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ON MAXIMUM MODULUS POINTS AND ZERO SETS OF ENTIRE FUNCTIONS OF REGULAR GROWTH

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ABSTRACT. Let f be an entire function. We denote by $R(w,f)$ the distance between a maximum modulus point w and the zero set of f. In a previous paper, the authors obtained asymptotical lower bounds for $R(w,f)$ as $|w|\to\infty$ for functions of finite positive order and regular growth. In this work we extend those results to functions of either zero or infinite order and show that our results are sharp in sense of order.

1. Introduction. Let f be an entire function. We call a point $w\in \mathbf{C}$ a *maximum modulus point* if

$$|f(w)| = M(|w|,f),$$

where

$$M(r,f) := \max_{|z|=r} |f(z)|.$$

We denote by $R(w,f)$ the distance between a maximum modulus point w and zero set of f, i.e.,

$$R(w,f) := \inf\{|w-z| : f(z)=0\}.$$

Lower estimates of $R(w,f)$ play an important role in Macintyre's version [**7**] of the Wiman-Valiron theory and its further generalizations, see [**4**, Chapter 1, Section 4] and [**11**].

Theorem A [**7**]. (i) *The following inequality holds*

$$\limsup_{|w|\to\infty} \frac{R(w,f)}{|w|}(\log M(|w|,f))^{1/2} > 0.$$

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(ii) *For each* $\varepsilon > 0$

$$\liminf_{\substack{|w|\to\infty \\ |w|\notin A_\varepsilon}} \frac{R(w,f)}{|w|}(\log M(|w|,f))^{(1/2)+\varepsilon} > 0,$$

where $\int_{A_\varepsilon} d\log t < \infty$.

The results of Macintyre are in terms of $\log M(r,f)$, and they are valid either for a sequence of values of $|w|$ or outside of an exceptional set. In [**8**] the authors considered functions of finite and positive order and obtained lower estimates for $R(w,f)$ valid for *all sufficiently large values of* w. The results are in terms of some smooth majorant of $\log M(r,f)$ and up to a constant factor unimprovable.

It is well known that, in almost all applications of the entire functions theory, see e.g., [**6**], one deals with functions of regular growth. It is worth mentioning that lower estimates of $R(w,f)$ for some special families of regularly growing entire functions of exponential type (so-called "grand partition sums") are important for the theory of phase transitions, see e.g., [**10**].

We proved in [**8**] that, for f of regular growth with respect to the majorant, the decay of $R(w,f)$ as $w\to\infty$ is slower than in the general case, and the slower the more regular the growth is. Nevertheless, the question about sharpness of the obtained results remained open. Moreover, in [**8**], we had restricted ourselves by functions of finite positive order. Functions of either infinite or zero order were considered later in [**12**] but without assumption of regular growth.

In this paper we extend the results of [**8**] related to regularly growing functions to the case of either infinite or zero order and also show that they can be considered as sharp in some sense.

Our results are stated in terms of proximate orders, therefore we need the related preliminaries.

2. Preliminaries. According to the classical definition of Valiron, see e.g., [**6**, page 32], a *proximate order* (*p.o.*) is a positive function $\rho(r)\in C^1(\mathbf{R}_+)$ satisfying conditions:

(i) there exists a finite limit $\rho = \lim_{r\to\infty}\rho(r)$;

(ii) $\rho'(r)r\log r\to 0$ as $r\to\infty$.

Further, we denote

$$V(r) = r^{\rho(r)}. \tag{1}$$

Valiron's theorem, see [**6**, page 35] shows that p.o.s form a scale of growth of entire functions of finite order in the following sense: For each function f of finite order ρ, there exists a p.o. $\rho(r) \to \rho$ such that

$$0 < \sigma := \limsup_{r\to\infty} \frac{\log M(r,f)}{V(r)} < \infty. \tag{2}$$

If (2) holds for f, then we shall write $f \in [\rho(r), \sigma]$.

For our purposes we need p.o.s having more regular asymptotic behavior than is required by Valiron's definition; nevertheless, they also form a scale of growth of entire functions of finite order. Moreover, we need analogues of p.o.s forming a scale of growth of entire functions of infinite order in the same sense as in Valiron's theorem. The smooth majorants mentioned above are functions V of the form (1).

For functions of finite order we use a subclass of the class of strong p.o.s introduced by Levin [**6**, pages 39–41]. According to Levin's definition, a *strong p.o.* is a function $\rho(r) \in C^2(\mathbf{R}_+)$ representable in the form

$$\rho(r) = \rho + \frac{\vartheta_1(\log r) - \vartheta_2(\log r)}{\log r}, \quad r \ge r_0 > 1, \tag{3}$$

where $0 < \rho < \infty$, and ϑ_j are *concave* functions of $C^2(\mathbf{R}_+)$ satisfying conditions, $(j = 1, 2)$:

$$\begin{gathered} \lim_{x\to\infty} \vartheta_j(x) = \infty; \qquad \lim_{x\to\infty} \vartheta_j(x)/x = 0; \\ \vartheta_j''(x) = o(\vartheta_j'(x)), \quad x \to \infty. \end{gathered} \tag{4}$$

For $0 < \rho < \infty$, we denote by $\mathcal{A}_\rho$ the class of all strong p.o.s $\rho(r)$ satisfying the additional condition

$$\lim_{x\to\infty} \vartheta_j'''(x) = 0. \tag{5}$$

For $\rho = 0$, we introduce more restrictions. Denote by $\mathcal{A}_0$ the class of all strong p.o.s representable by

$$\rho(r) = \frac{\vartheta(\log r)}{\log r}, \quad r \ge r_0 > 1, \tag{6}$$

with ϑ satisfying conditions:

$$\lim_{x\to\infty} x^{-1}e^{\vartheta(x)} = \infty, \qquad \lim_{x\to\infty} x^{-1}\vartheta(x) = 0; \tag{7}$$

$$\vartheta''(x) + (\vartheta'(x))^2 > 0, \quad x \geq x_0 > 0; \tag{8}$$

$$\vartheta'''(x) = O\left([\vartheta'(x)]^3\right), \quad x \to \infty. \tag{9}$$

We introduce (8) because in the case of zero order Levin's condition (4) is not sufficient for convexity of e^{ϑ}.

For studying entire functions f of zero order but not of too slow growth, namely such that

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{\log^2 r} = \infty, \tag{10}$$

we will also use the subclass $\mathcal{A}_0^*$ of $\mathcal{A}_0$ consisting of $\rho(r)$ satisfying additional conditions:

$$\begin{gathered} 2\vartheta''(x) + (\vartheta'(x))^2 > 0, \quad x \geq x_0 > 0; \\ \limsup_{x\to\infty} x^{-2}e^{\vartheta(x)} = \infty. \end{gathered} \tag{11}$$

Using arguments close to Levin's proof [**6**, pages 39–41] for strong p.o.s, it can be readily shown that the class $\mathcal{A}_\rho$, for $0 \leq \rho < \infty$, forms a scale of growth of entire functions of order ρ in the same sense as in Valiron's theorem. For $\mathcal{A}_0^*$ this is true for functions of zero order satisfying (10), see Lemma 5.1 below.

For the case of infinite order the following definition of p.o. naturally arises from results of Earl and Hayman [**2**]. A function $\rho(r) \in C^3(\mathbf{R}_+)$ is called an *infinite p.o.* if it is representable in the form (6), with ϑ a positive *convex* function satisfying conditions:

$$\lim_{x\to\infty} \vartheta'(x) = \infty; \quad \vartheta^{(j)}(x) = o\big([\vartheta'(x)]^j\big), \quad x \to \infty, \quad j = 2, 3. \tag{12}$$

We denote by $\mathcal{A}_\infty$ the class of all infinite p.o.s.

The main result of [**2**] implies that $\mathcal{A}_\infty$ forms a scale of growth of entire functions of infinite order in the same sense as in Valiron's theorem.

Definition 2.1. We say that a p.o. $\rho(r)$ is *admissible* if

$$\rho(r) \in \bigcup_{0\le\rho\le\infty} \mathcal{A}_\rho. \tag{13}$$

We say that a p.o. $\rho(r)$ is *strongly admissible* if it is admissible and, moreover,

$$\rho(r) \notin \mathcal{A}_0 \setminus \mathcal{A}_0^*. \tag{14}$$

Note that if a p.o. $\rho(r)$ is admissible, then function V defined by (1) is convex in $\log r$. Note also that admissible but not strongly admissible p.o.s, i.e., p.o.s belonging to $\mathcal{A}_0 \setminus \mathcal{A}_0^*$, form a rather small class connected with very slowly growing entire functions which do not satisfy condition (10).

3. Statement of results. The following theorem was proved in [**8**] when $0 < \rho < \infty$ and in [**12**] when $\rho = 0$ or $\rho = \infty$.

Theorem B. *Let $\rho(r)$ be an admissible p.o.*

(i) *If $f \in [\rho(r), \sigma]$, then*

$$\liminf_{|w|\to\infty} R(w,f)V'(|w|) \ge \frac{1}{e^2\sigma}.$$

(ii) *There exists $f \in [\rho(r), \sigma]$ such that*

$$\liminf_{|w|\to\infty} R(w,f)V'(|w|) \le \frac{\pi}{\sigma}.$$

The bound in Theorem B (i) is worse than Macintyre's bounds but it is valid for *all* sufficiently large values of w. In addition, part (ii) shows that this bound is sharp up to a constant factor and Macintyre's bounds cannot be valid without exceptional sets.

As it was mentioned in the introduction, in almost all applications of the entire function theory, one deals with functions of regular growth, i.e., functions for which the following limit exists:

$$\sigma = \lim_{r\to\infty} \frac{\log M(r,f)}{V(r)}.$$

Therefore, the problem of lower estimates of $R(w, f)$ for such functions f arises. For functions of finite and positive order, this problem was considered in [**8**].

Theorem C [**8**]. *Let $f \in [\rho(r), \sigma]$, $0 < \rho < \infty$, and let V be defined by* (1). *Assume that*

$$\log M(r, f) = \sigma\, V(r) + O(\theta(r)), \quad r \to \infty,$$

where $\theta(r) > 0$ is a nondecreasing function such that

(i) $\theta(r) = o(V(r))$, $r \to \infty$,

(ii) $\theta(2r) = O(\theta(r))$, $r \to \infty$.

Then

$$\liminf_{|w|\to\infty} \frac{R(w, f)}{|w|} \sqrt{\theta(|w|)V(|w|)} > 0. \tag{15}$$

Taking into account that, for $0 < \rho < \infty$,

$$V(r) = e^{\rho \log r + \vartheta_1(\log r) - \vartheta_2(\log r)}, \qquad rV'(r) = (\rho + o(1))V(r), \tag{16}$$

it can be readily seen that Theorem C gives a better estimate than Theorem B (i). Moreover, bound (15) depends on θ and the smaller θ is, the better the bound is. In particular, if $\theta(r) = O(1)$, $r \to \infty$, then the bound (15) is just Macintyre's bound in Theorem A (i) with $\limsup$ replaced by $\liminf$ and $\log M(|w|, f)$ replaced by $V(|w|)$. So, generally speaking, for functions of "very regular growth" Macintyre's bound is valid without any exceptional set.

We obtain here more general results dealing not only with functions of finite positive order, but also with functions of either zero or infinite order. To state our results we need the following definition.

Definition 3.1. We say that an entire function $f \in [\rho(r), \sigma]$ is a function of (V, θ)-*regular growth* if

$$\log M(r, f) = \sigma V(r) + O(\theta(r)), \tag{17}$$

where θ is a positive nondecreasing function on $\mathbf{R}_+$ satisfying conditions:

$$(i)\ \theta(r) = o(V(r)), \quad r \to \infty, \tag{18}$$

$$(ii)\ \theta\left(r \exp\{V(r)/(rV'(r))\}\right) = O(\theta(r)), \quad r \to \infty. \tag{19}$$

For each strongly admissible p.o. $\rho(r)$ and each θ satisfying (18)–(19), there exist functions of (V, θ)-regular growth. This is implied by the following result of Clunie and Kövari [**1**, Theorem 4, page 19].

For any function φ representable in the form

$$\varphi(r) = \int_1^r \psi(t)\, d\log t, \quad r \geq r_0 > 1, \tag{20}$$

where ψ is a positive increasing function satisfying condition

$$\psi(cr) - \psi(r) \geq 1, \quad \text{for some } c > 1, \text{ and for all} \quad r \geq r_0 > 1, \tag{21}$$

there exists an entire function f such that

$$\log M(r, f) = \varphi(r) + O(1), \quad r \to \infty. \tag{22}$$

It can readily be shown that, if $\rho(r)$ is a strongly admissible p.o., then condition (20)–(21) is satisfied by $\varphi(r) = V(r)$.

If $\rho(r)$ is admissible but not strongly, i.e., belongs to $\mathcal{A}_0 \backslash \mathcal{A}_0^*$, then (22) with $\varphi(r) = V(r)$ is not valid in general, see Theorem 3.4 below. Meanwhile, another result of [**1**, Theorem 2, page 13] implies existence of entire f of (V, θ)-regular growth under additional assumption $\theta(r) \geq 1/3 \log r$, $r \geq r_0 > 1$. In this case, f does not satisfy the condition (10), i.e., it is of very slow growth.

Our main result is the following theorem.

Theorem 3.1. *Let $\rho(r)$ be an admissible p.o., and let V be defined by* (1)*. If f is of (V, θ)-regular growth. Then, for all sufficiently large values of $|w|$, the following inequality holds*

$$\frac{R(w, f)}{|w|} \geq 1 - \exp\left\{-\frac{C}{|w|V'(|w|)}\sqrt{\frac{V(|w|)}{\theta(|w|)}}\right\}, \tag{23}$$

where C is a positive constant.

The following corollary is immediate.

Corollary 3.2. *If the conditions of Theorem* 3.1 *are satisfied and, moreover,*

$$\liminf_{r\to\infty} rV'(r)\sqrt{\frac{\theta(r)}{V(r)}} > 0, \tag{24}$$

then the following inequality holds

$$\liminf_{|w|\to\infty} R(w,f)V'(|w|)\sqrt{\frac{\theta(|w|)}{V(|w|)}} > 0. \tag{25}$$

We note that if $\rho(r)$ is strongly admissible then (24) holds for $\theta \equiv 1$ and hence for any nondecreasing positive θ. This is obvious when $\rho \neq 0$. When $\rho = 0$, condition (11) implies that $e^{\vartheta/2}$ is convex and $e^{\vartheta(x_n)/2}/x_n \to \infty$ for some sequence $x_n \to \infty$. Since $V(r) = e^{\vartheta(\log r)}$, we have

$$\frac{rV'(r)}{\sqrt{V(r)}} = 2r\left(e^{\vartheta(\log r)/2}\right)' \longrightarrow \infty, \quad \text{as} \quad r \to \infty.$$

Therefore Corollary 3.2 implies

Corollary 3.3. *If the conditions of Theorem* 3.1 *are satisfied and, moreover, $\rho(r)$ is a strongly admissible p.o., then* (25) *holds.*

In the case when f is of finite and positive order condition (19) can be written as $\theta(2r) = O(\theta(r))$, $r \to \infty$, and using (16) the inequality (25) can be written as (15). Thus, Theorem C is contained in Corollary 3.3.

We conjecture that admissibility of $\rho(r)$ implies (24) and therefore Corollary 3.3 remains true for all admissible (even nonstrongly) p.o.s $\rho(r)$. The reason is that entire functions of very slow growth cannot be "of very regular growth" in the sense that the function θ in (17) has growth restrictions from below. This shows the following result which is of independent interest.

Theorem 3.4. *Let $\rho(r)$ be an admissible p.o., and let f be an entire function satisfying condition*

$$\log M(r,f) = o(\log^2 r), \quad r \to \infty. \tag{26}$$

If f is of (V,θ)-regular growth, then

$$\limsup_{r\to\infty} rV'(r)\sqrt{\frac{\theta(r)}{V(r)}} > 0. \tag{27}$$

For example, if $V(r) = \log^\beta r$, $1 < \beta < 2$, then there is no entire function f of (V,θ)-regular growth with $\theta(r) = o(\log^{2-\beta} r)$, $r \to \infty$. For $\beta = 2$ there are functions of (V,θ)-regular growth with $\theta \equiv 1$, as the result of [**1**] mentioned above shows.

To consider the question whether the bound (25) is improvable or not, we need examples of entire functions f for which

(a) $|\log M(r,f) - \sigma V(r)|$ is relatively small,

(b) maximum modulus points of f are extremely close to its zero set.

For part (a) we can use results of Clunie and Kövari [**1**] mentioned above. Unfortunately, the method of these authors does not permit locating the positions of zeros and therefore provide (b).

Nevertheless, when $\theta(r)$ is not of very slow growth and has some special form, then we can prove that (25) is sharp.

Theorem 3.5. *Let $\rho(r)$ be a strongly admissible p.o., and let V be defined by* (1). *Given $1/3 \le \alpha < 1$, define*

$$\theta(r) = V(r)(rV'(r))^{\alpha-1}.$$

There exists an entire function f of (V,θ)-regular growth such that

$$\liminf_{|w|\to\infty} R(w,f)V'(|w|)\sqrt{\frac{\theta(|w|)}{V(|w|)}} \le \pi. \tag{28}$$

4. Proof of Theorem 3.1 (Case of infinite order).

Let $\rho(r)$ be a proximate order belonging to $\mathcal{A}_\infty$. We remind the reader that in this case $V(r) = r^{\rho(r)} = e^{\vartheta(\log r)}$, where $\vartheta(r)$ is a positive convex function satisfying condition (12). (We won't need the condition about the third derivative of ϑ, we will require it only when we prove Theorem 3.5.)

Note that since the function $r(g) := g\exp\{-1/\vartheta'(\log g)\}$ is an increasing function of g, there exists an inverse function such that

$$g(r) = r\exp\{1/\vartheta'(\log g(r))\}.$$

The following properties are easy to verify, see [**12**], and we omit their proofs:

$$g(r) > r \quad \text{and} \quad \lim_{r\to\infty} \frac{g(r)}{r} = 1, \tag{29}$$

$$\lim_{r\to\infty} \frac{\vartheta'(\log r)}{\vartheta'(\log g(r))} = 1, \tag{30}$$

$$\frac{V(g(r))}{V(r)} \leq e. \tag{31}$$

Lemma 4.1. *There exists a positive constant D such that if*

$$|R - r| \leq g(r) - r, \tag{32}$$

then

$$0 \leq V(R) - V(r) - (R-r)V'(r) \leq D\left(\frac{R}{r} - 1\right)^2 \vartheta'^2(\log r)V(r). \tag{33}$$

Proof. We assume $r \leq R$. Since by (12), $r^2V''(r) = \vartheta'^2(\log r)V(r)$ $(1 + o(1))$, for some c between r and R we have

$$\begin{aligned} V(R) - V(r) - (R-r)V'(r) &= \frac{1}{2}(R-r)^2V''(c) \\ &\leq \frac{1}{2}\left(\frac{R}{r} - 1\right)^2 \vartheta'^2(\log R)V(R)(1 + o(1)). \end{aligned}$$

Using (30) and (31), we obtain (33). When $r > R$, reasoning as before, we find

$$V(R) - V(r) - (R-r)V'(r) \leq \frac{1}{2}\left(\frac{R-r}{R}\right)^2 \vartheta'^2(\log r)V(r)(1+o(1)).$$

Because of (29), $R/r \to 1$ as $r \to \infty$. Thus, we obtain (33). □

We define the function $K(r,f)$ as

$$K(r,f) := r\frac{d}{dr}\log M(r,f) = \frac{d}{d\log r}\log M(r,f),$$

where, for definiteness, we take the right derivative. Note that $K(r,f)$ is nonnegative and nondecreasing.

Lemma 4.2. *If f is of (V,θ)-regular growth, then*

$$K(r,f) = \sigma\vartheta'(\log r)V(r) + O\left(\vartheta'(\log r)\sqrt{\theta(r)V(r)}\right), \quad r \to \infty.$$

Proof. Let

$$\delta(r) := \frac{1}{\vartheta'(\log g(r))}\sqrt{\frac{\theta(r)}{V(r)}}. \tag{34}$$

We set $R := r(1+\delta(r))$. By condition (19), for sufficiently large values of r and for some constant E_1, we have

$$\Big|[\log M(R,f) - \log M(r,f)] - [\sigma V(R) - \sigma V(r)]\Big| \leq E_1\theta(r). \tag{35}$$

Since

$$\log M(R,f) - \log M(r,f) = \int_r^R \frac{K(t,f)}{t}\,dt \geq K(r,f)\log\frac{R}{r},$$

we obtain

$$K(r,f) \leq \frac{1}{\log(R/r)}\left[\sigma V(R) - \sigma V(r) + E_1\theta(r)\right].$$

Because condition (32) is satisfied when r is sufficiently large, using Lemma 4.1 we find that

$$K(r,f) \leq \frac{1}{\log(1+\delta(r))}\Big\{\sigma\delta(r)\vartheta'(\log r)V(r) + \sigma C\delta^2(r)\vartheta'^2(\log r)V(r) + E_1\theta(r)\Big\}.$$

Thus,

(36)

$$\begin{aligned} K(r,f) &\leq \sigma\vartheta'(\log r)V(r) + O\left(\delta(r)\vartheta'^2(\log r)V(r)\right) + O\left(\theta(r)/\delta(r)\right) \\ &\overset{(30)}{=} \sigma\vartheta'(\log r)V(r) + O\left(\vartheta'(\log r)\sqrt{\theta(r)V(r)}\right), \quad r\to\infty. \end{aligned}$$

For the reverse inequality, we set

$$s := r(1-\delta(r)),$$

where $\delta(r)$ is defined in (34). We have

$$\log M(r,f) - \log M(s,f) = \int_s^r \frac{K(t,f)}{t}\,dt \leq K(r,f)\log\frac{r}{s}.$$

Using inequality similar to (35), for sufficiently large values of r and for some constant E_2, we find

$$K(r,f) \geq \frac{1}{\log(r/s)}[\sigma V(r) - \sigma V(s) - E_2\theta(r)].$$

Using Lemma 4.1 like previously, we obtain (36) with reversed inequality. □

We now prove Theorem 3.1 for $\rho(r) \in \mathcal{A}_\infty$. Let w be a maximum modulus point. We define, see [**7, 8**],

$$\Omega_w(z) := \frac{f(we^z)}{f(w)}e^{-K(|w|,f)z}, \tag{37}$$

and

$$P(h,w) := \max_{|z|\leq h}|\Omega_w(z)|. \tag{38}$$

If we set $|w| = r$ and $\operatorname{Re} z = t$, then

$$\log P(h,w) \le \max_{-h\le t\le h}\Big(\log M(re^t,f) - \log M(r,f) - K(r,f)t\Big). \tag{39}$$

We have

$$\begin{aligned}
&\log M(re^t,f) - \log M(r,f) - tK(r,f)\\
&= \int_r^{re^t} [K(u,f) - K(r,f)]\,\frac{du}{u}\\
&= \int_r^{re^t} [K(u,f) - \sigma\vartheta'(\log u)V(u)] - [K(r,f) - \sigma\vartheta'(\log r)V(r)]\,\frac{du}{u}\\
&\quad + \sigma\int_r^{re^t} [\vartheta'(\log u)V(u) - \vartheta'(\log r)V(r)]\,\frac{du}{u}.
\end{aligned}$$

Using Lemma 4.2, for some constant F_1 we obtain

(40)

$$\begin{aligned}
\log P(h,w) \le{}& \max_{-h\le t\le h}\left|\int_r^{re^t} F_1\vartheta'(\log u)\sqrt{V(u)\theta(u)}\,\frac{du}{u}\right|\\
&+ \max_{-h\le t\le h}\left|\int_r^{re^t} F_1\vartheta'(\log r)\sqrt{V(r)\theta(r)}\,\frac{du}{u}\right|\\
&+ \sigma\max_{-h\le t\le h}\left|V(re^t) - V(r) - t\vartheta'(\log r)V(r)\right|\\
=:{}& S_1 + S_2 + \sigma S_3.
\end{aligned}$$

We set

$$h = h_r = \frac{1}{\vartheta'(\log g(r))\sqrt{V(r)\theta(r)}}.$$

Let us estimate S_1, S_2 and S_3. Note that when r is large enough we have $re^{h_r} \le g(r)$. Hence, using (31) and (19) we obtain

$$S_1 \le F_1 h_r\vartheta'(\log re^{h_r})\sqrt{V(re^{h_r})\theta(re^{h_r})} = O(1), \quad r\to\infty.$$

Evidently,

$$S_2 = F_1 h_r\vartheta'(\log r)\sqrt{V(r)\theta(r)} = O(1), \quad r\to\infty.$$

Finally, since

$$S_3 = \max_{-h_r \le t \le h_r} \Big| V(re^t) - V(r) - (e^t - 1)\vartheta'(\log r)V(r) \\ + (e^t - t - 1)\vartheta'(\log r)V(r) \Big|,$$

using Lemma 4.1 we find

$$\begin{aligned} S_3 &\le \max_{-h_r \le t \le h_r} \big\{ D(e^t - 1)^2 \vartheta'^2(\log r)V(r) \\ &\qquad + |e^t - t - 1| \vartheta'(\log r)V(r) \big\} \\ &\le F_2 h_r^2 \vartheta'^2(\log r)V(r) = O(1), \quad r \to \infty. \end{aligned}$$

Hence,

$$\log P(h_r, w) = O(1), \quad r \to \infty. \tag{41}$$

The following lemma is implicitly contained in [**7**]. For the reader's convenience we repeat the proof here. Note that this lemma is applicable to an arbitrary entire function $f \not\equiv 0$, without any additional assumption.

Lemma 4.3. *Let $f \not\equiv 0$ be an entire function and let w be its arbitrary maximum modulus point. Define $P(h, w)$ by* (37) *and* (38). *The following inequality holds*:

$$R(w, f) \ge |w| \left(1 - \exp(-\frac{h}{P(h, w)}) \right).$$

Proof. We write $P := P(h, w)$. Let

$$\eta_w(z) := \frac{P(\Omega_w(z) - 1)}{P^2 - \Omega_w(z)}.$$

Evidently, $|\eta_w(z)| \le 1$ when $|z| \le h$ and $\eta_w(0) = 0$. Using Schwarz's lemma we find that $|\eta_w(z)| \le |z|/h$ when $|z| \le h$. Hence,

$$P|\Omega_w(z) - 1| \le \frac{|z|}{h} |P^2 - \Omega_w(z)| \le \frac{|z|}{h} (P^2 - 1 + |\Omega_w(z) - 1|), \quad |z| \le h.$$

This implies

$$|\Omega_w(z) - 1| \le \frac{|z|/h(P^2-1)}{P - |z|/h}, \quad |z| \le h. \tag{42}$$

When $|z| < h/P$, righthand side of (42) is less than 1. Therefore,

$$\Omega_w(z) \ne 0 \quad \text{for} \quad |z| < h/P.$$

Hence,

$$f(we^z) \ne 0 \quad \text{for} \quad |z| < h/P.$$

Thus,

$$R(w,f) \ge \min_{|z|=h/P} |w - we^z| = |w|\left(1 - \exp(-\frac{h}{P(h,w)})\right). \qquad \square$$

Using Lemma 4.3, (41) and (30), we obtain (23).

5. Proof of Theorem 3.1 (Case of zero order). The following lemma can be viewed as a supplement to Levin's result related to strong proximate order.

Lemma 5.1. (i) *Every transcendental entire function f of order zero has a p.o. $\rho(r) \in \mathcal{A}_0$.*

(ii) *Every transcendental entire function f of order zero satisfying* (10) *has a p.o. $\rho(r) \in \mathcal{A}_0^*$.*

Proof of (i). We follow the idea of Levin's proof [**6**, page 39]. We write $x = \log r$, $y = \varphi(x)$, where $\varphi(x) = \log\log M(e^x, f)$. Since

$$\limsup_{x\to\infty} \frac{\varphi(x)}{x} = 0,$$

for arbitrary $\varepsilon > 0$, the curve $y = \varphi(x)$ lies below the line $y = \varepsilon x$, for sufficiently large values of x. Consider the smallest convex domain containing all the points of the curve $y = \varphi(x)$ and the positive x-axis.

Let us denote by $y = \psi(x)$ the boundary of this domain. It is easy to see that ψ has the following properties:

(a) $\psi(x)$ is concave.

(b) $\lim_{x\to\infty} \psi(x)/x = 0$.

(c) $\varphi(x) \le \psi(x)$.

(d) $\lim_{x\to\infty} e^{\psi(x)}/x = \infty$.

(e) At extreme points, i.e., points not lying inside any line segment of the curve $y = \psi(x)$, one has $\varphi(x) = \psi(x)$.

(f) There exists a sequence of extreme points tending to infinity.

We now construct ϑ piece by piece by joining together some smooth majorants of ψ.

Let (l_0) be a line of support of ψ. On the line (l_0) take a point (x_0, y_0) and consider the curve

$$(43)\qquad (l_1):\ y = c_0^{(1)} + c_1^{(1)}(x - x_0) + \log\left(x - x_0 + c_2^{(1)}\right), \quad x \ge x_0,$$

that is tangent to the line (l_0) at the point (x_0, y_0). Then

$$c_2^{(1)} = \frac{1}{y_0' - c_1^{(1)}}, \qquad c_0^{(1)} = y_0 - \log\frac{1}{y_0' - c_1^{(1)}},$$

where y_0' is the slope of the line (l_0). The parameter $c_1^{(1)}$ will initially be chosen to be $y_0'/2$. We have

$$y' = c_1^{(1)} + \frac{1}{x - x_0 + c_2^{(1)}}, \qquad y'' = -\frac{1}{\left(x - x_0 + c_2^{(1)}\right)^2},$$

so that $y'' + y'^2 > 0$ when $x > x_0$.

If the abscissa x_0 is sufficiently large, then that part of the curve (l_1) lying to the right of x_0 is above the curve $y = \psi(x)$. Choosing (l_1) in this manner and then decreasing $c_1^{(1)}$ while keeping the point (x_0, y_0) fixed, we can find $c_1^{(1)} > 0$ such that this curve touches the curve $y = \psi(x)$ from above. To see this, consider the continuous function

$$g(c_1^{(1)}, x) := c_0^{(1)} + c_1^{(1)}(x - x_0) + \log\left(x - x_0 + c_2^{(1)}\right) - \psi(x),$$
$$0 \le c_1^{(1)} \le y_0'/2, \quad x \ge x_0.$$

If $c_1^{(1)} > 0$, then $g(c_1^{(1)}, x) \to \infty$ as $x \to \infty$, so that we can define

$$m(c_1^{(1)}) := \min_{x \geq x_0} g(c_1^{(1)}, x), \quad 0 < c_1^{(1)} \leq y_0'/2.$$

Clearly, m is continuous and $m(y_0'/2) > 0$. Also, because of (d), for arbitrary large values of M, we have

$$\psi(x) \geq M + \log x, \quad \text{when } x \text{ is large enough.} \tag{44}$$

Thus, $\lim_{x\to\infty} g(0,x) = -\infty$ and, by continuity of g, $\lim_{c_1^{(1)}\to 0} m(c_1^{(1)}) = -\infty$. Thus, there exists $c_1^{(1)}$, $0 < c_1^{(1)} < y_0'/2$, such that $m(c_1^{(1)}) = 0$ and therefore the curve (l_1) touches ψ at some point $(\tilde{x_0}, \tilde{y_0})$. Since (l_1) contains no line segments, the touching point $(\tilde{x}_0, \tilde{y}_0)$ must be an extreme point of ψ. This finishes the first step of the construction.

For the second step we initially set $c_1^{(2)} = c_1^{(1)}/2$ and choose a point (x_1, y_1), $x_1 > x_0 + 1$, on (l_1) far enough so that the part of the curve

$$(l_2) : y = c_0^{(2)} + c_1^{(2)}(x - x_1) + \log\left(x - x_1 + c_2^{(2)}\right), \quad x \geq x_1,$$

(this curve is tangent to the curve (l_1) at the point (x_1, y_1)), lying to the right of this point, lies above the curve $y = \psi(x)$. Then, without changing the point (x_1, y_1), we decrease $c_1^{(2)}$ so that (l_2) touches the curve ψ. As in the first step, we have $c_1^{(2)} > 0$. Next we set $c_1^{(3)} = c_1^{(2)}/2$ and take a point (x_2, y_2), $x_2 > x_1 + 1$, on (l_2), and form a curve (l_3), etc.

Now we form ϑ from the segments of the curves (l_0), $(l_1), \dots$ taken between the points of contact, i.e., $\vartheta(x) = l_j(x)$ when $x_{j-1} \leq x < x_j$, $j \geq 1$, and $\vartheta(x) = l_0(x)$ when $0 < x < x_0$. Clearly, ϑ satisfies (7). Also $\vartheta(x) \geq \psi(x)$, with equality holding for a sequence of extreme points of ψ tending to infinity. Using (c) and (e) we deduce that

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{e^{\vartheta(\log r)}} = 1.$$

Evidently, $\vartheta(x)$ is twice continuously differentiable and satisfies (8) and (9) except at the contact points x_j. It is easy to see that we can smooth

ϑ in such a way that all the properties mentioned above are preserved. This completes the proof of part (i).

Proof of (ii). The proof is similar to part (i) with a few modifications listed below. We change property (d) mentioned in the proof of part (i) with

(d′) $\limsup_{x\to\infty} e^{\psi(x)}/x^2 = \infty$. We change (43) with

$$(l_1) : y = c_0^{(1)} + c_1^{(1)}(x - x_0) + 2\log\left(x - x_0 + c_2^{(1)}\right), \quad x \geq x_0,$$

so that $2\vartheta''(x) + \vartheta'^2(x) > 0$. We change (43) with

$$\vartheta(x_n) \geq M + 2\log x_n, \quad \text{for some sequence} \quad x_n. \qquad \square$$

We proceed to prove Theorem 3.1 for the case $\rho(r) \in \mathcal{A}_0$. We remind that in this case $V(r) = r^{\rho(r)} = e^{\vartheta(\log r)}$, where ϑ satisfies (7), (8) and (9). (At this point we don't need condition (9); we will require it only when we prove Theorem 3.4.)

The following lemma is an analogue of Lemma 4.1.

Lemma 5.2. *There exists a positive constant D not dependent on r such that if*

$$\left|\log\frac{R}{r}\right| \leq \frac{1}{4\vartheta'(\log r)}, \tag{45}$$

then

$$\left|V(R) - V(r) - \log(R/r)\vartheta'(\log r)V(r)\right| \leq D\log^2(R/r)\vartheta'^2(\log r)V(r). \tag{46}$$

Proof. Assume first that $r \leq R$. Note that (8) implies

$$|\vartheta''(x)| < \vartheta'^2(x), \tag{47}$$

when x is large enough. We have, for some c between $\log r$ and $\log R$,
(48)

$$\begin{aligned}\left|\vartheta(\log R)-\vartheta(\log r)-\log(R/r)\vartheta'(\log r)\right| &= \frac{1}{2}\log^2(R/r)|\vartheta''(c)| \\ &\le \frac{1}{2}\log^2(R/r)\vartheta'^2(c) \\ &\le \frac{1}{2}\log^2(R/r)\vartheta'^2(\log r).\end{aligned}$$

Since (45) implies

$$\vartheta(\log R)-\vartheta(\log r)\le \log(R/r)\vartheta'(\log r)\le \frac{1}{4}, \tag{49}$$

there exists a positive constant D_1 such that

$$\begin{aligned}\left|e^{\vartheta(\log R)-\vartheta(\log r)}-1-\big(\vartheta(\log R)-\vartheta(\log r)\big)\right| & \\ \le D_1\Big(\vartheta(\log R)-(\vartheta(\log r))\Big)^2 & \\ \le D_1\log^2(R/r)\vartheta'^2(\log r).&\end{aligned} \tag{50}$$

Combining (48) and (50) we find

$$\left|e^{\vartheta(\log R)-\vartheta(\log r)}-1-\log(R/r)\vartheta'(\log r)\right|\le D_2\log^2(R/r)\vartheta'^2(\log r).$$

Noting that $V(r)=e^{\vartheta(\log r)}$ we obtain (46).

Now we assume that $r>R$ and (45) is satisfied. Since

$$r(h):=h\exp\{1/(2\vartheta'(\log h))\}$$

is an increasing function of h, there exists an inverse function $h(r)$ such that

$$h(r)<r \quad \text{and} \quad \log\frac{r}{h(r)}=\frac{1}{2\vartheta'(\log h(r))}. \tag{51}$$

We will need the following property:

$$\vartheta'(\log h(r))\le 2\vartheta'(\log r). \tag{52}$$

To see this, note that for some c between $\log h(r)$ and $\log r$, we have

$$\frac{\vartheta'(\log h(r)) - \vartheta'(\log r)}{\vartheta'(\log h(r))} = \frac{\log(r/h(r))|\vartheta''(c)|}{\vartheta'(\log h(r))} \overset{(51)}{=} \frac{|\vartheta''(c)|}{2\vartheta'^2(\log h(r))} \overset{(47)}{\leq} \frac{1}{2}.$$

The inequality

$$\log(r/R) \leq \frac{1}{4\vartheta'(\log r)} \leq \frac{1}{2\vartheta'(\log h(r))},$$

and (51) implies that $h(r) \leq R < r$. Therefore, using (52) we find

$$\vartheta'(\log R) \leq 2\vartheta'(\log r). \tag{53}$$

Thus, we have

$$\begin{aligned} \Big|\vartheta(\log R) - \vartheta(\log r) - \log(R/r)\vartheta'(\log r)\Big| &\leq \frac{1}{2}\log^2(R/r)\vartheta'^2(\log R) \\ &\leq 2\log^2(R/r)\vartheta'^2(\log r). \end{aligned} \tag{54}$$

Since

$$|\vartheta(\log R) - \vartheta(\log r)| \leq \log(r/R)\vartheta'(\log R) \leq 2\log(r/R)\vartheta'(\log r) \overset{(45)}{\leq} \frac{1}{2},$$

there exists a positive constant D_3 such that

$$\left|e^{\vartheta(\log R)-\vartheta(\log r)} - 1 - \Big(\vartheta(\log R) - \vartheta(\log r)\Big)\right| \leq D_3 \log^2(R/r)\vartheta'^2(\log r). \tag{55}$$

Using (54) and (55), we see that (46) is also valid in the case $R < r$. □

The following result is a counterpart of Lemma 4.2.

Lemma 5.3. *If f is of (V, θ)-regular growth, then*

$$K(r, f) = \sigma\vartheta'(\log r)V(r) + O\left(\vartheta'(\log r)\sqrt{V(r)\theta(r)}\right), \quad r \to \infty.$$

Proof. Let R be such that

$$\log\frac{R}{r} := \frac{1}{\vartheta'(\log r)}\sqrt{\frac{\theta(r)}{V(r)}}. \tag{56}$$

Using similar arguments as we did in the proof of Lemma 4.2, for sufficiently large values of r, we obtain

$$K(r,f) \leq \frac{1}{\log(R/r)}\Big(\sigma V(R) - \sigma V(r) + E_1\theta(r)\Big),$$

where E_1 is some constant. Applying Lemma 5.2 we find

$$K(r,f) \leq \frac{1}{\log(R/r)}\Big(\sigma \log(R/r)\vartheta'(\log r)V(r) \\ + D\log^2(R/r)\vartheta'^2(\log r)V(r) + E_1\theta(r)\Big).$$

Thus,

$$K(r,f) \leq \sigma\vartheta'(\log r)V(r) + O\left(\vartheta'(\log r)\sqrt{V(r)\theta(r)}\right), \quad r\to\infty.$$

For the reverse inequality, we set

$$\log\frac{r}{s} := \frac{1}{\vartheta'(\log r)}\sqrt{\frac{\theta(r)}{V(r)}},$$

and proceed like we did in the proof of Lemma 4.2. □

The remaining part of the proof is similar to the case $\rho(r) \in \mathcal{A}_\infty$. With w as a maximum modulus point, Ω_w and $P(h,w)$ as defined in (37) and (38) for some constant F, we have the following analogue of (40)

$$\begin{aligned}\log P(h,w) \leq & \max_{-h\leq t\leq h}\left|\int_r^{re^t} F\vartheta'(\log u)\sqrt{V(u)\theta(u)}\,\frac{du}{u}\right| \\ &+ \max_{-h\leq t\leq h}\left|\int_r^{re^t} F\vartheta'(\log r)\sqrt{V(r)\theta(r)}\,\frac{du}{u}\right| \\ &+ \sigma\max_{-h\leq t\leq h}\left|V(re^t) - V(r) - t\vartheta'(\log r)V(r)\right| \\ =:& \, T_1 + T_2 + \sigma T_3.\end{aligned}$$

We set

$$h = h_r = \frac{1}{\vartheta'(\log r)\sqrt{V(r)\theta(r)}}.$$

Note that $h_r \leq 1/(4\vartheta'(\log r))$ when r is sufficiently large. Thus, using (19), (49) and (53) we find

$$T_1 \leq F h_r \vartheta'(\log(re^{-h_r}))\sqrt{V(re^{h_r})\theta(re^{h_r})} = O(1), \quad r \to \infty.$$

Obviously, $T_2 = O(1)$. Finally, using Lemma 5.2, we find

$$T_3 \leq D h_r^2 \vartheta'^2(\log r) V(r) = O(1), \quad r \to \infty.$$

Thus, $\log P(h, w) = O(1)$ as $r \to \infty$. Applying Lemma 5.3 we obtain (23). This completes the proof of Theorem 3.1. □

6. Proof of Theorem 3.4. First of all we observe that it suffices to prove Theorem 3.4 for functions f with nonnegative Taylor coefficients. We use the following result of Erdös and Kövari:

Theorem [**3**]. *Let f be an entire function. There exists an entire function $\hat{f}$ with nonnegative Taylor coefficients such that*

$$\frac{1}{6} \leq \frac{M(r, f)}{M(r, \hat{f})} \leq 3, \quad r \geq 0.$$

By this theorem, $\log M(r, \hat{f}) = \log M(r, f) + O(1)$, $r \to \infty$, therefore (V, θ)-regular growth of f implies (V, θ)-regular growth of $\hat{f}$. Further, we assume that f has nonnegative Taylor coefficients.

It will be convenient to change variables to $r \mapsto e^x$, $x \geq 0$, and denote

(57)
$$h(x) = M(e^x, f), \ \kappa(x) = K(e^x, f), \quad v(x) = V(e^x), \quad t(x) = \theta(e^x).$$

The point of the proof of Theorem 3.4 is the assertion (ii) of the following Lemma 6.1. This assertion is a slight modification of a result by Hayman [**5**].

Lemma 6.1. *Let f be a transcendental entire function with nonnegative Taylor coefficients.*

(i) *The equations*

$$\kappa(a_k) = k + \varepsilon, \quad \kappa(b_k) = k + 1 - \varepsilon; \qquad k = k_0, k_0 + 1, k_0 + 2, \dots \tag{58}$$

where $\varepsilon \in (0, 1/2)$, $k_0 = [\kappa(0)] + 1$, *uniquely determine positive numbers*

$$a_{k_0} < b_{k_0} < a_{k_0+1} < b_{k_0+1} < \cdots < a_k < b_k < \cdots \longrightarrow \infty, \quad \textit{as} \quad k \to \infty. \tag{59}$$

(ii) *The inequality holds*

$$(\log h(x))'' \geq \varepsilon^2 \quad \textit{for} \quad x \in A_\varepsilon := \bigcup_{k=k_0}^{\infty} [a_k, b_k].$$

Proof. (i) Since f is transcendental, the function κ strictly increases to ∞. Therefore, the numbers a_k and b_k are uniquely determined by (58), and (59) holds.

(ii) We have

$$h(x) = f(e^x) = \sum_{k=0}^{\infty} d_k e^{kx}, \quad d_k \geq 0, \quad k = 0, 1, 2, \dots .$$

The following formula belongs to Rosenbloom [**9**]:

$$(\log h(x))'' = \sum_{k=0}^{\infty} (k - \kappa(x))^2 \, \frac{d_k e^{kx}}{h(x)}. \tag{60}$$

Its proof is a direct calculation:

$$\kappa(x) = (\log h(x))' = \frac{h'(x)}{h(x)} = \sum_{k=0}^{\infty} \frac{k \, d_k e^{kx}}{h(x)};$$

$$(\log h(x))'' = \frac{h''(x)}{h(x)} - \left(\frac{h'(x)}{h(x)} \right)^2 = \sum_{k=0}^{\infty} \frac{k^2 d_k e^{kx}}{h(x)} - \kappa(x) \sum_{k=0}^{\infty} \frac{k \, d_k e^{kx}}{h(x)}$$
$$= \sum_{k=0}^{\infty} (k - \kappa(x))^2 \, \frac{d_k e^{kx}}{h(x)}.$$

In order to derive the assertion (ii) from (60), it suffices to observe that for $x \in A_\varepsilon$ we have

$$\min_{k=0,1,2,\dots} |k - \kappa(x)| \geq \varepsilon.$$

Corollary 6.2. *Let f satisfy conditions of Lemma 6.1 and be of (V,θ)-regular growth. Then there exists a positive constant C such that*

$$Ct(b_k) \geq \frac{1}{2}\epsilon^2(b_k - a_k)^2 + \kappa(a_k)(b_k - a_k) - \sigma(v(b_k) - v(a_k)), \quad k \geq k_0. \tag{61}$$

Proof. By the Taylor formula, we have for some $c_k \in (a_k, b_k)$,

$$\log h(b_k) - \log h(a_k) = \kappa(a_k)(b_k - a_k) + \frac{1}{2}(\log h(c_k))''(b_k - a_k)^2.$$

Using Lemma 6.1, we get

$$\log h(b_k) - \log h(a_k) \geq \kappa(a_k)(b_k - a_k) + \frac{1}{2}\varepsilon^2(b_k - a_k)^2. \tag{62}$$

Since f is of (V,θ)-regular growth we have (in notations (57))

$$\log h(x) = \sigma v(x) + O(t(x)), \quad x \to \infty, \tag{63}$$

and therefore

$$\log h(b_k) - \log h(a_k) \leq \sigma v(b_k) - \sigma v(a_k) + Ct(b_k), \quad k \geq k_0,$$

where C is a positive constant. Juxtaposing with (62), we get (61). □

Now we assume that the assertion of Theorem 3.4 is wrong, that is,

$$\lim_{r\to\infty} rV'(r)\sqrt{\frac{\theta(r)}{V(r)}} = 0.$$

Using notations (57), we can rewrite this in the form

$$\lim_{x\to\infty} v'(x)\sqrt{\frac{t(x)}{v(x)}} = 0. \tag{64}$$

The proof that (64) leads to a contradiction will be divided into several steps.

Step 1. We show that

$$\lim_{x\to\infty} v''(x) = 0. \tag{65}$$

By the definition of admissible p.o., we have the representation

$$v(x) = e^{\vartheta(x)}, \tag{66}$$

where ϑ is a concave function. Therefore,

$$v''(x) = \big(\vartheta''(x) + \vartheta'^2(x)\big)\, v(x) \le \vartheta'^2(x)v(x) = \frac{v'^2(x)}{v(x)}. \tag{67}$$

Since $t(x)$ is a positive nondecreasing function, (64) implies that $v'^2(x)/v(x) \to 0$, as $x \to \infty$.

Step 2. We show that

$$\lim_{k\to\infty} (b_k - a_k) = \infty. \tag{68}$$

By Lemma 5.3 we have

$$\kappa(x) = \sigma v'(x) + O\left(v'(x)\sqrt{\frac{t(x)}{v(x)}}\right), \quad x \to \infty,$$

whence by (64),

$$\kappa(x) = \sigma v'(x) + o(1), \quad x \to \infty. \tag{69}$$

Therefore,

$$
\begin{aligned}
1-2\varepsilon = \kappa(b_k)-\kappa(a_k) &= \sigma v'(b_k)-\sigma v'(a_k)+o(1)\\
&= \sigma v''(c_k)(b_k-a_k)+o(1), \quad k\to\infty,
\end{aligned}
\tag{70}
$$

where $c_k \in (a_k, b_k)$. Using (65), we get (68).

Step 3. We show that

$$
\frac{(v'(b_k))^2}{v(b_k)}\,(b_k-a_k) \geq \frac{1-2\varepsilon}{\sigma}+o(1), \quad k\to\infty. \tag{71}
$$

Indeed, we have from (67) and (70) that

$$
\begin{aligned}
\sigma(b_k-a_k) &= \frac{1-2\varepsilon+o(1)}{v''(c_k)} \geq \frac{1-2\varepsilon+o(1)}{v'^2(c_k)}\,v(c_k)\\
&\geq \frac{1-2\varepsilon+o(1)}{v'^2(b_k)}\,[v(b_k)-v'(b_k)(b_k-c_k)]\\
&\geq \frac{v(b_k)}{v'^2(b_k)}\,(1-2\varepsilon+o(1))-\frac{1-2\varepsilon+o(1)}{v'(b_k)}\,(b_k-a_k)\\
&= \frac{v(b_k)}{v'^2(b_k)}\,(1-2\varepsilon+o(1))+o(b_k-a_k), \quad k\to\infty.
\end{aligned}
$$

Hence, (71) follows immediately.

Step 4. We show that

$$
\sigma v(b_k)-\sigma v(a_k)-\kappa(a_k)(b_k-a_k)=o((b_k-a_k)^2), \quad k\to\infty. \tag{72}
$$

Using (69), we see that for some $c_k \in (a_k, b_k)$, $c_k' \in (a_k, b_k)$

$$
\begin{aligned}
&\sigma v(b_k)-\sigma v(a_k)-\kappa(a_k)(b_k-a_k)\\
&\qquad = \sigma v'(c_k)(b_k-a_k)-(\sigma v'(a_k)+o(1))(b_k-a_k)\\
&\qquad = \sigma(v'(c_k)-v^{\omega}(a_k))(b_k-a_k)+o(b_k-a_k)\\
&\qquad = \sigma(c_k-a_k)v''(c_k')(b_k-a_k)+o(b_k-a_k).
\end{aligned}
$$

Taking into account (65) and (68), we obtain (72).

Step 5. Now we will complete the proof of Theorem 3.4. Taking into account (72), we derive from (61) that

$$Ct(b_k) \geq \left(\frac{1}{2}\varepsilon^2 + o(1)\right)(b_k - a_k)^2, \quad k \to \infty.$$

Hence, by (71), we obtain

$$\begin{aligned}\frac{v'^2(b_k)}{v(b_k)}\, t(b_k) &\geq \frac{1}{C}\left(\frac{1}{2}\varepsilon^2 + o(1)\right)\frac{v'^2(b_k)}{v(b_k)}\,(b_k - a_k)^2 \\ &\geq \frac{1}{C}\left(\frac{1}{2}\varepsilon^2 + o(1)\right)\left(\frac{1-2\varepsilon}{\sigma} + o(1)\right)(b_k - a_k), \quad k \to \infty.\end{aligned}$$

Using (68), we obtain a contradiction to (64).

7. Proof of Theorem 3.5. We will prove Theorem 3.5 when $\rho(r) \in \mathcal{A}_\rho$, $0 < \rho < \infty$. The cases $\rho(r) \in \mathcal{A}_0^*$ and $\rho(r) \in \mathcal{A}_\infty$ can be dealt with in a similar way, therefore we only outline corresponding proofs.

Given $1/3 \leq \alpha < 1$, we set

$$\beta := \frac{1+\alpha}{1-\alpha}. \tag{73}$$

Note that $\beta \geq 2$. We define

$$\nu^{\beta+1}(r) := rV'(r). \tag{74}$$

It is easy to see that $\nu(r)$ is an increasing function when r is large enough. By changing $V(r)$ on a finite interval, we may assume that $\nu(r)$ increases for all $r > 0$. Let

$$\lambda := \nu^{-1} \tag{75}$$

be the inverse function of ν. We set

$$f(z) := \prod_{k=1}^{\infty}\left(1 + \left(\frac{z}{\lambda(k)}\right)^{[k^\beta]}\right). \tag{76}$$

(i) We first show that f satisfies (28). This is rather simple. Indeed, since each point of the positive ray is a maximum modulus point, we have

$$R(\lambda(k), f) \leq 2\lambda(k) \sin \frac{\pi}{2[k^\beta]}.$$

Therefore,

$$\liminf_{k\to\infty} k^\beta \frac{R(\lambda(k), f)}{\lambda(k)} \leq \pi.$$

Noting that

$$k^{\beta+1} = \nu^{\beta+1}(\lambda(k)) = \lambda(k)V'(\lambda(k)), \tag{77}$$

we obtain the desired result.

(ii) Now we will prove that f is of (V, θ)-regular growth. This is much more cumbersome. We have

$$\begin{aligned}
\log M(r, f) &= \sum_{k=1}^{\infty} \log\left(1 + \left(\frac{r}{\lambda(k)}\right)^{[k^\beta]}\right) \\
&= \sum_{k=1}^{n} \log\left(\frac{r}{\lambda(k)}\right)^{[k^\beta]} + \sum_{k=1}^{n} \log\left(1 + \left(\frac{\lambda(k)}{r}\right)^{[k^\beta]}\right) \\
&\quad + \sum_{k=n+1}^{\infty} \log\left(1 + \left(\frac{r}{\lambda(k)}\right)^{[k^\beta]}\right) \\
&=: S_1 + S_2 + S_3, \\
&\qquad \lambda(n) \leq r < \lambda(n+1).
\end{aligned} \tag{78}$$

We will show that $S_2 = O(1)$ and $S_3 = O(1)$. For this we will first find an upper bound for $(\lambda(k)/\lambda(k+1))^k$. Since (4) implies

$$\nu'(r) = \frac{\nu(r)}{r}\left(\frac{\rho}{\beta+1} + o(1)\right), \tag{79}$$

we find

$$1 = \int_{\lambda(k)}^{\lambda(k+1)} \nu'(t)\, dt \leq [\lambda(k+1) - \lambda(k)] \frac{\nu(\lambda(k+1))}{\lambda(k)} \left(\frac{\rho}{\beta+1} + o(1)\right).$$

Therefore,

$$\frac{\lambda(k+1)}{\lambda(k)} \geq 1 + \frac{1}{(k+1)\left(\frac{\rho}{\beta+1} + o(1)\right)}.$$

Thus,

$$\liminf_{k\to\infty} k \log \frac{\lambda(k+1)}{\lambda(k)} \geq \frac{\beta+1}{\rho}. \tag{80}$$

We choose C_1 and C_2 such that

$$C_1 = C_2^2 \quad \text{and} \quad e^{-\frac{\beta+1}{\rho}} < C_1 < C_2 < 1.$$

Because of (80), there exists k_0 such that

$$\left(\frac{\lambda(k)}{\lambda(k+1)}\right)^k < C_1 \quad \text{when} \quad k > k_0. \tag{81}$$

Let $n > k_0$. We have (recall that $\beta \geq 2$)

$$\begin{aligned} S_2 &= \sum_{k=1}^{n} \log\left(1 + \left(\frac{\lambda(k)}{r}\right)^{[k^\beta]}\right) \leq \sum_{k=1}^{n} \left(\frac{\lambda(k)}{\lambda(n)}\right)^k \\ &= o(1) + \sum_{k=k_0+1}^{n} \left(\frac{\lambda(k)}{\lambda(n)}\right)^k. \end{aligned}$$

We divide the interval $[k_0+1, n]$ into two parts. There exists n' (depending on n) such that $\lambda(k)/\lambda(n) \leq C_2$ when $k_0 + 1 \leq k \leq n'$ and $\lambda(k)/\lambda(n) \geq C_2$ when $n' < k \leq n$. We have

$$S_2 \leq o(1) + \sum_{k=k_0+1}^{n'} \left(\frac{\lambda(k)}{\lambda(n)}\right)^k + \sum_{k=n'+1}^{n} \left(\frac{\lambda(k)}{\lambda(n)}\right)^k =: o(1) + T_1 + T_2.$$

Evidently,

$$T_1 \leq \sum_{k=k_0+1}^{n'} C_2^k \leq \sum_{k=0}^{\infty} C_2^k.$$

We use backward induction to show that the following inequality holds:

$$\left(\frac{\lambda(k)}{\lambda(n)}\right)^k \le C_2^{n-k}, \quad n'+1 \le k \le n-1.$$

The base of induction holds because of (81). We assume

$$\left(\frac{\lambda(n-j)}{\lambda(n)}\right)^{n-j} \le C_2^j, \quad j < n-(n'+1).$$

Since $n-j > n'+1$ we have $\lambda(n-j)/\lambda(n) \ge C_2$. Using (81) we obtain

$$\begin{aligned}\left(\frac{\lambda(n-j-1)}{\lambda(n)}\right)^{n-j-1} & \\ &= \left(\frac{\lambda(n-j-1)}{\lambda(n-j)}\right)^{n-j-1}\left(\frac{\lambda(n-j)}{\lambda(n)}\right)^{n-j}\frac{\lambda(n)}{\lambda(n-j)} \\ &\le C_1 . C_2^j . \frac{1}{C_2} = C_2^{j+1}.\end{aligned}$$

Therefore,

$$T_2 = \sum_{k=n'+1}^{n}\left(\frac{\lambda(k)}{\lambda(n)}\right)^k \le \sum_{k=0}^{\infty} C_2^k.$$

Thus, $S_2 = O(1)$,

Using similar arguments it is easy to show that, when $n > k_0$,

$$\left(\frac{\lambda(n+1)}{\lambda(k)}\right)^k \le C_1^{k-n-1}, \quad k \ge n+2.$$

Hence,

$$S_3 = \sum_{k=n+1}^{\infty} \log\left(1+\left(\frac{r}{\lambda(k)}\right)^{[k^\beta]}\right) \le \sum_{k=n+1}^{\infty}\left(\frac{\lambda(n+1)}{\lambda(k)}\right)^k \le \sum_{k=0}^{\infty} C_1^k.$$

Now, we are going to show that

$$S_1 = \frac{V(r)}{\beta+1} + O\left(V^\alpha(r)\right), \quad r \to \infty, \qquad \lambda(n) \le r < \lambda(n+1).$$

We have

$$S_1 = \sum_{k=1}^{n}[k^\beta]\log\frac{r}{\lambda(k)} = \sum_{k=1}^{n} k^\beta \log\frac{r}{\lambda(k)} - \sum_{k=1}^{n}(k^\beta - [k^\beta])\log\frac{r}{\lambda(k)} =: U_1 - U_2.$$

Let

$$u_1(t) := t^\beta \log\frac{r}{\lambda(t)}.$$

By Euler-Maclaurin sum formula, we have

$$U_1 = \sum_{k=1}^{n} u_1(k) = \int_1^n u_1(t)\,dt + \frac{1}{2}\left(u_1(n) + u_1(1)\right) + \frac{1}{12}\left(u_1'(n) - u_1'(1)\right) - \int_1^n u_1''(t)\frac{B_2(t-[t])}{2}\,dt, \tag{82}$$

where $B_2(t)$ is the second Bernoulli polynomial. We first evaluate $\int_1^n u_1(t)\,dt$. We have

$$\int_1^n u_1(t)\,dt = \frac{n^{\beta+1}}{\beta+1}\log r - \int_1^n t^\beta \log\lambda(t)\,dt + O(\log r). \tag{83}$$

Further,

$$\begin{aligned}\int_1^n t^\beta \log\lambda(t)\,dt &= \int_{\lambda(1)}^{\lambda(n)} \nu^\beta(s)\log s\,d\nu(s)\\ &= \frac{\nu^{\beta+1}(s)\log s}{\beta+1}\bigg|_{\lambda(1)}^{\lambda(n)} - \frac{1}{\beta+1}\int_{\lambda(1)}^{\lambda(n)}\frac{\nu^{\beta+1}(s)}{s}\,ds\\ &= \frac{n^{\beta+1}\log\lambda(n)}{\beta+1} - \frac{V(\lambda(n))}{\beta+1} + O(1).\end{aligned} \tag{84}$$

Hence, using (83) and (84) we find

$$\int_1^n u_1(t)\,dt = \frac{n^{\beta+1}}{\beta+1}\log\frac{r}{\lambda(n)} + \frac{V(r)}{\beta+1} - \frac{1}{\beta+1}\Big(V(r) - V(\lambda(n))\Big) + O(\log r). \tag{85}$$

We will need the following two properties (recall that $\lambda(n) \leq r < \lambda(n+1)$):

$$\frac{r}{\lambda(n)} - 1 = O\left(\frac{1}{n}\right), \quad n \to \infty, \tag{86}$$

and

$$\frac{V(r)}{n^{\beta+1}} \to \rho \quad \text{as} \quad n \to \infty. \tag{87}$$

To see (86) first note that $\lambda(n) \leq r < \lambda(n+1)$ implies $n = [\nu(r)]$. Also, using (79) we find that

$$\frac{\lambda'(r)}{\lambda(r)} = \frac{1}{r}\left(\frac{\beta+1}{\rho} + o(1)\right). \tag{88}$$

Therefore,

$$\begin{aligned}\frac{r-\lambda(n)}{r} = \frac{1}{r}\int_{[\nu(r)]}^{\nu(r)} \lambda'(t)\,dt &= \frac{1}{r}\int_{[\nu(r)]}^{\nu(r)} \left(\frac{\beta+1}{\rho} + o(1)\right)\frac{\lambda(t)}{t}\,dt \\ &\leq \frac{((\beta+1)/\rho + o(1))\,r}{nr}.\end{aligned}$$

We have (87) since

$$\begin{aligned}n^{\beta+1} = \nu^{\beta+1}(\lambda(n)) &\leq \nu^{\beta+1}(r) = V(r)(\rho + o(1)) \\ &\leq \nu^{\beta+1}(\lambda(n+1)) = (n+1)^{\beta+1}.\end{aligned}$$

We now estimate the term $V(r) - V(\lambda(n))$ that appeared in (85). Because

$$r^2 V''(r) = (\rho^2 - \rho + o(1))V(r),$$

we see that, for some c between $\lambda(n)$ and r,

$$\begin{aligned}&V(r) - V(\lambda(n)) - (r-\lambda(n))V'(\lambda(n)) \\ &\qquad = \frac{1}{2}\left(\frac{r}{\lambda(n)} - 1\right)^2 \lambda^2(n) V''(c) \\ &\qquad \overset{(86)}{=} O\left(\frac{1}{n^2}\right) O(V(r)) \\ &\qquad \overset{(87)}{=} O\left(V^{\alpha}(r)\right).\end{aligned} \tag{89}$$

Thus, using (77), (85) and (89) we obtain
(90)

$$\begin{aligned}\int_1^n u_1(t)\,dt &= \frac{n^{\beta+1}}{\beta+1}\left[\left(\frac{r}{\lambda(n)}-1\right)+O\left(\left(\frac{r}{\lambda(n)}-1\right)^2\right)\right]+\frac{V(r)}{\beta+1}\\ &\quad -\frac{1}{\beta+1}\left[n^{\beta+1}\left(\frac{r}{\lambda(n)}-1\right)+O\left(V^\alpha(r)\right)\right]\\ &= \frac{V(r)}{\beta+1}+O\left(V^\alpha(r)\right).\end{aligned}$$

We now estimate the remaining terms that appear on the righthand side of (82). It is easy to see that (4)–(5) implies

$$(\log\lambda(t))^{(j)} = O\left(\frac{1}{t^j}\right),\quad j=1,2. \tag{91}$$

Therefore, for $j=1,2$, we have
(92)

$$\begin{aligned}u_1^{(j)}(t) &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-j+1)}t^{\beta-j}\log\frac{r}{\lambda(t)}-\sum_{l=0}^{j-1}\binom{j}{l}(t^\beta)^{(l)}(\log\lambda(t))^{(j-l)}\\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-j+1)}t^{\beta-j}\log\frac{r}{\lambda(t)}+O(t^{\beta-j}).\end{aligned}$$

Hence, using (86) we find

$$u_1^{(j)}(n) = O(n^{\beta-j-1})+O(n^{\beta-j}) = O(n^{\beta-j}),\quad j=1,2. \tag{93}$$

Also,

$$u_1(n) = n^\beta\log\frac{r}{\lambda(n)} = O(n^{\beta-1}). \tag{94}$$

Finally, using (92) we find

$$\begin{aligned}&\left|\int_1^n u_1''(t)\frac{B_2(t-[t])}{2}\,dt\right| \\ &\qquad\le D_1\int_1^n t^{\beta-2}\log\frac{r}{\lambda(t)}\,dt+D_2\int_1^n t^{\beta-2}\,dt\\ &\qquad= D_1\frac{t^{\beta-1}}{\beta-1}\log\frac{r}{\lambda(t)}\bigg|_1^n+\frac{D_1}{\beta-1}\int_1^n t^{\beta-1}\frac{\lambda'(t)}{\lambda(t)}\,dt\\ &\qquad\quad+O(n^{\beta-1})\\ &\qquad\overset{(88)}{=} O(n^{\beta-1}).\end{aligned} \tag{95}$$

Thus, using (82), (90), (93)–(95), we obtain

$$U_1 = \frac{V(r)}{\beta+1} + O\left(V^\alpha(r)\right).$$

Now we are going to estimate U_2. Let

$$u_2(t) := \log \frac{r}{\lambda(t)}.$$

Then

$$\begin{aligned} U_2 \le \sum_{k=1}^{n} u_2(k) = \int_1^n u_2(t)\,dt + \frac{1}{2}\left(u_2(n)+u_2(1)\right) \\ + \int_1^n u_2'(t)\gamma(t)\,dt, \end{aligned} \tag{96}$$

where $\gamma(t) = t - [t] - 1/2$. We have

$$\begin{aligned} \int_1^n u_2(t)\,dt &= t\log\frac{r}{\lambda(t)}\,\Big|_1^n + \int_1^n \frac{t\lambda'(t)}{\lambda(t)}\,dt \\ &\overset{(88)}{=} n\log\frac{r}{\lambda(n)} + \int_1^n \left(\frac{\beta+1}{\rho} + o(1)\right) dt + O(\log r) \\ &\overset{(86)}{=} O(n). \end{aligned} \tag{97}$$

Thus, using (86), (91), (96) and (97), we obtain

$$U_2 = O(n) = O(n^{\beta-1}) = O\left(V^\alpha(r)\right).$$

This completes the proof of Theorem 3.5 for the case $\rho(r) \in \mathcal{A}_\rho$, $0<\rho<\infty$.

When $\rho(r)$ is in $\mathcal{A}_\infty$ or $\mathcal{A}_0^*$ we define ν, λ and f as in (74)–(76). In the case $\rho(r) \in \mathcal{A}_\infty$ to prove that f is of (V,θ)-regular growth we first write a formula similar to (78) and then show that

$$\begin{aligned} S_1 &= \frac{V(r)}{\beta+1} + O\left(\frac{V^\alpha(r)}{(\vartheta'(\log r))^{1-\alpha}}\right), \\ S_2 &= O\left(\frac{V^\alpha(r)}{(\vartheta'(\log r))^{1-\alpha}}\right), \\ S_3 &= O(1). \end{aligned}$$

In the estimation of S_1 we again use Euler-Maclaurin sum formula (82), but now it is necessary to change (91) with

$$(98)\qquad (\log\lambda(t))^{(j)} = O\left(\frac{1}{t^j\,\vartheta'(\log\lambda(t))}\right),\quad j=1,2,$$

which is a consequence of (12).

In the case $\rho(r)\in\mathcal{A}_0^*$ estimate (98) still holds because of (8)–(9). Using this we can show that

$$S_1 = \frac{V(r)}{\beta+1} + O\left(\frac{V^\alpha(r)}{(\vartheta'(\log r))^{1-\alpha}}\right),\quad S_2 = O(1),\quad S_3 = O(1)\qquad \square.$$

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