$ *be the maximal existence interval of the solution. Then*

$$0 < u(t) \leq M_1, \quad 0 < v(t), \qquad w(t) \leq M_2, \quad t \in [0, T),$$

where $M_1 = \max\{u(0), k\}$*,* $M_2 = \max\{u(0) + v(0) + w(0), ((r/l) + 1)M_1\}$*,* $l = \min\{D_1, D_2\}$ *and* $T = +\infty$*.*

Proof. It is easy to see that (1.2) has a unique positive local solution $(u(t), v(t), w(t))$. Let $T \in (0, +\infty]$ be the maximal existence time of the solution. By the first equation of (1.2), we have $u \leq M_1$.

Let $z = u + v + w$. Then

$$\frac{dz}{dt} = ru\left(1 - \frac{u}{k}\right) - D_1 v - D_2 w \leq (r + l)M_1 - lz,$$

where $l = \min\{D_1, D_2\}$. Thus, $z(t) \leq M_2$, $t \in [0, T)$, and $T = +\infty$. $\square$

Now we consider the stability of equilibrium points of (1.2). For simplicity, by rescaling the new variables,

$$u = \frac{1}{b_1}\overline{u}, \qquad v = \frac{1}{b_2}\overline{v}, \qquad w = \frac{a_1}{a_2 b_1}\overline{w}, \tag{2.1}$$

and using u, v and w instead of $\overline{u}$, $\overline{v}$ and $\overline{w}$, respectively, then system (1.2) reduces to

$$\begin{aligned} \frac{du}{dt} &= ru\left[1 - c_1 u - \frac{c_2 v}{1 + u}\right], \\ \frac{dv}{dt} &= mv\left[\frac{u}{1 + u} - \frac{w}{1 + v} - c_3\right], \\ \frac{dw}{dt} &= nw\left[\frac{v}{1 + v} - c_4\right], \end{aligned} \tag{2.2}$$

where

$$m = \frac{a_1}{b_1},\ \ n = \frac{a_2}{b_2}, \quad c_1 = \frac{1}{b_1 k},\ \ c_2 = \frac{a_1}{b_2 r},\ \ c_3 = \frac{D_1 b_1}{a_1},\ \ c_4 = \frac{D_2 b_2}{a_2}. \tag{2.3}$$

Then the conditions $E_i A_i > D_i$, $i = 1, 2$, become

(H_1) $0 < c_3 < 1$, $0 < c_4 < 1$.

System (2.2) has the following three trivial equilibrium points (spatial homogeneity and temporal invariance):

(i) the trivial equilibrium point $S_1(0, 0, 0)$;

(ii) the semi-trivial equilibrium point $S_2((1/c_1), 0, 0)$, corresponding to prey at carrying capacity and in the absence of predator and top-predator;

(iii) the semi-trivial equilibrium point $S_3((c_3/1 - c_3), (1 - c_3 - c_1 c_3)/((1 - c_3)^2 c_2), 0)$, which is positive for $1 - c_3 - c_1 c_3 > 0$ and corresponds to prey-predator coexistence in the absence of top-predator.

As for nontrivial equilibrium points, it is possible to show that at most two of them can be positive, namely, $P_1(u_1, v_1, w_1)$ and $P_2(u_2, v_2, w_2)$, where

$$\begin{aligned} u_1 &= \frac{1 - c_1 + \sqrt{(1 - c_1)^2 + 4c_1(1 - c_2 v_1)}}{2c_1}, \\ v_1 &= \frac{c_4}{1 - c_4}, \qquad w_1 = \left(\frac{u_1}{1 + u_1} - c_3\right)(1 + v_1), \end{aligned} \tag{2.4}$$

$$\begin{aligned} u_2 &= \frac{1 - c_1 - \sqrt{(1 - c_1)^2 + 4c_1(1 - c_2 v_2)}}{2c_1}, \\ v_2 &= \frac{c_4}{1 - c_4}, \qquad w_2 = \left(\frac{u_2}{1 + u_2} - c_3\right)(1 + v_2). \end{aligned} \tag{2.5}$$

It is easy to see that system (2.2) has a unique positive equilibrium point P_1, if

(H_2) $c_2 c_4 < 1 - c_4$, $1 - c_3 - c_1 c_3 > c_1$ and (2.2) have two positive equilibrium points P_1 and P_2, if

(H_3) $1/(c_2) < (c_4/1 - c_4) < ((1 + c_1)^2)/(4c_1 c_2)$, $(1 - c_1 - c_3 - c_1 c_3) - (1 - c_3)\sqrt{(c_1 + 1)^2 - (4c_1 c_2 c_4)/(1 - c_4)} > 0$.

Now we analyze the local geometric properties of the nonnegative equilibria of (2.2).

Theorem 2.2. *The trivial equilibrium point $S_1(0,0,0)$ of (2.2) is unstable.*

Theorem 2.3. *The semi-trivial equilibrium point $S_2((1/c_1),0,0)$ of (2.2) is locally asymptotically stable if $1-c_3-c_1c_3<0$. S_2 is unstable if $1-c_3-c_1c_3>0$.*

Theorems 2.2 and 2.3 are very obvious, so we omit their proofs.

Theorem 2.4. *Assume that $1-c_3-c_1c_3>0$ holds. For the semi-trivial equilibrium point S_3 of (2.2), we have*

(1) *If $1-c_3-c_1c_3<((1-c_3)^2c_2c_4)/(1-c_4)$ and $1-c_3-c_1c_3<c_1$, then S_3 is locally asymptotically stable*;

(2) *If $1-c_3-c_1c_3>((1-c_3)^2c_2c_4)/(1-c_4)$ or $1-c_3-c_1c_3>c_1$, then S_3 is unstable.*

Proof. The Jacobian matrix of the equilibrium S_3 is

$$J_1=\begin{pmatrix} \dfrac{cc_3r(1-c_3-c_1c_3-c_1)}{1-c_3} & -c_2c_3r & 0 \\ \dfrac{(1-c_3-c_1c_3)m}{c_2} & 0 & \dfrac{(1-c_3-c_1c_3)m}{(1-c_3)^2c_2+1-c_3-c_1c_3} \\ 0 & 0 & \dfrac{(1-c_3-c_1c_3)n}{(1-c_3)^2c_2+1-c_3-c_1c_3}-c_4n \end{pmatrix}.$$

The characteristic polynomial of J_1 is

$$\begin{aligned} f(\lambda)=&\left[\lambda-\frac{(1-c_3-c_1c_3)n}{(1-c_3)^2c_2+1-c_3-c_1c_3}+c_4n\right] \\ &\left[\lambda^2-\lambda\frac{c_3r}{1-c_3}(1-c_3-c_1c_3-c_1)+c_3(1-c_3-c_1c_3)rm\right]. \end{aligned}$$

If $1-c_3-c_1c_3<((1-c_3)^2c_2c_4)/(1-c_4)$ and $1-c_3-c_1c_3<c_1$, then $f(\lambda)$ has three negative real roots and S_3 is locally asymptotically

stable. If $1-c_3-c_1c_3 > ((1-c_3)^2c_2c_4)/(1-c_4)$ or $1-c_3-c_1c_3 > c_1$, then $f(\lambda)$ has a positive real root at least, and S_3 is unstable.

Theorem 2.5. *Assume that* (H_1), (H_2) *and*

(H_4) $c_4m < \dfrac{r(1-c_1)^2(1+c_1)}{1-c_3-c_1c_3}$,

$$\frac{r(1-c_1)^2[r(1-c_1)c_1^2c_2+(1-c_1-c_3)^2(1+c_1)^2(1-c_4)c_4m]}{(1-c_3-c_1c_3)(1-c_4)[r^2(1+c_1)+rmc_1^2+(1-c_3-c_1c_3)(1-c_4)c_4mn]} > 1$$

hold. Then the unique positive equilibrium point P_1 *of* (2.2) *is locally asymptotically stable.*

Proof. The Jacobian matrix of P_1 is

$$J_2 = \begin{pmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & m_{23} \\ 0 & m_{32} & 0 \end{pmatrix},$$

where

$$m_{11} = \frac{ru_1(1-c_1-2c_1u_1)}{1+u_1} < 0, \quad m_{12} = -\frac{rc_2u_1}{1+u_1} < 0,$$
$$m_{21} = \frac{mv_1}{(1+u_1)^2} > 0, \quad m_{22} = \left(\frac{u_1}{1+u_1} - c_3\right)c_4m > 0,$$
$$m_{23} = -c_4m < 0, \quad m_{32} = \left(\frac{u_1}{1+u_1} - c_3\right)(1-c_4)n > 0.$$

The characteristic polynomial is

$$\begin{aligned} f(\lambda) &= \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 \\ &= \lambda^3 - (m_{11}+m_{22})\lambda^2 + (m_{11}m_{22} - m_{12}m_{21} - m_{23}m_{32})\lambda \\ &\quad + m_{11}m_{23}m_{32}. \end{aligned}$$

According to the Routh-Hurwitz theorem [**23**], P_1 is locally asymptotically stable if $(H_1), (H_2)$ and (H_4) hold.

Remark. $c_1 = 0.8$, $c_2 = 300$, ${}_3 = 0.1$, $c_4 = 0.001$, $r = 0.1$, $m = 0.001$ and $n = 1$ satisfy (H_1), (H_2) and (H_4).

Theorem 2.6. *Assume that* (H_1), (H_3) *and*

$$(H_5)\quad \sqrt{(c_1+1)^2-\frac{4c_1c_2c_4}{1-c_4}} > \frac{(1+c_1)(1-c_1-c_3)c_4m}{r(1-c_1)},$$

$$\frac{(1+c_1)[r(1-c_1)c_1^2+(1-c_1-c_3-c_1c_3)c_4]}{r^2(1-c_1)^4+r(1-c_1)c_1m+(1-c_1-c_3)(1-c_4)c_4mn} > 1$$

hold. Then (1.2) *has two positive equilibrium points* P_1 *and* P_2. P_1 *is locally asymptotically stable and* P_2 *is unstable.*

Proof. If (H_1) and (H_3) hold, then (2.2) has two positive equilibrium points P_1 and P_2. According to the Routh-Hurwitz theorem [**23**], P_1 is locally asymptotically stable if (H_1), (H_3) and (H_5) hold.

The Jacobian matrix of P_2 is

$$J_3 = \begin{pmatrix} m'_{11} & m'_{12} & 0 \\ m'_{21} & m'_{22} & m'_{23} \\ 0 & m'_{32} & 0 \end{pmatrix},$$

where

$$m'_{11} = \frac{ru_2(1-c_1-2c_1u_2)}{1+u_2} > 0, \quad m'_{12} = -\frac{rc_2u_2}{1+u_2} < 0,$$

$$m'_{21} = \frac{mv_2}{(1+u_2)^2} > 0, \qquad m'_{22} = \left(\frac{u_2}{1+u_2} - c_3\right)c_4m > 0,$$

$$m'_{23} = -c_4m < 0, \qquad m'_{32} = \left(\frac{u_2}{1+u_2} - c_3\right)(1-c_4)n > 0.$$

The characteristic polynomial is

$$f(\lambda) = \lambda^3 - (m'_{11}+m'_{22})\lambda^2 + (m'_{11}m'_{22} - m'_{12}m'_{21} - m'_{23}m'_{32})\lambda + m'_{11}m'_{23}m'_{32}.$$

Noticing that $f(0) = m'_{11}m'_{23}m'_{32} < 0$ and $f(m'_{11}) = -m'_{11}m'_{12}m'_{21} > 0$, $f(\lambda)$ has a positive real root between 0 and m'_{11}. Hence, the positive equilibrium point P_2 is unstable.

In the following theorem, the global stability of the semi-trivial equilibrium points S_2 and S_3 is shown.

Theorem 2.7. (1) *If* $1 - c_3 - c_1c_3 < 0$, *then the semi-trivial equilibrium point* S_2 *of* (2.2) *is globally asymptotically stable.*

(2) *If* $1 - c_3 - c_1c_3 > 0$, $1 - c_3 - c_1c_3 < ((1-c_3)^2c_2c_4)/(1-c_4)$ *and* $1 - c_3 - c_1 < 0$, *then the semi-trivial equilibrium point* S_3 *of* (2.2) *is globally asymptotically stable.*

Proof. (1) Define the Lyapunov function

$$V(u,v,w) = u - \frac{1}{c_1} - \frac{1}{c_1}\ln c_1u + \rho_1 v + \delta_1 w,$$

where $\rho_1 = (rc_2)/(mc_1c_3)$ and $\delta_1 = (rc_2)/(nc_1c_3)$. Let (u,v,w) be the unique positive solution of (2.2). Then

$$\frac{dV}{dt} = -\frac{r}{c_1}(c_1u-1)^2 - \frac{rc_2v}{c_1c_3(1+u)}(c_1c_3+c_3-1) - \frac{rc_2c_4w}{c_1c_3}.$$

By the Lyapunov-LaSalle invariance principle [**15**], S_2 is globally asymptotically stable if $1 - c_3 - c_1c_3 < 0$.

(2) Define the Lyapunov function

$$V(u,v,w) = \left(u - u^* - u^*\ln\frac{u}{u^*}\right) + \rho_2\left(v - v^* - v^*\ln\frac{v}{v^*}\right) + \delta_2 w,$$

where $\rho_2 = (rc_2)/(m(1-c_3))$, $\delta_2 = (r(1-c_3-c_1c_3)/(nc_4(1-c_3)^3)$, $u^* = c_3/(1-c_3)$ and $v^* = (1-c_3-c_1c_3)/((1-c_3)^2c_2)$. Let (u,v,w) be the unique positive solution of system (2.2). Then

$$\begin{aligned}
\frac{dV}{dt} &\le -r\left(c_1 - \frac{c_2v^*}{1+u^*}\right)(u-u^*)^2 \\
&\quad - \frac{1}{1+u}\left(c_2r - \frac{m\rho}{1+u^*}\right)(u-u^*)(v-v^*) \\
&\quad - \frac{w}{1+v}[v(m\rho + n\delta c_4 - n\delta) + (n\delta c_4 - m\rho v^*)] \\
&\le -\frac{r}{1-c_3}(c_1+c_3-1)(u-u^*)^2 \\
&\quad - \frac{rc_2w(1-c_4)(1+u^*)}{c_4(1+v)}\left[\frac{c_4}{1-c_4} - \frac{1-c_3-c_1c_3}{(1-c_3)^2c_2}\right].
\end{aligned}$$

Hence global asymptotical stability of S_3 follows from the Lyapunov-LaSalle invariance principle [**15**].

3. Global behavior of the PDE system without cross diffusion. In this section we discuss the existence and uniform boundedness of global solutions and the stability of constant equilibrium solutions for the weakly coupled reaction-diffusion system (1.3). In particular, the instability results in Section 2 also hold for system (1.3) because solutions of (1.2) are also solutions of (1.3).

Let $f_1 = u[r(1-(u/k)) - (a_1 v)/(1+b_1 u)]$, $f_2 = v[(a_1 u)/(1+b_1 u) - (a_2 w)/(1+b_2 v - D_1)]$ and $f_3 = w[(a_2 v)/(1+b_2 v) - D_2]$. It is easy to see that $f_1,\ f_2,\ f_3 \in C^1(\overline{R}_+^3)$ with $\overline{R}_+^3 = \{(u,v,w) \mid u,v,w \geq 0\}$. Standard PDE theory [**21**] shows that (1.3) has unique solutions $(u,v,w) \in [C(\overline{\Omega}\times[0,T)) \cap C^{2,1}(\Omega\times(0,T))]^3$, where $T \leq +\infty$ is the maximal existence time. The following theorem shows that the solution of (1.3) is uniformly bounded, and thus $T=+\infty$.

Theorem 3.1. *Let* $(u,v,w) \in [C(\overline{\Omega}\times[0,T)) \cap C^{2,1}(\Omega\times(0,T))]^3$ *be a solution of* (1.3), T *the maximal existence time and* $(u_0(x),\ v_0(x),\ w_0(x)) \geq 0$. *Then* $0 \leq u(x,t) \leq M_1$, $0 \leq v(x,t), w(x,t) \leq M_2$, $t \in [0,T)$, *where* $M_1 = \max\{\|u_0(x)\|_{L^\infty(\Omega)}, k\}$, M_2 *is a positive constant depending on* Ω *and all the coefficients of equations* (1.3), $\|u_0\|_{L^\infty(\Omega)}$, $\|v_0\|_{L^\infty(\Omega)}$ *and* $\|w_0\|_{L^\infty(\Omega)}$. *Furthermore,* $T=+\infty$ *and* $(u(x,t),v(x,t),w(x,t)) > 0$ *for any* $t>0$ *if* $u_0 \geq (\not\equiv)0$, $v_0 \geq (\not\equiv)0$ *and* $w_0 \geq (\not\equiv)0$.

Proof. Let (u,v,w) be a solution of (1.3) with $(u_0(x),\ v_0(x)$ and $w_0(x)) \geq 0$. From the maximum principle for parabolic equations [**25**], it is not hard to verify that $(u,v,w) \geq 0$ for $(x,t) \in \Omega\times[0,T)$, where T is the maximal existence time of the solution (u,v,w). Furthermore, we know by the strong maximum principle that $(u,v,w) > 0$ for $t>0$ if $u_0 \geq (\not\equiv)0$, $v_0 \geq (\not\equiv)0$ and $w_0 \geq (\not\equiv)0$.

Now we prove that (u,v,w) is bounded on $\Omega\times[0,T)$. The maximum principle gives $u \leq \max\{\|u_0(x)\|_{L^\infty(\Omega)}, k\} := M_1$. Integrating the equations of (1.3) over Ω and adding up the results, we have that, by the Young inequality,

$$\begin{aligned}\frac{d}{dt}\int_\Omega (u+v+w)\,dx &= \int_\Omega \left(ru - \frac{r}{k}u^2 - D_1 v - D_2 w\right) dx\\ &\le \frac{k}{4r}(r+k)^2|\Omega| - \int_\Omega l(u+v+w)\,dx,\end{aligned}$$

where $l = \min\{D_1, D_2\}$. Therefore, $\|u(t)\|_{L^1(\Omega)}$, $\|v(t)\|_{L^1(\Omega)}$ and $\|w(t)\|_{L^1(\Omega)}$ are bounded in $[0,\infty)$. Using [**17**, Exercise 5, Section 3.5] we obtain that $\|u(t)\|_{L^\infty(\Omega)}$, $\|v(t)\|_{L^\infty(\Omega)}$ and $\|w(t)\|_{L^\infty(\Omega)}$ are also bounded in $[0,\infty)$. $\square$

To compare stability analysis for the ODE system (2.2), we discuss the following system

$$\begin{aligned}
&u_t = d_1\Delta u + ru\left[1 - c_1 u - \frac{c_2 v}{1+u}\right], \quad x\in\Omega,\ t>0,\\
&v_t = d_2\Delta v + mv\left[\frac{u}{1+u} - \frac{w}{1+v} - c_3\right], \quad x\in\Omega,\ t>0,\\
&w_t = d_3\Delta w + nw\left[\frac{v}{1+v} - c_4\right], \quad x\in\Omega,\ t>0,\\
&\partial_\eta u(x,t) = \partial_\eta v(x,t) = \partial_\eta w(x,t) = 0, \quad x\in\partial\Omega,\ t>0,\\
&u(x,0) = \bar{u}_0(x), \quad v(x,0) = \bar{v}_0(x), w(x,0) = \bar{w}_0(x), \quad x\in\Omega.
\end{aligned} \tag{3.1}$$

In order to establish global stability of the equilibrium solution, we first recall the following result which can be found in [**24**]:

Lemma 3.1. *Let a and b be positive constants. Assume that $\phi,\ \varphi \in C^1([a,\infty))$, $\varphi(t) > 0$ and ϕ is bounded from below. If $\phi'(t) \le -b\varphi(t)$ and $\varphi'(t) \le K$ in $[a,\infty)$ for some constant K, then $\lim_{t\to\infty}\varphi(t) = 0$.*

Let $0 = \mu_1 < \mu_2 < \mu_3 < \cdots$ be eigenvalues of the operator $-\Delta$ on Ω with the homogeneous Neumann boundary condition, and let $E(\mu_i)$ be the eigenspace corresponding to μ_i in $C^1(\overline{\Omega})$.

Let $X = \{U = (u, v, w) \in [C^1(\overline{\Omega})]^3 \mid \partial_\eta u = 0, x \in \partial\Omega\}$, $\{\phi_{ij},\ j = 1, 2, \dots, \dim E(\mu_i)\}$ be an orthonormal basis of $E(\mu_i)$ and $X_{ij} = \{C \cdot \phi_{ij} \mid C \in R^3\}$. Then $X_i = \oplus_{j=1}^{\dim E(\mu_i)} X_{ij}$, $X = \oplus_{i=1}^{\infty} X_i$ [**22, 24**].

Theorem 3.2. *The semi-trivial equilibrium point* S_2 *of* (3.1) *is globally asymptotically stable if* $1 - c_3 - c_1c_3 < 0$.

Proof. We present the proof in two steps:

Step 1. *Local stability.* Let $D = \operatorname{diag}(d_1, d_2, d_3)$ and $L = D\triangle + J_2$ (see Section 2). The linearization of (3.1) at S_2 is $U_t = LU$ $(i \geq 1)$. For each $i \geq 1$, X_i is invariant under the operator L, and λ is an eigenvalue of L on X_i if and only if it is an eigenvalue of the matrix $-\mu_i D + J_2$.

The characteristic polynomial of $-\mu_i D + J_2$ is given by

$$\varphi_i(\lambda) = (\lambda + \mu_i d_1 + r)\Big(\lambda + \mu_i d_2 - \frac{m}{1 + c_1} + c_3 m\Big)(\lambda + \mu_i d_3 + c_4 n).$$

Clearly, $-\mu_i d_1 - r$, $-\mu_i d_2 + (m/(1 + c_1)) - c_3 m$, $-\mu_i d_3 - c_4 n$ are the three roots of $\varphi_i(\lambda)$. Thus, the spectrum of L, consisting only of eigenvalues, lies in $\{\operatorname{Re}\lambda \leq -(1/2)\max\{r, c_3 m - (m/(1 + c_1)), c_4 n\}\}$ if $1 - c_3 - c_1 c_3 < 0$, and local stability of S_2 is obtained [**17**, Theorem 5.1.1].

Step 2. *Global stability.* Let (u, v, w) be the unique positive solution of system (3.1). It follows from Theorem A_2 in [**4**] and Theorem 3.1 that

$$\|u(\cdot, t)\|_{C^{2,\alpha}(\overline{\Omega})}, \quad \|v(\cdot, t)\|_{C^{2,\alpha}(\overline{\Omega})}, \quad \|w(\cdot, t)\|_{C^{2,\alpha}(\overline{\Omega})} \leq C \text{ for all } t \geq 1, \tag{3.2}$$

where $\alpha \in (0, 1)$ and C is a constant which does not depend on t.

Define the Lyapunov function

$$V(u, v, w) = \int_\Omega \left[\left(u - \frac{1}{c_1} - \frac{1}{c_1}\ln c_1 u\right) + \rho v + \delta w\right] dx,$$

where $\rho = (rc_2)/(mc_1c_3)$ and $\delta = (rc_2)/(nc_1c_3)$. By (3.1), we have

$$\begin{aligned}\frac{dV}{dt} = &-\int_\Omega \frac{d_1}{c_1 u^2}|\nabla u|^2\, dx - rc_1 \int_\Omega \left(u - \frac{1}{c_1}\right)^2 dx \\ &- \frac{rc_2(c_1c_3 + c_3 - 1)}{c_1c_3}\int_\Omega \frac{v}{1 + u}\, dx - \frac{rc_2c_4}{c_1c_3}\int_\Omega w\, dx.\end{aligned} \tag{3.3}$$

By Theorem 3.1, Lemma 3.1 and equations (3.2) and (3.3), we know that

$$\begin{aligned} &\lim_{t\to\infty}\int_\Omega |\nabla u|^2\,dx = 0, && \lim_{t\to\infty}\int_\Omega \left(u-\frac{1}{c_1}\right)^2 dx = 0, \\ &\lim_{t\to\infty}\int_\Omega v\,dx = 0, && \lim_{t\to\infty}\int_\Omega w\,dx = 0, \end{aligned} \tag{3.4}$$

if $1-c_3-c_1c_3<0$. From the Poincaré inequality, it follows that

$$\lim_{t\to\infty}\int_\Omega (u-\overline{u})^2\,dx = 0, \tag{3.5}$$

where $\overline{u}=(1/\Omega)\int_\Omega u\,dx$. By $\int_\Omega(\overline{u}-(1/c_1))^2\,dx \le 2\int_\Omega(\overline{u}-u)^2\,dx + 2\int_\Omega(u-(1/c_1))^2\,dx$, we have $\overline{u}(t)\to(1/c_1)$ as $t\to\infty$. Therefore, there exists a sequence $\{t_m\}$ with $t_m\to\infty$ such that $\overline{u}'(t_m)\to 0$ as $t_m\to\infty$.

Since $\overline{v}=(1/\Omega)\int_\Omega v\,dx\to 0$ and $\overline{w}=(1/\Omega)\int_\Omega w\,dx\to 0$ as $t\to\infty$, there exists a subsequence, still denoted by $\{t_m\}$, such that $\overline{v}'(t_m)\to 0$ and $\overline{w}'(t_m)\to 0$ as $t_m\to\infty$.

At $t=t_m$, from the first two equations of (3.1), we have

$$\begin{aligned} |\Omega|\overline{u}'(t_m) &= \int_\Omega ru\left[1-c_1u-\frac{c_2v}{1+u}\right]dx \longrightarrow 0, \\ |\Omega|\overline{v}'(t_m) &= \int_\Omega mv\left[\frac{u}{1+u}-\frac{w}{1+v}-c_3\right]dx \longrightarrow 0. \end{aligned} \tag{3.6}$$

Hence,

$$\lim_{m\to\infty}\overline{v}(t_m)=0, \qquad \lim_{m\to\infty}\overline{w}(t_m)=0. \tag{3.7}$$

It follows from (3.2) that a subsequence $\{t_m\}$ exists, still denoted by $\{t_m\}$, and nonnegative functions u_1, v_1 and $w_1\in C^2(\overline{\Omega})$ exist such that

$$\lim_{m\to\infty}\Big(\|u(\cdot,t_m)-u_1\|_{C^2(\overline{\Omega})}, \|v(\cdot,t_m)-v_1\|_{C^2(\overline{\Omega})}, \|w(\cdot,t_m)-w_1\|_{C^2(\overline{\Omega})}\Big) = 0.$$

Thus,

$$\lim_{m\to\infty}\left(\|u(\cdot,t_m)-\frac{1}{c_1}\|_{C^2(\overline{\Omega})}, \|v(\cdot,t_m)\|_{C^2(\overline{\Omega})}, \|w(\cdot,t_m)\|_{C^2(\overline{\Omega})}\right)=0.$$

The global asymptotical stability of S_2 follows from this together with the local stability of S_2.

Theorem 3.3. *Assume that* $c_1c_3 < 1 - c_3 < c_1$ *and* $1 - c_3 - c_1c_3 < ((1-c_3)^2c_2c_4)/(1-c_4)$ *hold. Then the semi-trivial equilibrium point* S_3 *of* (3.1) *is globally asymptotically stable.*

Proof. Let $L = D\triangle + J_3$ (see the proof of Theorem 3.2). The linearization of (3.1) at S_3 is $U_t = LU$. The characteristic polynomial of $-\mu_i D + J_3$ is given by

$$\begin{aligned}\varphi_i(\lambda) = \Big\{&\lambda^2 + \Big[\mu_i(d_1+d_2) - \frac{c_3 r}{1-c_3}(1-c_3-c_1c_3-c_1)\Big]\lambda \\ &+ \mu_i^2 d_1 d_2 + c_3(1-c_3-c_1c_3)rm \\ &- \mu_i d_2 \frac{c_3 r}{1-c_3}(1-c_3-c_1c_3-c_1)\Big\} \\ &\Big\{\lambda + \mu_i d_3 - \frac{(1-c_3-c_1c_3)n}{(1-c_3)^2c_2 + 1 - c_3 - c_1c_3} + c_4 n\Big\}.\end{aligned}$$

By direct computation we know that the three roots $\lambda_{i,1}$, $\lambda_{i,2}$ and $\lambda_{i,3}$ of $\varphi_i(\lambda)$ have negative real parts if $1-c_3-c_1c_3<\min\{c_1, ((1-c_3)^2c_2c_4)/(1-c_4)\}$.

Now, we prove that a positive constant δ exists such that

$$\text{(3.8)} \qquad \operatorname{Re}\{\lambda_{i,1}\}, \quad \operatorname{Re}\{\lambda_{i,2}\}, \quad \operatorname{Re}\{\lambda_{i,3}\} \le -\delta, \quad i \ge 1.$$

Let $\lambda = \mu_i\zeta$ and $\varphi_i(\zeta) \triangleq \widetilde{\varphi}_i(\lambda) = \mu_i^3\zeta^3 + A_i\mu_i^2\zeta^2 + B_i\mu_i\zeta + C_i$. Since $\mu_i \to \infty$ as $(i \to \infty)$, it follows that

$$\lim_{i\to\infty} \frac{\widetilde{\varphi}_i(\zeta)}{\mu_i^3} = \zeta^3 + (d_1+d_2+d_3)\zeta^2 + (d_1d_2+d_2d_3+d_3d_1)\zeta + d_1d_2d_3 \triangleq \widetilde{\varphi}(\zeta).$$

$\widetilde{\varphi}(\zeta)$ has the three roots $-d_1$, $-d_2$, $-d_3$. Thus, there exists an i_0 such that the three roots ζ_{i1}, ζ_{i2} and ζ_{i3} of $\widetilde{\varphi}_i(\zeta)$ satisfy $\operatorname{Re}\{\zeta_{i1}\}$, $\operatorname{Re}\{\zeta_{i2}\}$ and $\operatorname{Re}\{\zeta_{i3}\} \le -(d/2)$, $i \ge i_0$, where $d = \min\{d_1, d_2, d_3\}$. So $\operatorname{Re}\{\lambda_{i,1}\}$, $\operatorname{Re}\{\lambda_{i,2}\}$, $\operatorname{Re}\{\lambda_{i,3}\} \le -\mu_i(d/2) \le -(d/2)$, $i \ge i_0$. Let

$-\tilde{\delta} = \max_{1\le i\le i_0}\{\mathrm{Re}\,\{\lambda_{i,1}\}, \mathrm{Re}\,\{\lambda_{i,2}\}, \mathrm{Re}\,\{\lambda_{i,3}\}\}$. Then $\tilde{\delta} > 0$ and (3.8) holds for $\delta = \min\{\tilde{\delta}, (d/2)\}$.

Consequently, the spectrum of L, consisting only of eigenvalues, lies in $\{\mathrm{Re}\,\lambda \le -\delta\}$, and the local stability of S_2 is shown [**17**, Theorem 5.1.1].

Let (u, v, w) be the unique positive solution of system (3.1). Define the Lyapunov function

$$V(u,v,w) = \int_\Omega \left[\left(u - u^* - u^* \ln \frac{u}{u^*}\right) + \rho\left(v - v^* - v^* \ln \frac{v}{v^*}\right) + \delta w\right] dx,$$

where $\rho = (rc_2)/(m(1-c_3))$, $\delta = (r(1-c_3-c_1c_3)/(nc_4(1-c_3)^3)$, $u^* = (c_3)/(1-c_3)$ and $v^* = (1-c_3-c_1c_3)/((1-c_3)^2c_2)$. Then

$$\begin{aligned}\frac{dV}{dt} \le & -\int_\Omega \left(\frac{d_1u^*}{u^2}|\nabla u|^2 + \frac{d_2v^*}{v^2}|\nabla v|^2\right) dx \\ & - \frac{r}{1-c_3}(c_1+c_3-1)\int_\Omega (u-u^*)^2\, dx \\ & - \frac{rc_2(1-c_4)(1+u^*)}{c_4}\left[\frac{c_4}{1-c_4} - \frac{1-c_3-c_1c_3}{(1-c_3)^2c_2}\right]\int_\Omega \frac{w}{1+v}\, dx.\end{aligned} \tag{3.9}$$

From this, and using the similar argument in the proof of Theorem 3.2, we can show

$$\lim_{m\to\infty}\Big(\|u(\cdot,t_m) - u^*\|_{C^2(\overline{\Omega})}, \lim_{m\to\infty}\|v(\cdot,t_m) - v^*\|_{C^2(\overline{\Omega})}, \lim_{m\to\infty}\|w(\cdot,t_m) - 0\|_{C^2(\overline{\Omega})}\Big) = 0.$$

Global asymptotical stability of S_3 follows from this together with the local stability of S_3.

By some tedious calculations, the following theorem can be proved in a similar way as Theorem 2.5 and Theorem 3.3.

Theorem 3.4. *Assume that* (H_1), (H_2) *and* (H_4) *hold. Then the unique positive equilibrium point* P_1 *of* (3.1) *is locally asymptotically stable.*

4. Global existence of solution for the cross-diffusion system.

In this section, we will discuss the global existence of nonnegative classical solutions for the cross-diffusion system (1.4).

By [**1, 2**], (1.4) has a unique nonnegative solution $u, v, w \in C([0,T), W_p^1(\Omega)) \cap C^\infty((0,T), C^\infty(\Omega))$ if $u_0,\ v_0,\ w_0 \in W_p^1(\Omega)$ for $p > n$, where $T \in (0, +\infty]$ is the maximal existence time of the solution. If the solution (u, v, w) satisfies the estimate

(4.1)
$$\sup\{\|u(\cdot,t)\|_{W_p^1(\Omega)}, \|v(\cdot,t)\|_{W_p^1(\Omega)}, \|w(\cdot,t)\|_{W_p^1(\Omega)} : t \in (0,T)\} < \infty,$$

then $T = +\infty$. If, in addition, $u_0,\ v_0,\ w_0 \in W_p^2(\Omega)$, then $u,\ v,\ w \in C([0,\infty), W_p^2(\Omega))$.

To obtain L^∞-estimates of solutions of (1.4), some preparations are needed.

Lemma 4.1. *Let (u, v, w) be a solution of* (1.4), *$z = (d + \alpha u)u$, $Q_\tau = \Omega \times (0, \tau)$, $\tau < T$. Then there exists a positive constant $C(\tau)$ which depends on $\|u_0\|_{W_2^1(\Omega)}$ and $\|u_0\|_{L^\infty(\Omega)}$, such that*

$$\|z\|_{W_2^{2,1}(Q_\tau)} \le C(\tau). \tag{4.2}$$

Furthermore,

$$\nabla z \in V_2(Q_\tau), \qquad \nabla u \in L^{(2(n+2))/n}(Q_\tau). \tag{4.3}$$

Proof. Let (u, v, w) be a solution of (1.4). Then

$$z_t = (d + 2\alpha u)u_t = (d + 2\alpha u)\Delta z + \beta_1 - \beta_2 v,$$

where $\beta_1 = dru + (2\alpha r - (rd/k))u^2 - (2\alpha r/k)u^3$, $\beta_2 = [(a_1 u)/(1 + b_1 u)] \times (d + 2\alpha u)$. By [**6**, Lemma 2.2], one can obtain that $\|z\|_{W_2^{2,1}(Q_\tau)} \le C(\tau)$, so (4.3) is a standard embedding result. □

Combining Lemmas 2.3 and 2.4 of [**6**], we can prove the following lemma.

Lemma 4.2. *Let $p > 1$, $\tilde{p} = 2 + (4p)/(n(q + 1))$, and let w satisfy*

$$\sup_{0 \le t \le T} \|w\|_{L^{(2p)/(p+1)}(\Omega)} + \|\nabla w\|_{L^2(Q_T)} < \infty,$$

and there exist positive constants $\beta \in (0,1)$ and C_T such that

$$\int_\Omega |w(\cdot,t)|^\beta dx \le C_T \quad (\textit{for all } t \le T).$$

Then there exists a positive constant M' independent of w but which may depend upon n, Ω, p, β and C_T, such that

$$\|w\|_{L^{\tilde q}(Q_T)} \le M' \left\{ 1 + \left(\sup_{0\le t\le T} \|w(t)\|_{L^{(2p)/(p+1)}(\Omega)} \right)^{4p/(n(p+1)\tilde p)} \|\nabla w\|^{2/\tilde p}_{L^2(Q_T)} \right\}.$$

The main result in this section is as follows:

Theorem 4.1. *Assume that $u_0 \ge 0$, $v_0 \ge 0$ and $w_0 \ge 0$ satisfy zero Neumann boundary conditions and belong to $C^{2+\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, $\alpha_{11}\alpha_{22}\alpha_{33} \ge 0$ and $\alpha_{31} > 0$. Then (1.4) has a unique nonnegative solution u, v and $w \in C^{2+\alpha,1+(\alpha/2)}(\overline{\Omega}\times[0,\infty))$ if the space dimension $n \le 5$.*

Proof. Firstly, we establish L^1- and L^2-estimates of the solution (u,v,w) of (1.4).

From the maximum principle for parabolic equations, it is obvious that

$$0 \le u \le M_0, \quad 0 \le v, \quad 0 \le w, \qquad x \in \overline{\Omega},\ t \in [0,T), \tag{4.4}$$

where $M_0 = \max\{k, \|u_0\|_{L^\infty(\Omega)}\}$. Furthermore, we know by the strong maximum principle that $(u,v,w) > 0$ for $t > 0$ if $u_0 \ge (\not\equiv)0$, $v_0 \ge (\not\equiv)0$ and $w_0 \ge (\not\equiv)0$. In addition, there exist positive constants M_1 and M_2 depending upon k, Ω, the initial value u_0 and T such that

$$\|u\|_{L^1(Q_T)} \le M_1, \quad \|u\|_{L^2(Q_T)} \le M_2, \quad \text{for all } t \ge 0.$$

Integrating the first three equations of (1.4) over Ω and adding up the results, we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega (u+v+w)\,dx &= \int_\Omega (ru - \frac{r}{k}u^2 - D_1 v - D_2 w)\,dx \\ &\le \frac{k}{4r}(r+k)^2|\Omega| - \int_\Omega l(u+v+w)\,dx, \end{aligned}$$

where $l = \min\{D_1, D_2\}$. Therefore,

$$\begin{aligned}\|v(t)\|_{L^1(\Omega)}, &\|w(t)\|_{L^1(\Omega)} \\ &\le \max\left\{\frac{k}{4rl}(r+k)^2|\Omega|, \int_\Omega (u_0(x)+v_0(x)+w_0(x))\,dx\right\} \\ &= M_3'\end{aligned}$$

and

$$\|v\|_{L^1(Q_T)},\ \|w\|_{L^1(Q_T)} \le M_3'T = M_3, \quad \text{for all } t \ge 0.$$

Multiplying the second equation of (1.4) by v and integrating it over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^2\,dx \le -d_2\int_\Omega |\nabla v|^2\,dx + \frac{a_1}{b_1}\int_\Omega v^2\,dx.$$

By the Gagliardo-Nirenberg inequality, $|v|_2 \le C(|\nabla v|_2^{n/(n+2)}|v|_1^{2/(n+2)} + |v|_1)$, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^2\,dx \le -\frac{d_2}{CM_3'^{(2/n)}}\left[\int_\Omega v^2\,dx\right]^{(n+2)/n} + \frac{a_1}{b_1}\int_\Omega v^2\,dx + d_2 M_3'^2.$$

Therefore, a positive constant M_4' exists depending upon a_1, b_1, d_1, d_2 and k such that

$$\int_\Omega v^2\,dx \le M_4', \quad t > 0, \tag{4.5}$$

and $\|v\|_{L^2(Q_T)} \le M_4'T := M_4$.

Secondly, we will obtain L^q-estimates. Multiplying the second equation of (1.4) by qv^{q-1} $(q > 1)$ and integrating it over Ω, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega v^q\,dx = &-\frac{4(q-1)d_2}{q}\int_\Omega |\nabla(v^{q/2})|^2\,dx \\ &- \frac{8q(q-1)\alpha_{22}}{(q+1)^2}\int_\Omega |\nabla(v^{(q+1)/2})|^2\,dx + \frac{a_1}{b_1}q\int_\Omega v^q\,dx.\end{aligned}$$

Thus,

$$\begin{aligned}(4.6)\quad \int_\Omega v^q(x,T)\,dx &+ \frac{4(q-1)d_2}{q}\int_{Q_T}|\nabla(v^{q/2})|^2\,dx\,dt \\ &+ \frac{8q(q-1)\alpha_{22}}{(q+1)^2}\int_{Q_T}|\nabla(v^{(q+1)/2})|^2\,dx\,dt \\ &\qquad\le \int_\Omega v_0^q(x)\,dx + \frac{a_1}{b_1}q\int_{Q_T} v^q\,dx\,dt.\end{aligned}$$

Let $\overline{v} = v^{(q+1)/2}$. Then

$$\begin{aligned}\int_\Omega \overline{v}^{(2q)/(q+1)}(x,t)\,dx &+ \int_{Q_T}|\nabla\overline{v}|^2\,dx\,dt \\ &\le \int_\Omega v_0^q(x)\,dx + \frac{a_1}{b_1}q\|\overline{v}\|^{(2q)/(q+1)}_{L^{(2q)/(q+1)}(Q_T)} \\ &\le C_1\left(1+\|\overline{v}\|^{(2q)/(q+1)}_{L^{(2q)/(q+1)}(Q_T)}\right),\end{aligned}$$

and

$$(4.7)\qquad E \le C_1\left(1+\|\overline{v}\|^{(2q)/(q+1)}_{L^{(2q)/(q+1)}(Q_T)}\right),$$

where

$$E \equiv \sup_{0<t<T}\int_\Omega \overline{v}^{(2q)/(q+1)}(x,t)\,dx + \int_{Q_T}|\nabla\overline{v}|^2\,dx\,dt.$$

It follows from $(2q)/(q+1) < 2 < \tilde{q} = 2 + (4q)/(n(q+1))$ that

$$(4.8)\qquad E \le C_2\left(1+\|\overline{v}\|^{(2q)/(q+1)}_{L^{\tilde{q}}}\right).$$

Setting $\beta = (2/(q+1)) \in (0,1)$, from the L^1 estimate of v, we have

$$\|\overline{v}\|_{L^\beta(\Omega)} = \left(\int_\Omega |\overline{v}(\cdot,t)|^\beta\,dx\right)^{1/\beta} = \|v\|^{1/\beta}_{L^1(\Omega)} \le M_3'^{1/\beta}, \quad \text{for all } t \le T.$$

By Lemma 8 and (4.8), one can obtain that

$$
\begin{aligned}
E &\le C_2 \left[1 + \left(M' + M' \sup_{0<t<T} \|\overline{v}(\cdot,t)\|_{L^{(2q)/(q+1)}(\Omega)}^{(4q)/(n(q+1)\tilde{q})} \right.\right. \\
&\qquad\qquad \left.\left. \times \|\nabla \overline{v}\|_{L^2(Q_T)}^{2/\tilde{q}} \right)^{(2q)/(q+1)} \right] \\
&\le C_3 \left[1 + \left(\sup_{0<t<T} \|\overline{v}(\cdot,t)\|_{L^{(2q)/(q+1)}(\Omega)}^{(2q)/(q+1)} \right)^{(4q)/(n(q+1)\tilde{q})} \right. \\
&\qquad\qquad \left. \times \left(\|\nabla \overline{v}\|_{L^2(Q_T)}^2 \right)^{(2q)/((q+1)\tilde{q})} \right] \\
&\le C_3 \left(1 + E^{(4q)/(n(q+1)\tilde{q})} E^{(2q)/((q+1)\tilde{q})} \right).
\end{aligned} \tag{4.9}
$$

It follows from $[(4q)/(n(q+1)\tilde{q})] + [(2q)/((q+1)\tilde{q})] \in (0,1)$ and (4.9) that a positive constant M_5 exists such that $E \le M_5$. Thus, $\|\overline{v}\|_{L^{\tilde{q}}(Q_T)}$ is bounded and $v^{(q+1)/2} \in L^{\tilde{q}}(Q_T)$, that is,

$$
v \in L^{((q+1)\tilde{q})/2}(Q_T). \tag{4.10}
$$

Multiplying the third equation of (1.4) by qw^{q-1} ($q > 1$) and integrating it over Ω, we have

$$
\begin{aligned}
\frac{d}{dt} \int_\Omega w^q \, dx \le &- \frac{4(q-1)d_3}{q} \int_\Omega |\nabla (w^{q/2})|^2 \, dx \\
&- \frac{8q(q-1)\alpha_{33}}{(q+1)^2} \int_\Omega |\nabla (w^{(q+1)/2})|^2 \, dx \\
&- q(q-1)\alpha_{31} \int_\Omega w^{q-1} \nabla u \cdot \nabla w \, dx + \frac{a_2}{b_2} q \int_\Omega w^q \, dx.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int_\Omega w^q(x,T)\, dx &+ \frac{4(q-1)d_3}{q} \int_{Q_T} |\nabla (w^{q/2})|^2 \, dx\, dt \\
&+ \frac{8q(q-1)\alpha_{33}}{(q+1)^2} \int_{Q_T} |\nabla (w^{(q+1)/2})|^2 \, dx\, dt \\
&\le \int_\Omega w_0^q(x)\, dx - q(q-1)\alpha_{31} \\
&\cdot \int_{Q_T} w^{q-1} \nabla u \nabla w \, dx\, dt + \frac{a_2}{b_2} q \int_{Q_T} w^q \, dx\, dt.
\end{aligned} \tag{4.11}
$$

It follows from Lemma 4.1 that

$$
\begin{aligned}
(4.12)\quad & \left|q(q-1)\alpha_{31}\int_{Q_T} w^{q-1}\nabla u\cdot\nabla w\,dx\,dt\right| \\
&\le \frac{2q(q-1)\alpha_{31}}{q+1}\|w\|^{(q-1)/2}_{L^{((q-1)(n+2))/2}(Q_T)} \\
&\quad\times\|\nabla u\|_{L^{(2(n+2))/n}(Q_T)}\|\nabla(w^{(q+1)/2})\|_{L^2(Q_T)} \\
&\le C_4\|w\|^{(q-1)/2}_{L^{((q-1)(n+2))/2}(Q_T)}\|\nabla(w^{(q+1)/2})\|_{L^2(Q_T)} \\
&\le \frac{C_4\varepsilon_1}{2}\|\nabla(w^{(q+1)/2})\|^2_{L^2(Q_T)}+\frac{C_4}{2\varepsilon_1}\|w\|^{q-1}_{L^{((q-1)(n+2))/2}(Q_T)}.
\end{aligned}
$$

Choose ε_1 such that $(C_4\varepsilon_1)/2<[(8q(q-1)\alpha_{33})/((q+1)^2)]$. Then, by (4.11) and (4.12)

$$
\begin{aligned}
(4.13)\quad & \int_\Omega \overline{w}^{(2q)/(q+1)}(x,t)\,dx+\int_{Q_T}|\nabla\overline{w}|^2\,dx\,dt \\
&\le \int_\Omega w_0^q(x)\,dx+\frac{C_4}{2\varepsilon_1}\|\overline{w}\|^{(2(q-1))/(q+1)}_{L^{[(q-1)(n+2)]/(q+1)}(Q_T)}+\frac{a_2}{b_2}q\|\overline{w}\|^{(2q)/(q+1)}_{L^{(2q)/(q+1)}(Q_T)} \\
&\le C_5\left(1+\|\overline{w}\|^{(2(q-1))/(q+1)}_{L^{[(q-1)(n+2)]/(q+1)}(Q_T)}+\|\overline{w}\|^{(2q)/(q+1)}_{L^{(2q)/(q+1)}(Q_T)}\right),
\end{aligned}
$$

where $\overline{w}=w^{(q+1)/2}$. Let

$$
E\equiv\sup_{0<t<T}\int_\Omega\overline{w}^{(2q)/(q+1)}(x,t)\,dx+\int_{Q_T}|\nabla\overline{w}|^2\,dx\,dt.
$$

From (4.13), we have

$$
E\le C_5\left(1+\|\overline{w}\|^{(2(q-1))/(q+1)}_{L^{[(q-1)(n+2)]/(q+1)}(Q_T)}+\|\overline{w}\|^{(2q)/(q+1)}_{L^{(2q)/(q+1)}(Q_T)}\right).
$$

It is easy to see that $(2q)/(q+1)<2<\tilde{q}$ and $[(q-1)(n+2)]/(q+1)<\tilde{q}=2+(4q)/(n(q+1))$ if $q<(n(n+4))/(n^2-4)$. Thus,

$$
(4.14)\qquad E\le C_6\left(1+\|\overline{w}\|^{(2(q-1))/(q+1)}_{L^{\tilde{q}}(Q_T)}+\|\overline{w}\|^{(2q)/(q+1)}_{L^{\tilde{q}}}\right).
$$

Setting $\beta=2/(q+1)\in(0,1)$, from the L^1 estimate of w, we have

$$
\|\overline{w}\|_{L^\beta(\Omega)}=\left(\int_\Omega|\overline{w}(\cdot,t)|^\beta\,dx\right)^{1/\beta}=\|w\|^{1/\beta}_{L^1(\Omega)}\le M_3'^{1/\beta},
$$

$$
\text{for all } t\le T.
$$

It follows from Lemma 4.2 and (4.14) that

$$
\begin{aligned}
E &\le C_6 \Bigg[1 + \Big(M' + M' \sup_{0<t<T} \|\overline{w}(\cdot,t)\|_{L^{(2q)/(q+1)}(\Omega)}^{(4q)/(n(q+1)\tilde{q})} \\
&\qquad\qquad \times \|\nabla \overline{w}\|_{L^2(Q_T)}^{2/\tilde{q}} \Big)^{(2(q-1))/(q+1)} \\
&\qquad + \Big(M' + M' \sup_{0<t<T} \|\overline{w}(\cdot,t)\|_{L^{(2q)/(q+1)}(\Omega)}^{(4q)/(n(q+1)\tilde{q})} \\
&\qquad\qquad \times \|\nabla \overline{w}\|_{L^2(Q_T)}^{2/\tilde{q}} \Big)^{(2q)/(q+1)} \Bigg] \\
&\le C_7 \Bigg[1 + \left(\sup_{0<t<T} \|\overline{w}(\cdot,t)\|_{L^{(2q)/(q+1)}(\Omega)}^{(2q)/(q+1)} \right)^{(4(q-1))/(n(q+1)\tilde{q})} \\
&\qquad \times \left(\|\nabla \overline{w}\|_{L^2(Q_T)}^2 \right)^{(2(q-1))/((q+1)\tilde{q})} \\
&\qquad + \left(\sup_{0<t<T} \|\overline{w}(\cdot,t)\|_{L^{(2q)/(q+1)}(\Omega)}^{(2q)/(q+1)} \right)^{(4q)/(n(q+1)\tilde{q})} \\
&\qquad \times \left(\|\nabla \overline{w}\|_{L^2(Q_T)}^2 \right)^{(2q)/((q+1)\tilde{q})} \Bigg] \\
&\le C_7 \Big(1 + E^{(4(q-1))/(n(q+1)\tilde{q})} E^{(2(q-1))/((q+1)\tilde{q})} \\
&\qquad + E^{(4q)/(n(q+1)\tilde{q})} E^{(2q)/((q+1)\tilde{q})} \Big).
\end{aligned}
\tag{4.15}
$$

Since $[(4(q-1))/(n(q+1)\tilde{q})] + [(2(q-1))/((q+1)\tilde{q})] \in (0,1)$, $[4q/(n(q+1)\tilde{q})] + [2q/((q+1)\tilde{q})] \in (0,1)$, we know by (4.15) that a positive constant M_6 exists such that $E \le M_6$. This implies that $\|\overline{w}\|_{L^{\tilde{q}}(Q_T)}$ is bounded. Thus, $w \in L^{[(q+1)\tilde{q}]/2}(Q_T)$. From this and (4.15), we have

$$
v, w \in L^q(Q_T), \quad \text{for all } q \in \left(1, \frac{2(n+1)}{n-2}\right). \tag{4.16}
$$

Specifically, choose $q = 2$ in (4.6) and (4.11). Then there exists a positive constant M_7 for $n \le 5$ such that

$$
\|v\|_{V^2(Q_T)}, \qquad \|w\|_{V^2(Q_T)} \le M_7. \tag{4.17}
$$

Therefore,

$$
\|v\|_{L^{[2(n+2)]/n}(Q_T)}, \qquad \|w\|_{L^{[2(n+2)]/n}(Q_T)} \le M_7. \tag{4.18}
$$

Thirdly, to prove $v, w \in L^{\infty}(Q_T)$, we rewrite the second and the third equations of (1.4) as

$$v_t - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{ij}\frac{\partial v}{\partial x_i}) = v\left[\frac{a_1 u}{1+b_1 u} - \frac{a_2 w}{1+b_2 v} - D_1\right], \tag{4.19}$$

and

$$w_t - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(b_{ij}\frac{\partial w}{\partial x_i} + b_j w) = w\left[\frac{a_2 v}{1+b_2 v} - D_2\right], \tag{4.20}$$

where $a_{ij} = (d_2 + 2\alpha_{22}v)\delta_{ij}$, $b_{ij} = (d_3 + \alpha_{31}u + 2\alpha_{33}w)\delta_{ij}$, $b_j = \alpha_{31}u_{x_j}$ and δ_{ij} are Kronecker symbols.

To apply the maximum principle [**21**, Theorem 9.1, pages 341–342] to conclude that $v, w \in L^{\infty}(Q_T)$, we need to verify the following conditions: (1) $\|w\|_{V^2(Q_T)}$ is bounded; (2) $\sum_{i,j=1}^{n} b_{ij}\xi_i\xi_j \geq \nu \sum_{i=1}^{n} \xi_i^2$ (ν is a positive constant); (3) $\| \sum_{j=1}^{n} b_j^2,\ w((a_2 v)/(1+b_2 v) - D_2)\|_{L^{q,r}(Q_T)} \leq \mu_1$ (μ_1 is a positive constant), where q and r satisfy

$$\begin{gathered}\frac{1}{r} + \frac{n}{2q} = 1 - \chi, \quad 0 < \chi < 1, \\ q \in \left[\frac{n}{2(1-\chi)}, +\infty\right), \qquad r \in \left[\frac{1}{1-\chi}, +\infty\right), \quad n \geq 2.\end{gathered} \tag{4.21}$$

We next verify conditions (1)–(3) in turn. We know that condition (1) is true from (4.17). One can choose $\nu = d_3$ in condition (2). Let $u_2 = (d_1 + \alpha_{11}u)u$. It follows from Lemma 4.1 and fundamental estimates of the parabolic equation [**21**] that a positive constant M_8 exists such that

$$\|u_2\|_{W_q^{2,1}(Q_T)} \leq M_8, \quad q \in \left(\frac{n+2}{2}, \frac{2(n+1)}{n-2}\right).$$

By the Sobolev embedding theorem, it follows that

$$\nabla u_2 \in L^{((n+2)q)/(n+2-q)}(Q_T).$$

Solving equation $u_2 = (d_1 + \alpha_{11}u)u$ for u, we know that a positive constant M_9 exists such that $\|\nabla u\|_{L^{((n+2)q)/(n+2-q)}} \leq M_9$ for all $q \in (((n+2)/2), (2(n+1))/(n-2))$. Hence,

$$b_j^2 \in L^{((n+2)q)/(2(n+2-q))}(Q_T) \subset L^q(Q_T), \quad w\left[\frac{a_2 v}{1+b_2 v} - D_2\right] \in L^q(Q_T).$$

Selecting $q = r = ((n+2)q)/(2(n+2-q))$, the above three conditions are all satisfied. It follows from [**21**, Theorem 9.1] that a positive constant M_{10} exists such that

$$\|w\|_{L^\infty(Q_T)} \leq M_{10}. \tag{4.22}$$

Similarly, a positive constant exists, still denoted by M_{10}, such that

$$\|v\|_{L^\infty(Q_T)} \leq M_{10}. \tag{4.23}$$

It follows from (4.4), (4.22) and (4.23) that a positive constant M_{11} exists for all $T > 0$ such that

$$\|u\|_{L^\infty(Q_T)}, \qquad \|v\|_{L^\infty(Q_T)}, \qquad \|w\|_{L^\infty(Q_T)} \leq M_{11}.$$

Finally, we prove that the solution (u, v, w) of (1.4) is classical in Q_T for any $T > 0$. The first equation of (1.4) can be written as

$$u_t - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(\overline{a}_{ij}\frac{\partial u}{\partial x_i}) = u\left[r\left(1 - \frac{u}{k}\right) - \frac{a_1 v}{1 + b_1 u}\right], \tag{4.24}$$

where $\overline{a}_{ij}(x,t) = (d_1 + 2\alpha_{11}u)\delta_{ij}$.

From Lemma 4.1 and equation (4.4), we know that (4.24) satisfies all conditions of [**21**, Theorem 10.1] for $n = 2,\ 3,\ 4,\ 5$ and $q = r \in (((n+2)/2), (2(n+1))/(n-2))$. Thus, a positive constant M_{12} and some $\beta \in (0,1)$ exist such that

$$\|u\|_{C^{\beta,(\beta/2)}(\overline{Q}_T)} \leq M_{12}. \tag{4.25}$$

From $u_2 = (d_1 + \alpha_{11}u)u \in W_q^{2,1}(Q_T)$ and the Sobolev embedding theorem, we have

$$u_2 \in C^{1+\alpha,((1+\alpha)/2)}(\overline{Q}_T) \text{ and } u = \left(-d_1 + \sqrt{d_1^2 + 4\alpha_{11}u_2}\right)/(2\alpha_{11}),$$

so

$$u \in C^{1+\alpha,((1+\alpha)/2)}(\overline{Q}_T). \tag{4.26}$$

Now we return to (4.19) and (4.20). From the above proof, we know that $\|v\|_{V^2(Q_T)}$ and $\|w\|_{V^2(Q_T)}$ are bounded for $n = 2, 3, 4, 5$, and $v(((a_1u)/(1+b_1u)) - ((a_2w)/(1+b_2v)) - D_1)$, $w(((a_2v)/(1+b_2v)) - D_2) \in L^\infty(Q_T)$. One can obtain by Schauder estimate [**21**, Theorem 10.1] that $\alpha^* \in (0,1)$ exists such that

$$v, w \in C^{\alpha^*,(\alpha^*/2)}(\overline{Q}_T). \tag{4.27}$$

By (4.25)–(4.27), we have $(d_1 + 2\alpha_{11}u)\delta_{ij}$, $-2\alpha_{11}(\partial u/\partial x_j)$, $u[r(1 - (u/k)) - ((a_1v)/(1+b_1u))] \in C^{\sigma,(\sigma/2)}(\overline{Q}_T)$, where $\sigma = \min\{\alpha, \alpha^*\}$. Therefore, applying the Schauder estimate [**41**, Theorem 5.3] for (4.24), we have

$$u \in C^{2+\sigma,1+(\sigma/2)}(\overline{Q}_T). \tag{4.28}$$

Let

$$v_2 = (d_2 + \alpha_{22}v)v, w_2 = (d_3 + \alpha_{31}u + \alpha_{33}w)w. \tag{4.29}$$

Then

$$v_{2t} = (d_2+2\alpha_{22}v)\Delta v_2+v(d_2+2\alpha_{22}v)\left(\frac{a_1u}{1+b_1u}-\frac{a_2w}{1+b_2v}-D_1\right), \tag{4.30}$$

$$\begin{aligned} w_{2t} &= (d_3 + \alpha_{31}u + 2\alpha_{33}w)\Delta w_2 \\ &\quad + w(d_3 + \alpha_{31}u + 2\alpha_{33}w)\left(\frac{a_2v}{1+b_2v} - D_2\right) \\ &\quad + \alpha_{31}wu_t. \end{aligned} \tag{4.31}$$

It follows from (4.28) that all the coefficients of (4.30) and (4.31) are in $C^{\sigma,(\sigma/2)}(\overline{Q}_T)$. Thus, by the Schauder estimate [**21**, Theorem 5.3],

$$v_2, w_2 \in C^{2+\sigma,1+(\sigma/2)}(\overline{Q}_T).$$

Furthermore, by solving equation (4.29) for v and w, we have

$$v, w \in C^{2+\sigma,1+(\sigma/2)}(\overline{Q}_T). \tag{4.32}$$

In particular, to conclude, $u, v, w \in C^{2+\alpha,1+(\alpha/2)}(\overline{Q}_T)$ for the case $\sigma < \alpha$. We need to repeat the above bootstrap arguments. Since T is arbitrary and Theorem 4.1 can be proved in a similar way as Theorem 2 in [**11**] when the space dimension $n = 1$, the proof of Theorem 4.1 is completed.

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