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MODELING DISEASE SPREAD VIA TRANSPORT-RELATED INFECTION BY A DELAY DIFFERENTIAL EQUATION

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ABSTRACT. A delayed SIS model is developed to describe the effect of transport-related infection, where time delay arises very naturally and the basic reproduction number R_0 can be calculated. It is shown that this number characterizes the disease transmission dynamics: if $R_0 < 1$, there exists only the disease-free equilibrium which is globally asymptotically stable; and if $R_0 > 1$, then there is a disease endemic equilibrium and the disease persists. Analysis of the dependence of R_0 on the transport-related infection parameters shows that an outbreak can arise purely due to this transport-related infection.

1. Introduction. Much has been done in terms of modeling spatial spread of diseases. For example, Wang and Ruan [**14**] studied the global spread pattern of the 2002–03 SARS outbreak. Rvachev and Longini [**6, 7**] used a discrete time difference equation in a continuous state space to study the global spread of influenza. Sattenspiel and Dietz [**8**] introduced a model with travel between populations, where they identified parameters in the case of the transmission of measles in the Caribbean island of Dominica and numerically studied the dynamics of the model. Sattenspiel and Herring [**9**] used the same type of model for the consideration of travel between populations in the Canadian subarctic, which can be thought of as a closed population where travelers can be easily quarantined. Extensions to include quarantine were given in [**10**] with an application to the data of the 1918–1919 influenza epidemic in the center of Canada. Metapopulation models involving multi-patches have also been recently studied in [**1, 13, 15, 16**].

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Transport-related infection was not considered until the work [**2**] of Cui, Takeuchi and Saito, where the following SIS model was formulated and analyzed:

$$(1.1)\quad \begin{cases} \dfrac{dS_1}{dt} = A - dS_1 - \dfrac{\beta S_1 I_1}{S_1 + I_1} + \delta I_1 - \alpha S_1 + \alpha S_2 - \dfrac{\gamma\alpha S_2 I_2}{S_2 + I_2}, \\ \dfrac{dI_1}{dt} = \dfrac{\beta S_1 I_1}{S_1 + I_1} + \dfrac{\gamma\alpha S_2 I_2}{S_2 + I_2} - (c + \delta + \alpha) I_1 + \alpha I_2, \\ \dfrac{dS_2}{dt} = A - dS_2 - \dfrac{\beta S_2 I_2}{S_2 + I_2} + \delta I_2 - \alpha S_2 + \alpha S_1 - \dfrac{\gamma\alpha S_1 I_1}{S_1 + I_1}, \\ \dfrac{dI_2}{dt} = \dfrac{\beta S_2 I_2}{S_2 + I_2} + \dfrac{\gamma\alpha S_1 I_1}{S_1 + I_1} - (c + \delta + \alpha) I_2 + \alpha I_1. \end{cases}$$

This model was generalized by Liu and Takeuchi [**5**] to an SIQS model to incorporate entry screening and exit screening. The global dynamics of model (1.1) was also investigated in [**12**].

Our work is inspired by the aforementioned studies [**2, 5, 12**]. Our focus is to rigorously describe the disease dynamics through an SIS model during the transportation, which gives a precise replacement of the term $(\gamma\alpha S_i I_i)/(S_i + I_i)$, $i = 1, 2$, in model (1.1) which involves a delay representing the time needed to complete the use of the transportation. Such a replacement seems to be natural and necessary, for otherwise solutions of (1.1) may become negative if the infection rate γ during the use of the transportation is sufficiently large.

In the next section, we will construct a delay SIS epidemic model with transport-related infection and obtain an explicit formula for the basic reproduction number R_0 and for an endemic equilibrium (if it exists). We show in Section 3 that $R_0 < 1$ implies the global stability of the disease-free equilibrium. We then consider the case when $R_0 > 1$. In this case, we obtain the local stability of a positive (endemic) equilibrium (in Section 4), and we show the persistence of the disease (in Section 5). We conclude with some discussions in Section 6 about the role of transport-related infection in a disease outbreak.

2. Model derivation. We consider two patches connected by same sort of transportation and assume, for the sake of simplicity, a typical user of the transportation needs τ-units to complete a one-way transport between two patches. Denote by $S_i(t)$ and $I_i(t)$ the number of susceptibles and infectives in patch i, and let $N_i(t) = S_i(t) + I_i(t)$. Also, let $s_{21}(\theta, t-\tau)$ and $i_{21}(\theta, t-\tau)$ be the number of susceptibles and infectives at time θ which left patch 2 for patch 1, where $t-\tau \leq \theta \leq t$. Therefore, if α is the per capita rate an individual leaves patch 2 to patch 1, then $s_{21}(t-\tau, t-\tau) = \alpha S_2(t-\tau)$ and $i_{21}(t-\tau, t-\tau) = \alpha I_2(t-\tau)$.

During the transport from patch 2 to patch 1 between time $t-\tau$ and time t, we have

$$
(2.0)\quad \begin{cases} \dfrac{\partial}{\partial\theta} s_{21}(\theta, t-\tau) = -\gamma \dfrac{s_{21}(\theta, t-\tau) i_{21}(\theta, t-\tau)}{n_{21}(\theta, t-\tau)}, \\ \dfrac{\partial}{\partial\theta} i_{21}(\theta, t-\tau) = \gamma \dfrac{s_{21}(\theta, t-\tau) i_{21}(\theta, t-\tau)}{n_{21}(\theta, t-\tau)}, \\ n_{21}(\theta, t-\tau) = s_{21}(\theta, t-\tau) + i_{21}(\theta, t-\tau), \quad \theta \in [t-\tau, t], \end{cases}
$$

where γ is the transmission rate during the use of the transportation. Thus $(\partial/\partial\theta) n_{21}(\theta, t-\tau) = 0$, and hence $n_{21}(\theta, t-\tau) = \alpha(S_2(t-\tau) + I_2(t-\tau)) = K$ is a constant, independent of $\theta \in [t-\tau, t]$. From the second equation of (2.0), we get a logistic equation

$$
\frac{\partial}{\partial\theta} i_{21}(\theta, t-\tau) = \gamma \left[1 - \frac{i_{21}(\theta, t-\tau)}{K}\right] i_{21}(\theta, t-\tau).
$$

Therefore,

$$
i_{21}(t, t-\tau) = \frac{K i_{21}(\theta, t-\tau)}{[K - i_{21}(\theta, t-\tau)] e^{-\gamma(t-\theta)} + i_{21}(\theta, t-\tau)}.
$$

In particular,

$$
i_{21}(t, t-\tau) = \frac{\alpha I_2(t-\tau)[S_2(t-\tau) + I_2(t-\tau)]}{e^{-\gamma\tau} S_2(t-\tau) + I_2(t-\tau)}
$$

gives the inflow to the infective class of patch 1. Therefore,

$$
\begin{aligned} s_{21}(t, t-\tau) &= K - i_{21}(t, t-\tau) \\ &= \alpha[S_2(t-\tau) + I_2(t-\tau)] - i_{21}(t, t-\tau) \\ &= \frac{\alpha e^{-\gamma\tau} S_2(t-\tau)[S_2(t-\tau) + I_2(t-\tau)]}{e^{-\gamma\tau} S_2(t-\tau) + I_2(t-\tau)}, \end{aligned}
$$

and $s_{21}(t, t-\tau)$ is the input rate to the susceptible class of patch 1.

By symmetry, we can get $s_{12}(t, t-\tau)$ and $i_{12}(t, t-\tau)$.

Consequently, we obtain the following delay differential system for the disease transmission between two patches involving transport-related infection

$$(2.1)\quad \begin{cases} \dfrac{dS_1}{dt} = A - dS_1 - \dfrac{\beta S_1 I_1}{S_1 + I_1} + \delta I_1 - \alpha S_1 + s_{21}(t, t-\tau), \\ \dfrac{dI_1}{dt} = \dfrac{\beta S_1 I_1}{S_1 + I_1} + i_{21}(t, t-\tau) - (d + \delta + \alpha) I_1, \\ \dfrac{dS_2}{dt} = A - dS_2 - \dfrac{\beta S_2 I_2}{S_2 + I_2} + \delta I_2 - \alpha S_2 + s_{12}(t, t-\tau), \\ \dfrac{dI_2}{dt} = \dfrac{\beta S_2 I_2}{S_2 + I_2} + i_{12}(t, t-\tau) - (d + \delta + \alpha) I_2, \end{cases}$$

with

$$(2.2)\quad \begin{aligned} s_{21}(t, t-\tau) &= \frac{\alpha e^{-\gamma\tau} S_2(t-\tau)}{e^{-\gamma\tau} S_2(t-\tau) + I_2(t-\tau)} [S_2(t-\tau) + I_2(t-\tau)], \\ i_{21}(t, t-\tau) &= \frac{\alpha I_2(t-\tau)}{e^{-\gamma\tau} S_2(t-\tau) + I_2(t-\tau)} [S_2(t-\tau) + I_2(t-\tau)], \\ s_{12}(t, t-\tau) &= \frac{\alpha e^{-\gamma\tau} S_1(t-\tau)}{e^{-\gamma\tau} S_1(t-\tau) + I_1(t-\tau)} [S_1(t-\tau) + I_1(t-\tau)], \\ i_{12}(t, t-\tau) &= \frac{\alpha I_1(t-\tau)}{e^{-\gamma\tau} S_1(t-\tau) + I_1(t-\tau)} [S_1(t-\tau) + I_1(t-\tau)]. \end{aligned}$$

Here, and in what follows, A is the newborn rate (into the susceptible class), d is the natural death rate, β is the disease transmission rate within a patch, δ is the recovery rate.

It should be emphasized that the concept of patch should be understood in a much broader sense than a geographical location such as a city. For example, we can consider here a city with two major regions connected by public transportation. The model (2.1)–(2.2) implicitly assumes that individuals claim their residency in a patch as long as they are physically there, and the behaviors including the use of transportation are the same for all the individuals within the patch.

The initial conditions for system (2.1) take the form of

$$(2.3)\quad \begin{aligned} &S_1(\theta) = \phi_1(\theta) \geq 0, I_1(\theta) = \phi_2(\theta) \geq 0, \\ &S_2(\theta) = \psi_1(\theta) \geq 0, I_2(\theta) = \psi_2(\theta) \geq 0, \quad \theta \in [-\tau, 0], \end{aligned}$$

where $\Phi = (\phi_1, \phi_2, \psi_1, \psi_2) \in C^+([-\tau, 0], \mathbf{R}^4_+)$, the space of continuous functions mapping $[-\tau, 0]$ into $\mathbf{R}^4_+$. Note that time lag τ arises very naturally.

It is straightforward to show, see [**11**], that

Lemma 2.1. *The solution of system* (2.1) *with an initial condition* (2.3) *is nonnegative for all* $t \geq 0$.

Lemma 2.2. *There exists an* $M > 0$ *such that for any solution of system* (2.1) *with initial condition* (2.3), *there must be a* $T > 0$ *such that* $S_i(t) \leq M$ *and* $I_i(t) \leq M$ *for* $i = 1, 2$ *and* $t \geq T$.

Proof. For $V(t) = S_1(t) + I_1(t) + S_2(t) + I_2(t)$, we have

$$\dot{V}(t) = 2A - (d + \alpha)V(t) + \alpha V(t - \tau). \tag{2.4}$$

It follows that

$$\lim_{t\to\infty} V(t) = \frac{2A}{d}. \tag{2.5}$$

This completes the proof. □

It is easy to check that $E_0 = ((A/d), 0, (A/d), 0)$ is the disease-free equilibrium of (2.1), and it exists for all nonnegative values of the parameters. The basic reproduction number is defined as

$$R_0 = \frac{\beta + \alpha e^{\gamma\tau}}{d + \delta + \alpha}.$$

We can also show that if $R_0 > 1$, then system (2.1) also has a unique endemic equilibrium $E_+ = (S_*, I_*, S_*, I_*)$ given by

$$I_* = \frac{A}{d} - S_*, \qquad S_* = \frac{-B - \sqrt{B^2 - 4C}}{2\beta(1 - e^{-\gamma\tau})} \frac{A}{d}, \tag{2.6}$$

with

$$B = (d + \delta + \alpha)e^{-\gamma\tau} - (d + \delta + \alpha + \beta), \qquad C = \beta(d + \delta)(1 - e^{-\gamma\tau}).$$

In fact, the total populations in both patches are described by the equation

$$\begin{cases} \dfrac{dN_1(t)}{dt} = A - (d+\alpha)N_1(t) + \alpha N_2(t-\tau), \\ \dfrac{dN_2(t)}{dt} = A - (d+\alpha)N_2(t) + \alpha N_1(t-\tau), \end{cases} \tag{2.7}$$

which has a unique equilibrium $(N_1^*, N_2^*) = ((A/d), (A/d))$. Suppose $E_+ = (S_1^*, I_1^*, S_2^*, I_2^*)$ is an endemic equilibrium of (2.1), then the symmetry, implies $S_1^* = S_2^* \triangleq S_*$, $I_1^* = I_2^* \triangleq I_*$. Then $S_* + I_* = A/d$, and S_* satisfies

$$f(S_*) = \frac{d\beta}{A}(1 - e^{-\gamma\tau})S_*^2 + BS_* + \left(1 + \frac{\delta}{d}\right)A = 0.$$

It is easy to see that $f(S_*)$ has two positive roots

$$S_{1*} = \frac{-B - \sqrt{B^2 - 4C}}{2\beta(1 - e^{-\gamma\tau})}(A/d),$$
$$S_{2*} = \frac{-B + \sqrt{B^2 - 4C}}{2\beta(1 - e^{-\gamma\tau})}(A/d).$$

To obtain the endemic equilibrium, we need $0 < S_* < (A/d)$. On the other hand, $S_{2*} \geq (A/d)$, for otherwise, we have $S_{2*} < (A/d)$, that is,

$$2\beta(1 - e^{-\gamma\tau}) + B > \sqrt{B^2 - 4C}, \tag{2.8}$$

which implies

$$e^{-\gamma\tau}(d + \delta + \alpha - 2\beta) > d + \delta + \alpha - \beta. \tag{2.9}$$

Squaring (2.8), we get $d + \delta + \alpha > \beta + \alpha e^{\gamma\tau}$. Thus, (2.9) implies $e^{-\gamma\tau} > (d + \delta + \alpha - \beta)/(d + \delta + \alpha - 2\beta) > 1$, a contradiction.

Since $B < 0$, then it follows that a unique endemic equilibrium exists if and only if $f(A/d) < 0$. Hence, a unique endemic equilibrium exists if and only if $R_0 > 1$.

3. Global Stability of E_0.

Theorem 3.1. *The disease-free equilibrium $E_0 = ((A/d), 0, (A/d), 0)$ is unstable if $R_0 > 1$ and is globally asymptotically stable if $R_0 < 1$.*

Proof. The characteristic equation of system (2.1) at E_0 is

$$\det \begin{pmatrix} \lambda I - X & -Y \\ -Y & \lambda I - X \end{pmatrix} = 0, \tag{3.1}$$

where

$$X = \begin{pmatrix} -(d+\alpha) & -\beta+\delta \\ 0 & \beta-(d+\delta+\alpha) \end{pmatrix},$$

and

$$Y = \begin{pmatrix} \alpha e^{-\lambda\tau} & \alpha(1-e^{\gamma\tau})e^{-\lambda\tau} \\ 0 & \alpha e^{\gamma\tau}e^{-\lambda\tau} \end{pmatrix}.$$

From [**2**], the roots of (3.1) are identical to those of $\det(\lambda I - X - Y) = 0$ and $\det(\lambda I - X + Y) = 0$. On the other hand, we have

$$\begin{aligned} \det(\lambda I - X - Y) = (\lambda + d + \alpha - \alpha e^{-\lambda\tau}) \\ (\lambda - \beta + d + \delta + \alpha - \alpha e^{\gamma\tau}e^{-\lambda\tau}) = 0, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \det(\lambda I - X + Y) = (\lambda + d + \alpha + \alpha e^{-\lambda\tau}) \\ (\lambda - \beta + d + \delta + \alpha + \alpha e^{\gamma\tau}e^{-\lambda\tau}) = 0. \end{aligned} \tag{3.3}$$

Let $f(\lambda) = \lambda - \beta + d + \delta + \alpha - \alpha e^{\gamma\tau}e^{-\lambda\tau}$, $g(\lambda) = \lambda - \beta + d + \delta + \alpha + \alpha e^{\gamma\tau}e^{-\lambda\tau}$. We know from [**3**] that all roots of

$$\lambda + d + \alpha \pm \alpha e^{-\lambda\tau} = 0 \tag{3.4}$$

have negative real parts.

Note also that if $R_0 = (\beta + \alpha e^{\gamma\tau})/(d + \delta + \alpha) > 1$, then $f(0) = d + \delta + \alpha - (\beta + \alpha e^{\gamma\tau}) < 0$, and $f(+\infty) = \infty$. Hence, $f(\lambda) = 0$ has at least one positive root and E_0 is unstable.

If $R_0 < 1$, then $d + \delta + \alpha > \beta + \alpha e^{\gamma\tau}$. Let $\lambda = u + iv$ with $u, v \in \mathbf{R}$ be a root of $f(\lambda) = 0$. Then we have

$$(u + d + \delta + \alpha - \beta)^2 + v^2 = (\alpha e^{\gamma\tau}e^{-u\tau})^2.$$

If $u \geq 0$, then

$$(u + d + \delta + \alpha - \beta)^2 + v^2 > (\alpha e^{\gamma\tau})^2 \geq (\alpha e^{\gamma\tau} e^{-u\tau})^2,$$

a contradiction. This shows that all roots of $f(\lambda) = 0$ must have negative real parts. Similarly, we can show that all roots of $g(\lambda) = 0$ have negative real parts. Therefore, E_0 is asymptotically stable.

To complete the proof of Theorem 3.1, we only need to show that E_0 is globally attractive under the condition $R_0 < 1$. Let $V(t) = I_1(t)+I_2(t)$. We have

$$\begin{aligned}\dot{V}(t) &= \beta\frac{S_1 I_1}{S_1 + I_1} + \beta\frac{S_2 I_2}{S_2 + I_2} - (d + \delta + \alpha)(I_1 + I_2)\\ &\quad + \frac{\alpha I_1(t-\tau)}{e^{-\gamma\tau}S_1(t-\tau) + I_1(t-\tau)}[S_1(t-\tau) + I_1(t-\tau)]\\ &\quad + \frac{\alpha I_2(t-\tau)}{e^{-\gamma\tau}S_2(t-\tau) + I_2(t-\tau)}[S_2(t-\tau) + I_2(t-\tau)]\\ &\leq \alpha e^{\gamma\tau} V(t-\tau) - (d + \delta + \alpha - \beta)V(t).\end{aligned}$$

But, for the equation

$$\dot{u}(t) = \alpha e^{\gamma\tau} u(t-\tau) - (d + \delta + \alpha - \beta)u(t),$$

if $R_0 < 1$, then $\lim_{t\to\infty} u(t) = 0$ (again, see [**3**]). Therefore, an application of the standard comparison argument yields $\lim_{t\to\infty} V(t) = 0$, which implies that

$$\lim_{t\to\infty} I_1(t) = 0, \qquad \lim_{t\to\infty} I_2(t) = 0. \tag{3.5}$$

On the other hand, the total populations satisfy

$$\begin{cases} \dfrac{dN_1(t)}{dt} = A - (d + \alpha)N_1(t) + \alpha N_2(t-\tau), \\ \dfrac{dN_2(t)}{dt} = A - (d + \alpha)N_2(t) + \alpha N_1(t-\tau). \end{cases} \tag{3.6}$$

This is a cooperative and irreducible system of delay differential equations with a unique locally asymptotically stable equilibrium $((A/d),(A/d))$. Therefore,

$$\lim_{t\to\infty}(N_1(t), N_2(t)) = \left(\frac{A}{d}, \frac{A}{d}\right),$$

completing the proof. □

4. Local stability of E_+. In this section, we consider the local stability of positive equilibrium E_+ under the condition $R_0 > 1$.

Theorem 4.1. *If $R_0 = (\beta + \alpha e^{\gamma\tau})/(d + \delta + \alpha) > 1$, then the positive equilibrium E_+ is locally asymptotically stable.*

Proof. For simplicity, we consider the following equations:

$$\tag{4.1} \begin{cases} \dfrac{dI_1(t)}{dt} = \dfrac{\beta[N_1(t) - I_1(t)]I_1(t)}{N_1(t)} + i_{21}(t, t-\tau) - (d + \delta + \alpha)I_1(t), \\ \dfrac{dN_1(t)}{dt} = A - (d+\alpha)N_1(t) + \alpha N_2(t-\tau), \\ \dfrac{dI_2(t)}{dt} = \dfrac{\beta[N_2(t) - I_2(t)]I_2(t)}{N_2(t)} + i_{12}(t, t-\tau) - (d + \delta + \alpha)I_2(t), \\ \dfrac{dN_2(t)}{dt} = A - (d+\alpha)N_2(t) + \alpha N_1(t-\tau), \end{cases}$$

with

$$\tag{4.2} \begin{aligned} i_{21}(t, t-\tau) &= \frac{\alpha I_2(t-\tau)}{e^{-\gamma\tau}N_2(t-\tau) + (1 - e^{-\gamma\tau})I_2(t-\tau)} N_2(t-\tau), \\ i_{12}(t, t-\tau) &= \frac{\alpha I_1(t-\tau)}{e^{-\gamma\tau}N_1(t-\tau) + (1 - e^{-\gamma\tau})I_1(t-\tau)} N_1(t-\tau). \end{aligned}$$

Linearizing system (4.1) at the equilibrium $\widehat{E}_+ = (I_*, N_*, I_*, N_*)$, where I_* is defined by (2.6) and $N_* = A/d$, we get the characteristic equation as follows

$$\tag{4.3} \det \begin{pmatrix} \lambda I - \widetilde{X} & -\widetilde{Y} \\ -\widetilde{Y} & \lambda I - \widetilde{X} \end{pmatrix} = 0,$$

with

$$\widetilde{X} = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}, \qquad \widetilde{Y} = \begin{pmatrix} b_{11}e^{-\lambda\tau} & b_{12}e^{-\lambda\tau} \\ 0 & b_{22}e^{-\lambda\tau} \end{pmatrix}$$

and $a_{11} = \beta(1 - (2I_*/N_*)) - (d + \delta + \alpha)$, $a_{12} = (\beta I_*^2)/(N_*^2)$, $a_{22} = -(d + \alpha)$,

$$b_{11} = \frac{\alpha e^{-\gamma\tau} N_*^2}{[e^{-\gamma\tau}N_* + (1 - e^{-\gamma\tau})I_*]^2}, \qquad b_{12} = \frac{\alpha(1 - e^{-\gamma\tau})I_*^2}{[e^{-\gamma\tau}N_* + (1 - e^{-\gamma\tau})I_*]^2},$$

$b_{22} = \alpha$. Using a similar argument to that for Theorem 3.1, we note that the roots of equation (4.3) coincide with the roots of $\det(\lambda I - \widetilde{X} - \widetilde{Y}) = 0$ and $\det(\lambda I - \widetilde{X} + \widetilde{Y}) = 0$.

Firstly, we prove that all roots of $\det(\lambda I - \widetilde{X} - \widetilde{Y}) = 0$ have negative real parts, namely, all roots of

$$[\lambda - (a_{11} + b_{11}e^{-\lambda\tau})][\lambda - (a_{22} + b_{22}e^{-\lambda\tau})] = 0, \tag{4.4}$$

have negative real parts. We start with the equation

$$\lambda - (a_{11} + b_{11}e^{-\lambda\tau}) = 0. \tag{4.5}$$

Assume $\lambda = x + iy$ with $x \geq 0$, $x, y \in \mathbf{R}$. Then (4.5) implies

$$x = a_{11} + b_{11}e^{-x\tau}\cos y\tau,$$
$$y = -b_{11}e^{-x\tau}\sin y\tau.$$

As $b_{11} > 0$, we have

$$\begin{aligned} x &\leq a_{11} + b_{11} \\ &= \beta\left(1 - \frac{2I_*}{N_*}\right) - (d + \delta + \alpha) + \frac{\alpha e^{-\gamma\tau}N_*^2}{(e^{-\gamma\tau}N_* + (1 - e^{-\gamma\tau})I_*)^2} \\ &= -\frac{\beta I_*}{N_*} - \frac{\alpha N_*}{e^{-\gamma\tau}N_* + (1 - e^{-\gamma\tau})I_*} + \frac{\alpha e^{-\gamma\tau}N_*^2}{(e^{-\gamma\tau}N_* + (1 - e^{-\gamma\tau})I_*)^2} \\ &= -\frac{\beta I_*}{N_*} - \frac{\alpha(1 - e^{-\gamma\tau})N_* I_*}{(e^{-\gamma\tau}N_* + (1 - e^{-\gamma\tau})I_*)^2} \\ &< 0, \end{aligned}$$

a contradiction. Hence, all roots of (4.5) have negative real parts. Next, we consider the equation

$$\lambda - (a_{22} + b_{22}e^{-\lambda\tau}) = 0. \tag{4.6}$$

Assume $\lambda = x + iy$ with $x \geq 0$, $x, y \in \mathbf{R}$. Then (4.6) implies

$$x = a_{22} + b_{22}e^{-x\tau}\cos y\tau,$$
$$y = -b_{22}e^{-x\tau}\sin y\tau.$$

Since $b_{22} > 0$, we have

$$x \le a_{22} + b_{22} = -(d+\alpha) + \alpha = -d < 0,$$

again a contradiction. Thus all roots of (4.6) have negative real parts. We conclude that all roots of (4.4) have negative real parts.

Similarly, we can show that all roots of $\det(\lambda I - \widetilde{X} + \widetilde{Y}) = 0$ have negative real parts.

Then we have proved that all roots of (4.3) have negative real parts. Hence, the positive equilibrium is locally asymptotically stable. This completes the proof. □

5. Permanence. We now consider the issue of disease persistence.

Theorem 5.1. *Let $R_0 = (\beta + \alpha e^{\gamma\tau})/(d+\delta+\alpha) > 1$. Then there exists an $\epsilon > 0$ such that every solution $(S_1(t), I_1(t), S_2(t), I_2(t))$ of system* (2.1) *with initial conditions $\phi_1(\theta) \ge 0$, $\psi_1(\theta) \ge 0$, $\phi_2(\theta) \ge 0$, $\psi_2(\theta) \ge 0$ and $\phi_2(\theta_0) + \psi_2(\theta_0) \ne 0$ for some $\theta_0 \in [-\tau, 0]$ satisfies*

$$\liminf_{t\to\infty} S_i(t) \ge \epsilon, \qquad \liminf_{t\to\infty} I_i(t) \ge \epsilon, \quad i = 1, 2.$$

In order to prove Theorem 5.1, we need the uniform persistence theorem for infinite dimensional systems from [**4**]. Let X be a complete metric space. Suppose that X^0 is open and dense in X, with $X^0 \cup X_0 = X$, $X^0 \cap X_0 = \varnothing$. Assume that $T(t)$ is a C^0 semi-group on X satisfying

$$T(t) : X^0 \longrightarrow X^0, \qquad T(t) : X_0 \longrightarrow X_0. \tag{5.1}$$

Let $T_b(t) = T(t)|_{X_0}$, and let A_b be the global attractor for $T_b(t)$. Assume further that

(i) There is a $t_0 \ge 0$ such that $T(t)$ is compact for $t > t_0$;

(ii) $T(t)$ is point dissipative in X;

(iii) $\tilde{A}_b = \cup_{x \in A_b} \omega(x)$ is isolated and has an acyclic covering M, where
$$M = \cup_{i=1}^{k} M_k;$$

(iv) $W^s(M_i) \cap X^0 = \varnothing$ for $i = 1, 2, \dots, k$.

Then X_0 is a uniform repeller with respect to X^0, i.e., there is an $\epsilon > 0$ such that for any $x \in X^0$, $\liminf_{t\to+\infty} d(T(t)x, X_0) \geq \epsilon$, where d is the distance of $T(t)x$ from X_0.

We now are able to present the proof of Theorem 5.1.

Proof of Theorem 5.1. From (2.1), we have

$$\frac{dS_i(t)}{dt} \geq A - (\beta + d + \alpha)S_i(t), \quad i = 1, 2.$$

Hence $S_i(t)$ is ultimately bounded below by some positive constant (for example, $m_s = (A/2(\beta + d + \alpha)))$, which is independent of the initial conditions. We need to prove that $\liminf_{t\to\infty} I_i(t) \geq \epsilon$, $i = 1, 2$.

Let

$$\begin{aligned} X &= C^+([-\tau, 0], \mathbf{R}^4_+), \\ X^0 &= \{(\phi_1, \phi_2, \psi_1, \psi_2) \in X|\ \phi_2(\theta^*) > 0 \text{ or } \psi_2(\theta^*) > 0 \\ &\qquad \text{for some } \theta^* \in [-\tau, 0]\}, \\ X_0 &= \{(\phi_1, \phi_2, \psi_1, \psi_2) \in X|\ \phi_2(\theta) \equiv 0, \\ &\qquad \psi_2(\theta) \equiv 0 \text{ for all } \theta \in [-\tau, 0]\}. \end{aligned}$$

It suffices to show that there exists an $\epsilon_0 > 0$ such that, for any solution u_t of system (2.1) initiating from X^0, $\liminf_{t\to\infty} d(u_t, X_0) \geq \epsilon_0$. To this end, we verify below that the conditions of the above uniform persistence theorem are satisfied. Firstly, we show that X^0 is positively invariant. By (2.1), we have

$$\begin{aligned} I_1(t) = e^{-(d+\delta+\alpha)t}\Bigg\{ I_1(0) + \int_0^t \Bigg[&\frac{\beta S_1(\rho) I_1(\rho)}{S_1(\rho) + I_1(\rho)} \\ &+ \frac{\alpha I_2(\rho - \tau)}{e^{-\gamma\tau} S_2(\rho - \tau) + I_2(\rho - \tau)} \\ &\quad (S_2(\rho - \tau) + I_2(\rho - \tau))\Bigg] e^{(d+\delta+\alpha)\rho} d\rho \Bigg\}, \end{aligned} \tag{5.2}$$

and

$$I_2(t) = e^{-(d+\delta+\alpha)t}\left\{ I_2(0) + \int_0^t \left[\frac{\beta S_2(\rho) I_2(\rho)}{S_2(\rho)+I_2(\rho)} + \frac{\alpha I_1(\rho-\tau)}{e^{-\gamma\tau}S_1(\rho-\tau)+I_1(\rho-\tau)} (S_1(\rho-\tau)+I_1(\rho-\tau))\right] e^{(d+\delta+\alpha)\rho} d\rho \right\}. \tag{5.3}$$

Then $I_i(t) > 0$ for all $t > 0$, $i = 1, 2$ if $\Phi = (\phi_1, \phi_2, \psi_1, \psi_2) \in X^0$. This implies that X^0 is positively invariant.

By (2.1), we have

$$\left.\frac{dI_i(t)}{dt}\right|_{(\phi_1,\phi_2,\psi_1,\psi_2)\in X_0} \equiv 0, \quad i = 1, 2;$$

thus, $I_i(t) \equiv 0$ for all $t \geq 0$, $i = 1, 2$. Hence, X_0 is positively invariant, and condition (5.1) is satisfied. We have verified the point dissipativeness of the semi-flow of system (2.1) in Lemma 2.2.

Denote the ω-limit set of the solution of system (2.1) starting in $(\phi_1, \phi_2, \psi_1, \psi_2) \in X$ by $\omega(\phi_1, \phi_2, \psi_1, \psi_2)$. Let

$$\Omega = \cup\{\omega(\phi_1, \phi_2, \psi_1, \psi_2) \mid (\phi_1, \phi_2, \psi_1, \psi_2) \in X_0\}.$$

Restricting system (2.1) to X_0 gives

$$\begin{cases} \dfrac{dS_1(t)}{dt} = A - (d+\alpha)S_1(t) + \alpha S_2(t-\tau), \\ \dfrac{dS_2(t)}{dt} = A - (d+\alpha)S_2(t) + \alpha S_1(t-\tau). \end{cases} \tag{5.4}$$

It is easy to verify that system (5.4) has a unique equilibrium $E_1 = (S_0, S_0)$, where $S_0 = A/d$, which is globally asymptotically stable. Thus $\Omega = \{E_0\}$, and E_0 is a covering of Ω, which is isolated (since E_0 is the unique equilibrium) and is acyclic (since there exists no solution in X_0 which links E_0 to itself). It remains to show that

$$W^s(E_0) \cap X^0 = \varnothing, \tag{5.5}$$

where $W^s(E_0)$ denotes the stable manifold of E_0. Suppose (5.5) does not hold; then there exists a solution $(S_1(t), I_1(t), S_2(t), I_2(t)) \in X^0$, $t \geq 0$, of (2.1) with initial conditions in X^0, such that

$$\lim_{t\to\infty} S_i(t) = S_0, \qquad \lim_{t\to\infty} I_i(t) = 0. \tag{5.6}$$

Since $R_0 = (\beta + \alpha e^{\gamma\tau})/(d+\delta+\alpha) > 1$, we can choose sufficiently small $\varepsilon > 0$ so that $S_0 - \varepsilon > 0$ and

$$m_\delta \triangleq (\beta + \alpha e^{\gamma\tau})(S_0 - \varepsilon)/S_0 - (d+\delta+\alpha) > 0. \tag{5.7}$$

Define $V(t) = I_1(t) + I_2(t)$. For $\varepsilon > 0$, by (5.6), there exists a $t_1 > 0$ such that

$$S_0 - \varepsilon < S_i(t) < S_0 + \varepsilon, \quad 0 < I_i(t) < \frac{1}{2}e^{-\gamma\tau}\varepsilon, \text{ for all } t \geq t_1, \quad i = 1, 2.$$

Hence, by (2.1),

$$\begin{aligned}
\dot{V}(t) &= \dot{I}_1(t) + \dot{I}_2(t)\\
&= \beta\frac{S_1 I_1}{S_1 + I_1} + \beta\frac{S_2 I_2}{S_2 + I_2} - (d+\delta+\alpha)(I_1 + I_2)\\
&\quad + \frac{\alpha I_1(t-\tau)}{e^{-\gamma\tau}S_1(t-\tau) + I_1(t-\tau)}[S_1(t-\tau) + I_1(t-\tau)]\\
&\quad + \frac{\alpha I_2(t-\tau)}{e^{-\gamma\tau}S_2(t-\tau) + I_2(t-\tau)}[S_2(t-\tau) + I_2(t-\tau)]\\
&\geq \beta\left(\frac{S_0-\varepsilon}{S_0-\varepsilon+\varepsilon}I_1 + \frac{S_0-\varepsilon}{S_0-\varepsilon+\varepsilon}I_2\right)\\
&\quad - (d+\delta+\alpha)(I_1(t) + I_2(t))\\
&\quad + \frac{\alpha I_1(t-\tau)}{e^{-\gamma\tau}(S_0-\varepsilon) + e^{-\gamma\tau}\varepsilon}(S_0-\varepsilon)\\
&\quad + \frac{\alpha I_2(t-\tau)}{e^{-\gamma\tau}(S_0-\varepsilon) + e^{-\gamma\tau}\varepsilon}(S_0-\varepsilon)\\
&= \alpha e^{\gamma\tau}\left(\frac{S_0-\varepsilon}{S_0}\right)V(t-\tau)\\
&\quad - \left(d+\delta+\alpha-\beta\frac{S_0-\varepsilon}{S_0}\right)V(t), \quad t \geq t_1 + \tau.
\end{aligned}$$

Since $R_0 > 1$ implies $\alpha e^{\gamma\tau} + \beta > d + \delta + \alpha$, we can choose $\varepsilon > 0$ sufficiently small so that $\alpha e^{\gamma\tau}(S_0 - \varepsilon)/(S_0) > d+\delta+\alpha-\beta(S_0 - \varepsilon)/(S_0)$. Then the solution of the linear delay differential equation $\dot{u}(t) = \alpha e^{\gamma\tau}(S_0 - \varepsilon)/(S_0)u(t-\tau)-(d+\delta+\alpha-\beta(S_0 - \varepsilon)/(S_0))u(t)$ with positive initial condition must converge to infinity as $t \to \infty$. Therefore, a comparison argument yields $V(t) \to \infty$ as $t \to \infty$, a contradiction to (5.6). Thus (5.5) holds. This completes the proof. □

6. Discussions. We have developed a delay SIS model that precisely accounts for the transport-related infection. We have then shown that the model is well-posed, even when the infection acquired during the use of the transportation is large, while models developed in existing literature do not allow large transport-related infection. We have shown that the basic reproduction number characterizes the disease transmission dynamics: if $R_0 < 1$, there exists only the disease-free equilibrium which is globally asymptotically stable; and if $R_0 > 1$, then there is a disease endemic equilibrium and the disease persists.

The basic reproduction number $R_0 = (\beta + \alpha e^{\gamma\tau})/(d + \delta + \alpha)$ clearly describes the contribution of transport-related infection to a disease outbreak. If an outbreak occurs when patches are isolated from each other ($\beta > d + \delta$), then it also does when the two patches are connected by transport ($\beta + \alpha e^{\gamma\tau} > d + \delta + \alpha$). As $(\partial R_0)/(\partial\alpha) = (e^{\gamma\tau}(d + \delta) - \beta)/((d + \delta + \alpha)^2)$, we note that transportation increases R_0 only when $\beta < e^{\gamma\tau}(d + \delta)$. In particular, if $\beta > d + \delta$ and $\gamma\tau < \ln(\beta/d + \delta)$, then increases in transportation decreases R_0. Finally, we remark that an outbreak can arise purely due to the transport-related infection. Namely, if $\beta < d + \delta$ (no outbreak when both patches are isolated) and if the transport-related infection $e^{\gamma\tau} > 1+[(d + \delta - \beta)/\alpha]$, then there will be a disease outbreak in both patches.

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