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TIME-LIMITED MANAGEMENT STRATEGIES OF A SINGLE-SPECIES WITH ALLEE EFFECT

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ABSTRACT. Two kinds of time-limited management strategies of a single-species with Allee effect, described by the impulsive differential equation with initial and boundary value conditions, are presented according to the initial density of the species. By means of the comparison principle and the methods of upper and lower solutions, boundary value problems of impulsive management models are discussed. According to the initial density of the species, there are two kinds of models: the model with impulsive release and the model with impulsive harvesting. The corresponding sufficient conditions under which the corresponding model has a solution or no solution are obtained. If the models have a solution, the corresponding management strategy can be performed successfully. For the model with impulsive release, if other parameters are given, the population of release can be estimated. For the model with impulsive harvesting, the times of impulsive harvesting can also be estimated. Finally, some discussions and corresponding numerical simulations about the results obtained in this paper are given.

1. Introduction.

Many biological and mathematical models suppose that the density of a species always increases if the density doesn't reach the carrying capacity of environment no matter how exiguous it is. But it isn't true for some cases because, for a lot of species (such as white-flag dolphin (*Lipotes vexillifer*) and Chinese sturgeon (*Aclpenser Sinensis Grdy*)), the population density of species will decrease and tend to zero when the density reaches a very low level. The causations from outside are usually attributed to the over-exploitation of biological resources (especially to those with great economic value) and the destruction of the natural habitats of rare

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species by human and other activity. Some causations also exist inside the species. First, some animals are not social. The opportunity of copulation is small when density is exiguous. This will make the density of the species decrease and tend to zero. Second, the sparsity of the species results in reproduction by inbreeding, depresses the quality of species and even leads the species to be extinct. Besides, even though the species has strong ability of reproduction, due to human activities its population decreases rapidly. For example, the sparrow (*Passer montanus saturatus*) has a stronger ability of reproduction and were dispersed widely in China a few decades ago. But, because the sparrows feed on corn, they are considered pests and have been heavily killed. This make the sparrow almost extinct in some areas. On the other hand, the sparrow plays an important role in controlling pests, especially *Lepidoptera*. So, for such a species, we need not only control the population but also prevent it from extinction according to the practice. Therefore, we should consider the Allee effect (Sparse effect) when we investigate management strategies of a single-species.

This paper aims to study management strategy of a single-species with Allee effect by the mathematical method. Many researchers have investigated systems with the Allee effect (for example, [**3**–**5**, **8**, **12**–**14**]) and obtained lots of excellent results. Traditional mathematical models always suppose that the control process is continuous. With the advance of impulsive differential equations, impulsive differential equations are used to describe the evolving process of species which makes the models more reasonable (for example, [**7**, **10**, **11**]). These models are considered in infinite time. But, in practice, when initial population density of the species is less than a critical value, the density will decrease to zero, which requires us to take protective strategies to restore the species in a given time. On the other hand, when the initial population density is larger than the critical value, the density will increase and has a trend to infinite or carrying capacity of the environment, the species might be harmful to the environment, industry and agriculture, so we need to control its density to a lower level at which the species doesn't harm the environment or to the level which is lower than economic injury level (EIL) (or economical threshold ET) in a given time. However, there have been just a few studies on finite time.

The main purpose of this paper is to discuss two kinds of different management strategies for a single-species with Allee effect in a given time (or in a finite time) according to the initial density of the species, that is to say, how the species reach a certain level and live through the sparse term by artificial release and how to control the density so that the species don't become harmful in finite time.

The present paper is organized as follows. In Section 2, two single-species models with Allee effect and artificial control (impulsive harvesting and impulsive release at fixed moments) are formulated according to the initial density. In Section 3, by using the comparison principle and the method of upper and lower solution, the existence and nonexistence of solutions of the models given in Section 2 are investigated, and the corresponding sufficient conditions are given. In Section 4, some discussions and numerical simulations on the two control measures are provided.

2. Model formulation. We first suppose that our object is one kind of single species with Allee effect. The single species model with Allee effect can be written as follows [**2**]:

$$\frac{dx}{dt} = x(b(x) - d(x)),$$

where x is the density of the species, $b(x)$ represents the birth rate of the species, and $d(x)$ is the death rate. Since the natural birth rate is usually the function of the density, we consider $b(x) = bx/(a + x)$ and $d(x) = d$ in this paper. Then the above model can be rewritten as

$$\frac{dx}{dt} = x\left(\frac{bx}{a+x} - d\right).$$

It is clear that $x = 0$ and $x = ad/(b-d)$ are two equilibria and $x(t) \geq 0$, $t \geq 0$, since $dx/dt = 0$. $x = ad/(b - d)$ is a positive equilibrium when $b > d$; otherwise, the system has no positive equilibrium. Furthermore, $x = 0$ is stable and $x = ad/(b - d)$ is unstable when $b > d$. Obviously, the density $x(t)$ decreases and tends to zero as $t \to \infty$ when the initial density $x(0)$ is less than h, $h = ad/(b - d)$, and increases when $x(0)$ is larger than h. If we don't take any measures, the species will be extinct when the initial density is less than h (the sparse term). In view of biodiversity, artificial release is required to restore the species in this case. When the initial density is larger than h, the species will increase. Though there exists intracompetition and the population does not reach the carrying capacity of environment, the species might have been a harmful one. So corresponding control measures should be taken to decrease the population of the species.

For the case of $x(0) < h$, if no artificial measure is taken, the density of the species will be decreasing. The reduction of the per capita growth rate of a population of a biological species at densities smaller than a critical value, e.g., h, is known as the Allee effect [**5**]. Our aim is to increase the density by impulsive release at fixed moments such that the species begins to increase after a given time T, which is finite. The process can be described by the following model:

$$\begin{cases} (dx(t)/dt) = bx^2(t)/(a + x(t)) - dx(t) & t \neq i\tau, \\ \Delta x(t) = p & t = i\tau, \\ x(0) = A < h, \ x(T) + p > h, & \\ i = 1, 2, \dots, n, \ n\tau \leq T, & \end{cases} \tag{2.1}$$

where p is the population of release at fixed moments $i\tau$ and τ is the period of artificial reproduction and release. $x(T) + p > h$ represents that, if there have been n times releases before the moment T, we only add one time release at the moment T and can make the density $x(T^+)$ larger than h. This implies that the population of the species will increase and the goal of restoring the species will be achieved.

For the case of $x(0) > h$ and $x(0)$ larger than a value (for example, ET or EIL) at which the species might be considered a harmful one, we want to decrease the density or population by impulsive harvesting at fixed moments and make $x(T_1) \leq B$. The process can be described by the following model:

$$\begin{cases} (dx(t)/dt) = bx^2(t)/(a + x(t)) - dx(t) & t \neq i\tau_1, \\ \Delta x(t) = -Ex(t) & t = i\tau_1, \\ x(0) = C > B > h, x(T_1) \leq B, & \\ i = 1, 2, \dots, n, \ n\tau_1 \leq T_1, & \end{cases} \tag{2.2}$$

where $0 < E < 1$ represents the fraction of the density which decreases due to a certain artificial measure. τ_1 is the period of impulsive control. In order not to make the density drop back to below h when

$t \in [0, T_1]$, E should be less than $1 - h/B$. $T_1 > 0$ is a given moment. Equation (2.2) shows that, after the time of control experiences from 0, $x(0) = A$, to T_1, the density of the species $x(T_1)$ is less than or equal to B, $B > h$.

In the following, we will discuss the existence or nonexistence of solutions of (2.1) and (2.2) under the condition $b > d$.

We will use a basic comparison result in [**6**]. For convenience, we state it in our notations.

Lemma 2.1 [**6, 9**]. *Let $V : R_+ \times R_+ \to R_+$ and $V \in V_0$. Assume that*

$$\begin{cases} D^+V(t,x) \leq g(t, V(t,x)) & t \neq n\tau, \\ V(t, x(t^+)) \leq \psi_n(V(t, x(t)) & t = n\tau, \end{cases} \tag{2.3}$$

where $g : R_+ \times R_+ \mapsto R$ is continuous in $(n\tau, (n+1)\tau] \times R_+$ and for $v \in R_+$ and $n \in Z_+$,

$$\lim_{(t,y)\to(n\tau^+,v)} g(t,y) = g(n\tau^+, v)$$

exists and $\psi_n : R_+ \to R_+$ is nondecreasing. Let $r(t)$ be the maximal solution of the scalar impulsive differential equation

$$\begin{cases} \dot{u}(t) = g(t, u(t)) & t \neq n\tau, \\ u(t^+) = \psi_n(u(t)) & t = n\tau, \\ u(0^+) = u_0, \end{cases} \tag{2.4}$$

existing on $[0, \infty)$. Then $V(0^+, x_0) \leq u_0$ implies that $V(t, x(t)) \leq r(t)$, $t \geq 0$, where $x(t)$ is any solution of (2.3).

A similar result can be obtained when all the directions of the inequalities in the lemma are reversed. Note that if we have some smoothness conditions of $g(t)$ to guarantee the existence and uniqueness of solutions for (2.3), then $r(t)$ is exactly the unique solution of (2.3).

3. The existence of solutions. Since the expression of the solution of (2.1) or (2.2) is complex, which adds difficulty to the study on the boundary value problem especially in the process of iteration in finite time, we will use the comparison principle and the method of upper and lower solution to discuss and give some sufficient conditions under which (2.1) or (2.2) has a solution or no solution.

3.1. The existence of solutions of (2.1). We first introduce the following theorem:

Theorem 3.1. *Let* $\underline{m}_1 = A\exp(-d\tau)$, $\underline{r}_2 = b\underline{m}_1/(a+\underline{m}_1)-d$, $b > d$.

1) *Suppose* $T = n\tau$, *if* $\underline{m}_1 + p > A$ *and*

$$\underline{m}_1 \exp(\underline{r}_2(T-\tau)) + p\frac{1-\exp(\underline{r}_2 T)}{1-\exp(\underline{r}_2\tau)} > h. \tag{3.1}$$

Then (2.1) *has a solution which satisfies the boundary value problem.*

2) *Suppose* $T > n\tau$, *if* $\underline{m}_1 + p > A$ *and*

$$\underline{m}_1 \exp(\underline{r}_2(T-\tau)) + p\frac{\exp(\underline{r}_2(T-n\tau)) - \exp(\underline{r}_2 T)}{1-\exp(\underline{r}_2\tau)} + p > h. \tag{3.2}$$

Then (2.1) *has a solution which satisfies the boundary value problem.*

Proof. Since $f(x) = bx/(a+x) - d$ is an increasing function, then (2.1) has a series of lower solutions $\underline{x}_i(t)$ when $t \in ((i-1)\tau, i\tau]$, $i = 1, 2, \dots, n$. For simplification, we denote $b\underline{m}_{i-1}/(a+\underline{m}_{i-1}) - d$ by $\underline{r}_i$, $i = 2, \dots, n$ and $\underline{r}_1 = -d$.

In fact, when $t \in [0, \tau]$, $x(t) \geq \underline{x}_1(t)$, where $\underline{x}_1(t)$ satisfies

$$\begin{cases} (d\underline{x}_1(t)/dt) = -d\underline{x}_1(t) = \underline{r}_1\underline{x}_1(t), \\ \underline{x}_1(0^+) = x(0) = A, \end{cases} \tag{3.3}$$

and

$$x(\tau) \geq \underline{x}_1(\tau) = \underline{m}_1 = A\exp(-d\tau), \quad t \in [0, \tau].$$

When $A\exp(-d\tau)+p > A$, it is clear that $\underline{m}_1+p > A$. When $t \in (\tau, 2\tau]$, $x(t) \geq \underline{x}_2(t)$, where $\underline{x}_2(t)$ satisfies

$$\begin{cases} (d\underline{x}_2(t)/dt) = (b\underline{m}_1/(a+\underline{m}_1) - d)\underline{x}_2(t) = \underline{r}_2\underline{x}_2(t), \\ \underline{x}_2(\tau^+) = \underline{m}_1 + p, \end{cases} \tag{3.4}$$

and

$$x(2\tau) \geq \underline{x}_2(2\tau) = \underline{m}_2 = (\underline{m}_1 + p)\exp(\underline{r}_2\tau), \quad t \in (\tau, 2\tau].$$

Since $\underline{r}_2 > \underline{r}_1$ and $A\exp(-d\tau) + p > A$, we have $\underline{m}_2 + p > \underline{m}_1 + p > A$. Similarly, we can obtain that when $t \in ((n-1)\tau, n\tau]$, then $x(t) \geq \underline{x}_n(t)$, where $\underline{x}_n(t)$ satisfies

$$\begin{cases} (d\underline{x}_n(t)/dt) = (b\underline{m}_{n-1}/(a + \underline{m}_{n-1}) - d)\underline{x}_n(t) = \underline{r}_n\underline{x}_n(t), \\ \underline{x}_n((n-1)\tau^+) = \underline{m}_{n-1} + p, \end{cases} \tag{3.5}$$

and

$$x(n\tau) \geq \underline{x}_2(n\tau) = \underline{m}_n = (\underline{m}_{n-1} + p)\exp(\underline{r}_n\tau), \; t \in ((n-1)\tau, n\tau].$$

Since $\underline{r}_n > \cdots > \underline{r}_2 > \underline{r}_1$ and $A\exp(-d\tau) + p > A$, we have $\underline{m}_n + p > \cdots > \underline{m}_2 + p > \underline{m}_1 + p > A$ and $x(n\tau) \geq \underline{x}_n(n\tau) = \underline{m}_n$.

$$\begin{aligned} \underline{m}_n &= (\underline{m}_{n-1} + p)\exp(\underline{r}_n\tau) = \underline{m}_{n-1}\exp(\underline{r}_n\tau) + p\exp(\underline{r}_n\tau) \\ &= \cdots \\ &= \underline{m}_1\exp((\underline{r}_2 + \underline{r}_3 + \cdots + \underline{r}_n)\tau) + p\exp((\underline{r}_2 + \underline{r}_3 + \cdots + \underline{r}_n)\tau) \\ &\quad + p\exp((\underline{r}_3 + \cdots + \underline{r}_n)\tau) + \cdots + p\exp(\underline{r}_n\tau) \\ &\geq \underline{m}_1\exp((n-1)\underline{r}_2\tau) + p\exp((n-1)\underline{r}_2\tau) + p\exp((n-2)\underline{r}_2\tau) \\ &\quad + \cdots + p\exp(\underline{r}_2\tau). \end{aligned}$$

If $T = n\tau$, when the conditions of the theorem hold and we have

$$x(T) + p \geq \underline{m}_1\exp(\underline{r}_2(T - \tau)) + p\frac{1 - \exp(\underline{r}_2 T)}{1 - \exp(\underline{r}_2\tau)} > h. \tag{3.6}$$

Then (2.1) has a solution.

If $T > n\tau$, then $x(t) \geq \underline{x}_{n+1}(t), \; t \in (n\tau, T]$, where $\underline{x}_{n+1}(t)$ satisfies

$$\begin{cases} (d\underline{x}_{n+1}(t)/dt) = (b\underline{m}_n/(a + \underline{m}_n) - d)\underline{x}_{n+1} = \underline{r}_{n+1}\underline{x}_{n+1}, \\ \underline{x}_{n+1}(n\tau^+) = \underline{m}_n + p, \end{cases}$$

and then

$$x(T) \geq \underline{x}_{n+1}(T) = (\underline{m}_n + p)\exp(\underline{r}_{n+1}(T - n\tau)).$$

Furthermore,

$$x(T)+p \geq \underline{m}_1 \exp(\underline{r}_2(T-\tau))+p\frac{\exp(\underline{r}_2(T-n\tau))-\exp(\underline{r}_2 T)}{1-\exp(\underline{r}_2\tau)}+p > h.$$

The proof is completed. □

Remark 3.1. When other parameters except p are given firstly, from (3.1) and (3.2) we can estimate the population of impulsive release:

$$p > p_1 = \frac{h(1-\exp(\underline{r}_2\tau))+\underline{m}_1(\exp(\underline{r}_2 T)-\exp(\underline{r}_2(T-\tau))}{1-\exp(\underline{r}_2 T)}, \quad T = n\tau; \tag{3.7}$$

$$p > p_1' = \frac{h(1-\exp(\underline{r}_2\tau))+\underline{m}_1(\exp(\underline{r}_2 T)-\exp(\underline{r}_2(T-\tau))}{\exp(\underline{r}_2(T-n\tau))-\exp(\underline{r}_2 T)+1-\exp(\underline{r}_2\tau)}, \quad T > n\tau; \tag{3.8}$$

equation (3.7) (or (3.8)) implies $x(T)+p > h$ when $p > p_1$ (or $p > p_1'$), that is to say, after the given moment T, the density of the species will be larger than h if the density of impulsive release p is larger than p_1 (or p_1').

Similar to the discussion above, we have the following theorem:

Theorem 3.2. *Let* $\overline{r}_1 = bA/(a+A)-d$, $\overline{m}_1 = A\exp(\overline{r}_1\tau)$, $b > d$.

1) *Suppose* $T = n\tau$. *If* $\underline{m}_1 + p > A$ *and*

$$\overline{m}_1 + np < h, \tag{3.9}$$

then (2.1) *has no solution which satisfies the boundary value problem.*

2) *Suppose* $T > n\tau$. *If* $\underline{m}_1 + p > A$ *and*

$$\overline{m}_1 + (n+1)p < h, \tag{3.10}$$

then (2.1) *has no solution which satisfies the boundary value problem.*

Proof. Similar to the proof of Theorem 3.1, (2.1) has a series of upper solutions $\overline{x}_i(t)$ when $t \in ((i-1)\tau, i\tau]$, $i = 1, 2, \ldots, n$. For simplification, we denote $b(\overline{m}_{i-1}+p)/(a+\overline{m}_{i-1}+p)-d$ by $\overline{r}_i$, $i = 2, \ldots, n$.

In fact, when $t \in [0, \tau]$, $x(t) \leq \overline{x}_1(t)$, where $\overline{x}_1(t)$ satisfies

$$\begin{cases} (d\bar{x}_1(t)/dt) = (bA/(a+A) - d)\bar{x}_1(t) = \bar{r}_1\bar{x}_1(t), \\ \bar{x}_1(0^+) = x(0) = A, \end{cases} \tag{3.11}$$

and

$$x(\tau) \leq \bar{x}_1(\tau) = \overline{m}_1 = A\exp(\bar{r}_1\tau), \quad t \in [0, \tau].$$

When $A\exp(-d\tau)+p > A$, it is clear that $\overline{m}_1+p > A$. When $t \in (\tau, 2\tau]$, $x(t) \leq \bar{x}_2(t)$, where $\bar{x}_2(t)$ satisfies

$$\begin{cases} (d\bar{x}_2(t)/dt) = (b(\overline{m}_1+p)/(a+\overline{m}_1+p) - d)\bar{x}_2(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \bar{r}_2\bar{x}_2(t), \\ \bar{x}_2(\tau^+) = \overline{m}_1 + p, \end{cases} \tag{3.12}$$

and

$$x(2\tau) \leq \bar{x}_2(2\tau) = \overline{m}_2 = (\overline{m}_1+p)\exp(\bar{r}_2\tau), \quad t \in (\tau, 2\tau].$$

Since $\bar{r}_2 > \bar{r}_1$ and $A\exp(-d\tau)+p > A$, we have $\overline{m}_2+p > \overline{m}_1+p > A$. Similarly, we can obtain that when $t \in ((n-1)\tau, n\tau]$, then $x(t) \leq \bar{x}_n(t)$, where $\bar{x}_n(t)$ satisfies

$$\begin{cases} (d\bar{x}_n(t)/dt) = (b(\overline{m}_{n-1}+p)/(a+\overline{m}_{n-1}+p) - d)\bar{x}_n(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \bar{r}_n\bar{x}_n(t), \\ \bar{x}_n((n-1)\tau^+) = \overline{m}_{n-1} + p, \end{cases} \tag{3.13}$$

and

$$x(n\tau) \leq \bar{x}_2(n\tau) = \overline{m}_n = (\overline{m}_{n-1}+p)\exp(\bar{r}_n\tau), \ t \in ((n-1)\tau, n\tau].$$

Since $0 > \bar{r}_n > \cdots > \bar{r}_2 > \bar{r}_1$ and $A\exp(-d\tau)+p > A$, we have $\overline{m}_n + p > \cdots > \overline{m}_2 + p > \overline{m}_1 + p > A$ and $x(n\tau) \leq \bar{x}_n(n\tau) = \overline{m}_n$.

$$\begin{aligned} \overline{m}_n &= (\overline{m}_{n-1}+p)\exp(\bar{r}_n\tau) = m_{n-1}\exp(\bar{r}_n\tau) + p\exp(\bar{r}_n\tau) \\ &= \cdots \\ &= \overline{m}_1\exp((\bar{r}_2+\bar{r}_3+\cdots+\bar{r}_n)\tau) + p\exp((\bar{r}_2+\bar{r}_3+\cdots+\bar{r}_n)\tau) \\ &\quad + p\exp((\bar{r}_3+\cdots+\bar{r}_n)\tau) + \cdots + p\exp(\bar{r}_n\tau) \\ &\leq \overline{m}_1\exp((n-1)\bar{r}_n\tau) + p\exp((n-1)\bar{r}_n\tau) + p\exp((n-2)\bar{r}_n\tau) \\ &\quad + \cdots + p\exp(\bar{r}_n\tau). \end{aligned}$$

If $T = n\tau$, when the conditions of the theorem hold, we have

$$x(T) + p \leq \overline{m}_1 + np < h. \tag{3.14}$$

Then (2.1) has no solution.

If $T > n\tau$, then $x(t) \leq \overline{x}_{n+1}(t)$, $t \in (n\tau, T]$, and $\overline{x}_{n+1}(t)$ satisfies

$$\begin{cases} (d\bar{x}_{n+1}(t)/dt) = (b(\overline{m}_n + p)/(a + \overline{m}_n + p) - d)\bar{x}_{n+1} = \bar{r}_{n+1}\bar{x}_{n+1}, \\ \bar{x}_{n+1}(n\tau^+) = \overline{m}_n + p, \end{cases}$$

and then

$$x(T) \leq \bar{x}_{n+1}(T) = (\overline{m}_n + p)\exp(\bar{r}_{n+1}\tau).$$

Furthermore, when the conditions of the theorem hold, we have

$$x(T) + p \leq \overline{m}_1 + (n+1)p < h.$$

The proof is completed. □

Remark 3.2. When other parameters except p are given firstly, from (3.9) and (3.10) we can also estimate another population of impulsive release:

$$p < p_2 = \frac{h - \overline{m}_1}{n}, \quad T = n\tau; \tag{3.15}$$

$$p < p_2' = \frac{h - \overline{m}_1}{n+1}, \quad T > n\tau; \tag{3.16}$$

(3.15) (or 3.16) implies $x(T) + p < h$ when $p < p_2$ (or $p < p_2'$), that is to say, after the given moment T, the density of the species will be less than h if the density of impulsive release p is less than p_2 (or p_2'). Furthermore, our aims cannot be achieved.

Remark 3.3. From Theorems 3.1 and 3.2 and Remarks 3.1 and 3.2, we know that if other parameters are given, there must exist a p_0 ($p_2 \leq p_0 \leq p_1$) such that $x(T) + p > h$ when the population of impulsive release $p \geq p_0$ and $T = n\tau$. Similarly, there must exist a p_0' ($p_2' \leq p_0' \leq p_1'$) such that $x(T) + p > h$ when the population of impulsive release $p \geq p_0'$ and $T > n\tau$. Therefore, we can estimate the population of impulsive release from Theorems 3.1 and 3.2.

3.2. The existence of solutions of (2.2). In the following, we discuss the cases of $x(0) > h$ and $x(0) = C$ at which the species might be harmful for either the environment or humans. It is a requirement that the density should be controlled to a lower level in a given time T_1 because it will have no practical meaning if the population is controlled in infinite time. On the other hand, in order not to make the density below h again when $t \in [0, T_1]$, we restrict $E < 1 - h/B$.

Theorem 3.3. *Let* $\underline{s} = b(1-E)B/(a + (1-E)B) - d$, $\underline{M}_1 = C\exp(\underline{s}\tau)$, $E < 1 - h/B$, $b > d$.

1) *Suppose* $T_1 = n\tau_1$, *if* $(1-E)\underline{M}_1 < C$ *and*

$$(1-E)^{n-1}C\exp(\underline{s}T_1) > B. \tag{3.17}$$

Then (2.2) *has no solution which satisfies the boundary value problem.*

2) *Suppose* $T_1 > n\tau_1$. *If* $(1-E)\underline{M}_1 < C$ *and*

$$(1-E)^n C\exp(\underline{s}T_1) > B, \tag{3.18}$$

then (2.2) *has no solution which satisfies the boundary value problem.*

Proof. Similarly, (2.2) also has a series of lower solutions $\underline{y}_i(t)$ when $t \in ((i-1)\tau_1, i\tau_1]$, $i = 1, 2, \dots, n$. For simplification, we denote $b(1-E)\underline{M}_{i-1}/(a + (1-E)\underline{M}_{i-1}) - d$ by $\underline{s}_i$, $i = 2, \dots, n$.

When $t \in [0, \tau_1]$, $x(t) \geq \underline{y}_1(t)$, where $\underline{y}_1(t)$ satisfies

$$\begin{cases} (d\underline{y}_1(t)/dt) = (bC/(a+C) - d)\underline{y}_1(t) = \underline{s}_1\underline{y}_1(t), \\ \underline{y}_1(0^+) = x(0) = C, \end{cases} \tag{3.19}$$

and

$$x(\tau_1) \geq \underline{y}_1(\tau_1) = \underline{M}_1 = C\exp(\underline{s}_1\tau_1), \quad t \in [0, \tau_1].$$

Since $(1-E)\underline{M}_1 < C$, then when $t \in (\tau_1, 2\tau_1]$, $x(t) \geq \underline{y}_2(t)$, where $\underline{y}_2(t)$ satisfies

$$\begin{cases} (d\underline{y}_2(t)/dt) = (b(1-E)\underline{M}_1/(a + (1-E)\underline{M}_1) - d)\underline{y}_2(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \underline{s}_2\underline{y}_2(t), \\ \underline{y}_2(\tau_1^+) = (1-E)\underline{M}_1, \end{cases} \tag{3.20}$$

and

$$x(2\tau_1) \geq \underline{y}_2(2\tau_1) = \underline{M}_2 = (1-E)\underline{M}_1 \exp(\underline{s}_2\tau_1), \quad t \in (\tau_1, 2\tau_1].$$

Since $(1-E)\underline{M}_1 < C$, then $(1-E)\underline{M}_2 < (1-E)\underline{M}_1 < C$ and $\underline{s}_2 < \underline{s}_1$. Similarly, we can obtain that when $t \in ((n-1)\tau_1, n\tau_1]$, then $x(t) \geq \underline{y}_n(t)$, where $\underline{y}_n(t)$ satisfies

$$(3.21)\quad \begin{cases} (d\underline{y}_n(t)/dt) = (b(1-E)\underline{M}_{n-1}/(a+(1-E)\underline{M}_{n-1}) - d)\underline{y}_n(t) \\ \qquad\qquad = \underline{s}_n\underline{y}_n(t), \\ \underline{y}_n((n-1)\tau_1^+) = (1-E)\underline{M}_{n-1}, \end{cases}$$

and

$$x(n\tau_1) \geq \underline{y}_2(n\tau_1) = \underline{M}_n = (1-E)\underline{M}_{n-1} \exp(\underline{s}_n\tau_1),\ t \in ((n-1)\tau_1, n\tau_1].$$

Since $(1-E)\underline{M}_1 < C$, then $\underline{s}_n < \cdots < \underline{s}_2 < \underline{s}_1$, $(1-E)\underline{M}_n < \cdots < (1-E)\underline{M}_2 < (1-E)\underline{M}_1 < C$ and $x(n\tau_1) \geq \underline{y}_n(n\tau_1) = \underline{M}_n$.

$$\begin{aligned} \underline{M}_n &= (1-E)\underline{M}_{n-1} \exp(\underline{s}_n\tau_1) \\ &= (1-E)^{n-1}\underline{M}_1 \exp(\underline{s}_2 + \underline{s}_3 + \cdots + \underline{s}_n)\tau_1) \\ &\geq (1-E)^{n-1}\underline{M}_1 \exp((n-1)\underline{s}_n\tau_1) \\ &\geq (1-E)^{n-1}\underline{M}_1 \exp((n-1)\underline{s}\tau_1). \end{aligned}$$

If $T_1 = n\tau_1$, when the conditions of the theorem hold, we have

$$(3.22)\qquad x(T) \geq (1-E)^{n-1}\underline{M}_1 \exp((n-1)\underline{s}\tau_1) > B.$$

Then (2.2) has no solution.

If $T_1 > n\tau_1$, then $x(t) \geq \underline{y}_{n+1}(t)$, $t \in (n\tau_1, T_1]$, and $\underline{y}_{n+1}(t)$ satisfies

$$\begin{cases} (d\underline{y}_{n+1}(t)/dt) = (b(1-E)\underline{M}_n/(a+(1-E))\underline{M}_n) - d)\underline{y}_{n+1} \\ \qquad\qquad = \underline{s}_{n+1}\underline{y}_{n+1}, \\ \underline{y}_{n+1}(n\tau_1^+) = (1-E)\underline{M}_n, \end{cases}$$

and then

$$x(T_1) \geq \underline{y}_{n+1}(T_1) = (1-E)\underline{M}_n \exp(\underline{s}_{n+1}(T - n\tau_1)).$$

Furthermore, when the conditions of the theorem hold, we have

$$x(T_1) \geq (1-E)^n C \exp(\underline{s} T_1) > B.$$

The proof is completed. □

Remark 3.4. When other parameters except n are given firstly, from (3.17) and (3.18) we can obtain the times of impulsive release:

$$n < n_1 = \left[\frac{\underline{s} T_1 - \ln(B/C)}{\ln(1/1-E)}\right] + 2, \quad T_1 = n\tau_1; \tag{3.23}$$

$$n < n_1' = \left[\frac{\underline{s} T_1 - \ln(B/C)}{\ln(1/1-E)}\right] + 1, \quad T_1 > n\tau_1. \tag{3.24}$$

where $[\cdot]$ represents the maximum integer which isn't larger than "$\cdot$." Equation (3.23) (or (3.24)) implies $x(T) > B$ when $n < n_1$ (or $n < n_1'$)), that is to say, at the given moment T_1, the density of the species will be larger than B if the impulsive times n is less than n_1 (or n_1'). Here, we should notice that the period of impulsive control τ_1 varies with n since $T_1 = n\tau_1$ (or $T_1 > n\tau_1$) (here, we only give the estimation of n).

Similar to the discussion above, we have the following theorem:

Theorem 3.4. *Let* $M = Ce^{(b-d)\tau_1}$, $\overline{s}_1 = bM/(a+M) - d$, $b > d$, $\overline{M}_1 = C\exp(\overline{s}_1\tau_1)$ *and* $E < 1 - h/B$.

1) *Suppose* $T_1 = n\tau_1$. *If* $(1-E)\overline{M}_1 < C$ *and*

$$(1-E)^{n-1} C \exp(\overline{s}_1 T_1) \leq B, \tag{3.25}$$

then (2.2) *has a solution which satisfies the boundary value problem.*

2) *Suppose* $T_1 > n\tau_1$. *If* $(1-E)\overline{M}_1 < C$ *and*

$$(1-E)^{n} C \exp(\overline{s}_1 T_1) \leq B, \tag{3.26}$$

then (2.2) *has a solution which satisfies the boundary value problem.*

Proof. There exist a series of upper solutions $\overline{y}_i(t)$ of (2.2) when $t \in ((i-1)\tau_1, i\tau_1]$, $i = 1, 2, \ldots, n$. For simplification, we denote

$b\overline{M}_{i-1}/(a+\overline{M}_{i-1})-d$ by $\overline{s}_i$, $i=2,\dots,n$. Since $bx/(a+x)<b$, then $x(t)<y(t)$, $\in[0,\tau_1]$, where $y(t)$ satisfies

$$\begin{cases}(dy(t)/dt)=(b-d)y(t),\\ y(0^+)=x(0)=C,\end{cases}$$

and $x(t)\le y(t)\le M=Ce^{(b-d)\tau_1}$.

Therefore, when $t\in[0,\tau_1]$ and $x(t)\le\overline{y}_1(t)$, where $\overline{y}_1(t)$ satisfies

$$\begin{cases}(d\overline{y}_1(t)/dt)=(bM/(a+M)-d)\overline{y}_1(t)=\overline{s}_1\overline{y}_1(t),\\ \overline{y}_1(0^+)=x(0)=C,\end{cases}\tag{3.27}$$

and

$$x(\tau_1)\le\overline{y}_1(\tau_1)=\overline{M}_1=C\exp(\overline{s}_1\tau_1),\quad t\in[0,\tau_1].$$

When $t\in(\tau_1,2\tau_1]$ and $x(t)\le\overline{y}_2(t)$, where $\overline{y}_2(t)$ satisfies

$$\begin{cases}(d\overline{y}_2(t)/dt)=(b\overline{M}_1/(a+\overline{M}_1)-d)\overline{y}_2(t)=\overline{s}_2\overline{y}_2(t),\\ \overline{y}_2(\tau_1^+)=(1-E)\overline{M}_1,\end{cases}\tag{3.28}$$

and

$$x(2\tau_1)\le\overline{y}_2(2\tau_1)=\overline{M}_2=(1-E)\overline{M}_1\exp(\overline{s}_2\tau_1),\quad t\in(\tau_1,2\tau_1].$$

Since $\overline{s}_2<\overline{s}_1$ and $(1-E)\overline{M}_1<C$, we have $(1-E)\overline{M}_2<(1-E)\overline{M}_1<C$. Similarly, we can obtain that when $t\in((n-1)\tau_1,n\tau_1]$, then $x(t)\le\overline{y}_n(t)$, where $\overline{y}_n(t)$ satisfies

$$\begin{cases}(d\overline{y}_n(t)/dt)=(b\overline{M}_{n-1}/(a+\overline{M}_{n-1})-d)\overline{y}_n(t)=\overline{s}_n\overline{y}_n(t),\\ \overline{y}_n((n-1)\tau_1^+)=(1-E)\overline{M}_{n-1},\end{cases}\tag{3.29}$$

and

$$x(n\tau_1)\le\overline{y}_n(n\tau_1)=\overline{M}_n=(1-E)\overline{M}_{n-1}\exp(\overline{s}_n\tau_1),\ t\in((n-1)\tau_1,n\tau_1].$$

Since $\overline{s}_n<\cdots<\overline{s}_2<\overline{s}_1$ and $(1-E)\overline{M}_1<C$, we have $(1-E)\overline{M}_n<\cdots<(1-E)\overline{M}_2<(1-E)\overline{M}_1<C$ and $x(n\tau_1)\le\overline{y}_n(n\tau_1)=\overline{M}_n$.

$$\begin{aligned}\overline{M}_n&=(1-E)\overline{M}_{n-1}\exp(\overline{s}_n\tau_1)\\&=\cdots\\&=(1-E)^{n-1}\overline{M}_1\exp((\overline{s}_2+\overline{s}_3+\cdots+\overline{s}_n)\tau_1)\\&=(1-E)^{n-1}C\exp((\overline{s}_1+\overline{s}_2+\cdots+\overline{s}_n)\tau_1)\\&\le(1-E)^{n-1}C\exp(n\overline{s}_1\tau_1).\end{aligned}$$

If $T_1 = n\tau_1$, when the conditions of the theorem hold, we have

$$x(T) \le \overline{M}_n \le (1-E)^{n-1} C \exp(\overline{s}_1 T_1) \le B. \tag{3.30}$$

Then (2.2) has a solution.

If $T_1 > n\tau_1$, then $x(t) \le \overline{y}_{n+1}(t)$, $t \in (n\tau_1, T_1]$, and $\overline{y}_{n+1}(t)$ satisfies

$$\begin{cases} (d\overline{y}_{n+1}(t)/dt) = ((b\overline{M}_n/a + \overline{M}_n) - d)\overline{y}_{n+1} = \overline{s}_{n+1}\overline{y}_{n+1}, \\ \overline{y}_{n+1}(n\tau_1^+) = (1-E)\overline{M}_n, \end{cases}$$

and then

$$x(T_1) \le \overline{y}_{n+1}(T_1) = (1-E)\overline{M}_n \exp(\overline{s}_{n+1}(T_1 - n\tau_1)).$$

Furthermore,

$$x(T_1) \le (1-E)^n C \exp(\overline{s}_1 T_1) \le B.$$

The proof is completed. $\square$

Remark 3.5. When other parameters except n are given firstly, from (3.25) and (3.26) we can obtain another time of impulsive release:

$$n \ge n_2 = \left[\frac{\overline{s}_1 T_1 - \ln(B/C)}{\ln(1/1-E)}\right] + 2, \quad T_1 = n\tau_1, \tag{3.31}$$

$$n \ge n_2' = \left[\frac{\overline{s}_1 T_1 - \ln(B/C)}{\ln(1/1-E)}\right] + 1, \quad T_1 > n\tau_1, \tag{3.32}$$

where $[\cdot]$ represents the maximum integer which isn't larger than "$\cdot$." Equation (3.31) (or (3.32)) implies $x(T_1) \le B$ when $n \ge n_2$ (or $n \ge n_2'$), that is to say, at the given moment T_1, the density of the species will be less than or equal to B if the impulsive times n is larger than or equal to n_2 (or n_2').

Remark 3.6. From Theorems 3.3 and 3.4 and Remarks 3.4 and 3.5, we can know that if other parameters are given, there exists an n_0 ($n_1 \le n_0 \le n_2$) such that $x(T_1) \le B$ when the times of impulsive release $n \ge n_0$ and $T_1 = n\tau_1$. Similarly, there exists an n_0' ($n_1' \le n_0' \le n_2'$) such that $x(T_1) \le B$ when the times of impulsive release $n \ge n_0'$ and $T_1 > n\tau_1$.

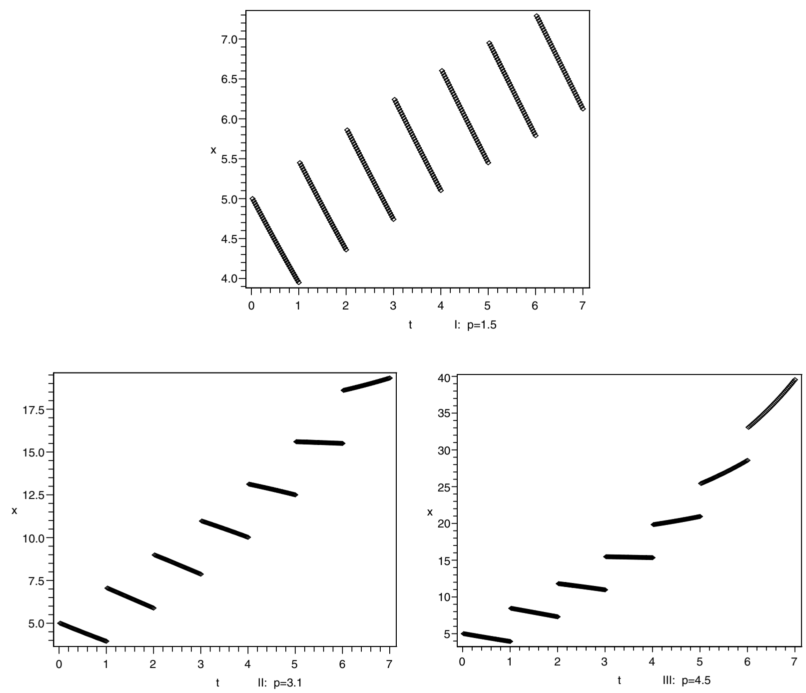

FIGURE 1. Time series of (2.1) when $b = 0.9$, $T = 6$, $a = 20$, $A = 5$, $d = 0.4$ and $\tau = 1$. I: $p = 1.5 < p_2$; II: $p_2 < p = 3.1 < p_1$; III. $p = 4.5 > p_1$.

4. Discussion. Two kinds of time-limited management strategies of a single-species with Allee effect, described by the initial and boundary problem of impulsive differential equation, are discussed by means of the comparison principle and the methods of upper and lower solution in this paper. Theorem 3.1 gives sufficient conditions for the existence of the solution of (2.1) when the initial density is less than h, which implies that the goal of management can be achieved and the species can be restored. When the conditions of Theorem 3.2 hold, we cannot achieve our aims. From Remarks 3.1, 3.2 and 3.3, we can estimate the population of impulsive release. For example, we take $b = 0.9$, $T = 6$, $a = 20$, $A = 5$, $d = 0.4$ and $\tau = 1$, then $h = 16$, $p_1 = 4.471182708$ and $p_2 = 1.712486573$. Figure 1 (I, II and III) shows the times series of (2.1). There must exist a p_0 ($p_2 \leq p_0 \leq p_1$) such that $x(T) + p > h$ when $p \geq p_0$.

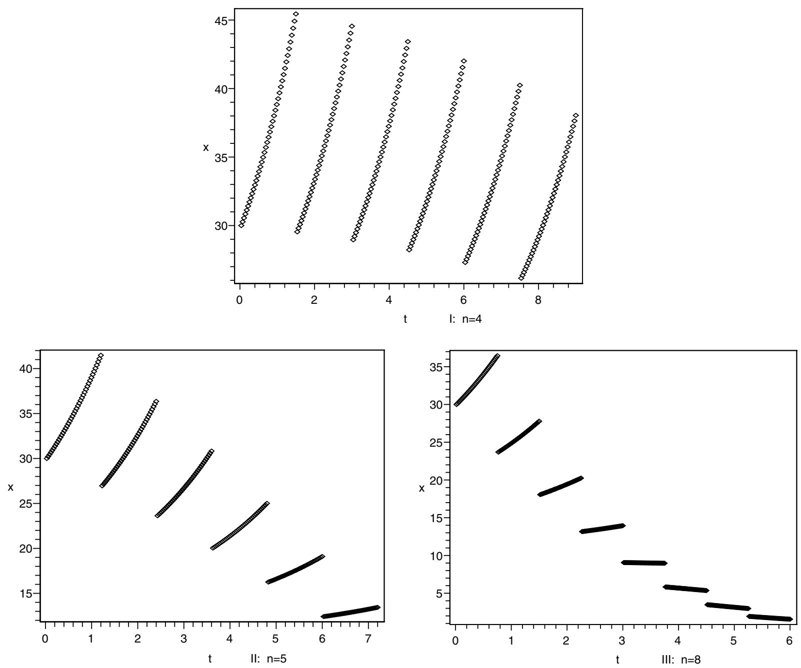

FIGURE 2. Time series of (2.2) when $b = 0.9$, $T_1 = 6$, $a = 12$, $C = 30$, $d = 0.4$, $E = 0.35$ and $B = 15$. I: $n = 4$, II: $n = 5$, III: $n = 8$.

Theorem 3.3 shows that the density of the species cannot be controlled if the conditions of Theorem 3.3 hold. Theorem 3.4 gives sufficient conditions for the existence of the solution of (2.2) when the initial density is larger than h. From Remarks 3.4, 3.5 and 3.6, we can also estimate the times of impulsive harvesting. For example, we take $b = 0.9$, $T_1 = 6$, $a = 12$, $C = 30$, $d = 0.4$, $E = 0.35$ and $B = 15$, then $h = 9.6$, $n_1 = 4$ and $n_2 = 8$. Numerical simulations can be seen in Figure 2 (I, II, III) which shows the time series of (2.2). There must exist an n_0 ($n_1 \leq n_0 \leq n_2$) such that $x(T) \leq B$ when $n \geq n_0$.

The results obtained here can be verified by corresponding numerical simulations (Figures 1, 2). We still have a lot of problems on the subject which should be explained in mathematical terms. For example, how can the species be restored if people cannot be provided with enough species' population to release? Does a periodic solution exist for long-term management, etc.? These will be the subjects of our future work.

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