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ON THE OSCILLATION OF FIRST ORDER DELAY DYNAMIC EQUATIONS WITH VARIABLE COEFFICIENTS

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ABSTRACT. In this paper we obtain some new oscillation and nonoscillation criteria for the first order delay dynamic equation with variable coefficients

$$y^{\Delta}(t) + \sum_{i=1}^{n} p_i(t)y(\tau_i(t)) = 0$$

on a time scale $\mathbf{T}$. Moreover, a new sufficient condition for oscillation of

$$y'(t) + \sum_{i=1}^{n} p_i(t)y(\tau_i(t)) = 0$$

is obtained.

1. Introduction. In recent years, the theory of time scales, which was introduced by Stefan Hilger in his Ph.D. thesis in 1988 in order to unify continuous and discrete analysis, has received a lot of attention, see [**8**]. In fact, there has been much research concerning the oscillation and nonoscillation of solutions of differential equations on time scales (or measure chains). We refer the reader to recent papers [**1, 5, 6, 10**] and the references cited therein. A book on the subject of time scales, by Bohner and Peterson [**2**], summarizes and organizes much of time scales calculus, see also the book by Bohner and Peterson [**3**] for advances in dynamic equations on time scales.

In this paper, we are concerned with oscillation and nonoscillation of the first order delay dynamic equation with variable coefficients

$$y^{\Delta}(t) + \sum_{i=1}^{n} p_i(t)y(\tau_i(t)) = 0 \tag{1.1}$$

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on a time scale $\mathbf{T}$. Since we are interested in asymptotic behavior of solutions, we will suppose that the time scale $\mathbf{T}$ under consideration is not bounded above, i.e., it is a time scale interval of the form $[t_0, \infty)$. Throughout this paper, we assume that:

(H_1) $p_i(t)$, $i = 1, 2, \dots, n$, are nonnegative real valued rd-continuous functions defined on $\mathbf{T}$,

(H_2) $\tau_i(\cdot) : \mathbf{T} \to \mathbf{T}$ and $\tau_i(t) < t$ for all $t \in \mathbf{T}$, $i = 1, 2, \dots, n$.

By a solution of equation (1.1), we mean a nontrivial real valued function $y(t)$ which satisfies equation (1.1) for all $t \geq t_y$. Our attention is restricted to those solutions of equation (1.1) which exist on some half line $[t_x, \infty)$ and satisfy $\sup\{x|(t)| : t > t_1\} > 0$ for any $t_1 > t_y$.

A solution $y(t)$ of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

We note that, if $\mathbf{T} = \mathbf{R}$, then $\sigma(t) = t$, $\mu(t) = 0$, $f^{\Delta}(t) = f'(t)$ and then equation (1.1) becomes the first order delay differential equation with several delays

$$y'(t) + \sum_{i=1}^{n} p_i(t) y(\tau_i(t)) = 0, \quad t \in [t_0, \infty) \tag{1.2}$$

which has been the subject of much investigation, see for example [**4, 7, 9**].

2. Some preliminaries on time scales. A time scale $\mathbf{T}$ is an arbitrary nonempty closed subset of the real number $\mathbf{R}$. On any time scale $\mathbf{T}$ we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s \in \mathbf{T} : s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbf{T} : s < t\}. \tag{2.1}$$

A point $t \in \mathbf{T}$, $t > \inf \mathbf{T}$ is said to be left-dense if $\rho(t) = t$, right-dense if $t > \sup \mathbf{T}$ and $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. The graininess function $\mu : \mathbf{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$.

For a function $f : \mathbf{T} \to \mathbf{R}$ the (delta) derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t) - f(t))}{\sigma(t) - t}. \tag{2.2}$$

f is said to be differentiable if its derivative exists. A useful formula is

$$f^{\sigma} = f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t). \tag{2.3}$$

If f, g are differentiable, then fg and the quotient f/g (where $gg^{\sigma} \neq 0$) are differentiable with

$$(fg)^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta} = f g^{\Delta} + f^{\Delta} g^{\sigma} \tag{2.4}$$

and

$$\left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta} g - f g^{\Delta}}{g g^{\sigma}}. \tag{2.5}$$

If $f^{\Delta}(t) \geq 0$, then f is nondecreasing.

A function $f : [a, b] \to \mathbf{R}$ is said to be right-dense continuous if it is right continuous at each right-dense point, and there exists a finite left limit at all left-dense points.

A function $f : \mathbf{T} \to \mathbf{R}$ is called regressive if $1 - \mu(t) f(t) \neq 0$ for all $t \in \mathbf{T}$. The set of all functions $f : \mathbf{T} \to \mathbf{R}$ which are regressive and rd-continuous will be denoted by $\mathbf{R} = \mathbf{R}(\mathbf{T}) = \mathbf{R}(\mathbf{T}, \mathbf{R})$. We define the set $\mathbf{R}^{+}$ of all positively regressive elements of $\mathbf{R}$ by $\mathbf{R}^{+} = \{f \in \mathbf{R} : 1 - \mu(t) f(t) \neq 0,\ t \in \mathbf{T}\}$.

A function F with $F^{\Delta} = f$ is called an antiderivative of f and then we define

$$\int_a^b f(t) \Delta t = F(b) - F(a) \tag{2.6}$$

where $a, b \in \mathbf{T}$. It is well known that rd-continuous functions possess antiderivatives. A simple consequence of formula (2.3) is

$$\int_{\tau}^{\sigma(t)} f(s) \Delta s = \mu(t) f(t) \tag{2.7}$$

and infinite integrals are defined as

$$\int_a^{\infty} f(t) \Delta t = \lim_{b \to \infty} \int_a^b f(t) \Delta t. \tag{2.8}$$

If $p \in \mathbf{R}$, then the exponential function is defined in [**8**] as

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\nu)}(-p(v))\Delta v\right), \tag{2.9}$$

for all $s,t \in \mathbf{T}$, where $\xi_h(z)$ is the cylinder transformation, which is given by

$$\xi_h = \begin{cases} (\log(1+hz))/h & h \neq 0, \\ z & h = 0. \end{cases} \tag{2.10}$$

For properties of this exponential function, see [**2**].

Before stating the main results in this paper, we need the following lemmas.

Lemma 1 (see [**8, 10**]). *If*

$$y^{\Delta}(t) + \left(\sum_{i=1}^{n} p_i(t)\right)y(t) \leq 0,$$

where $t \in [t_0,\infty)$, $p_i(t) > 0$, $i = 1,2,\dots,n$, *then the following inequality holds*:

$$y(t) \leq y(t_0)\exp\left\{\int_{t_0}^t \xi_{\mu(s)}\left(-\sum_{i=1}^{n} p_i(s)\right)\Delta s\right\}. \tag{2.11}$$

Lemma 2. *If* $t \in [t_0,\infty), p_i(t) > 0, \tau_i(t) < t$, $i = 1,2,\dots,n$, *and* $\mu(t)\sum_{i=1}^n p_i(t) < 1$, *we have*

$$1 - \sum_{i=1}^{n}\int_{\tau_i(t)}^{t} p_i(s)\Delta s \leq \frac{\sum_{i=1}^{n} p_i(t)}{\sum_{i=1}^{n} p_i(t)\exp\left(-\int_{\tau_i(t)}^{t}\xi_{\mu(s)}(-p_i(s))\Delta s\right)}.$$

Proof. From [**10**, Lemma 3], we have

$$1 - \int_{\tau^*(t_1)}^{t}\left(\sum_{i=1}^{n} p_i(s)\right)\Delta s \leq \exp\left(\int_{\tau^*(t_1)}^{t}\xi_{\mu(s)}\left(-\sum_{i=1}^{n} p_i(s)\right)\Delta s\right), \tag{2.12}$$

where

$$t \in \bigcap_{i=1}^{n} [\tau_i(t_1), t] = [\tau^*(t_1), t], \quad t > t_1 > \tau^*(t_1).$$

Since

$$1 - \sum_{i=1}^{n} \int_{\tau_i(t)}^{t} p_i(s)\Delta s \le 1 - \int_{\tau^*(t_1)}^{t} \Big(\sum_{i=1}^{n} p_i(s) \Big) \Delta s, \tag{2.13}$$

and

$$\exp \int_{\tau^*(t_1)}^{t} \xi_{\mu(s)} \Big(-\sum_{i=1}^{n} p_i(s) \Big) \Delta s \le \frac{\sum_{i=1}^{n} p_i(t)}{\sum_{i=1}^{n} p_i(t) \exp \Big(-\int_{\tau_i(t)}^{t} \xi_{\mu(s)} \Big(-\sum_{i=1}^{n} p_i(s) \Big) \Delta s \Big)}. \tag{2.14}$$

From (2.12), (2.13) and (2.14) the result of this lemma follows.

3. Main results. In this section, we obtain a sufficient condition for the oscillation and another sufficient condition for nonoscillation for equation (1.1), by extending the technique in [**10**] to be suitable for first order delay dynamic equations with several delays. Moreover, we get a new sufficient condition for the oscillation of equation (1.2).

In the following theorem we get a sufficient condition for the existence of positive solutions of equation (1.1).

Theorem 3.1. *Assume that H_1, H_2 hold and that there exists*

$$\lambda \in E = \Big\{ \lambda > 0 : 1 - \lambda \Big(\sum_{i=1}^{n} p_i(t) \Big) \mu(t) > 0 \Big\}$$

and $A > 0$ such that

$$\frac{\lambda \sum_{i=1}^{n} p_i(t)}{\sum_{i=1}^{n} p_i(t) \exp\Big(- \int_{\tau_i(t)}^{t} \xi_{\mu(s)}\Big(- \lambda \sum_{i=1}^{n} p_i(s)\Big) \Delta s\Big)} \geq 1, \tag{3.1}$$

where $t > A$. Then equation (1.1) *has a positive solution.*

Proof. We shall construct a positive solution of equation (1.1) as follows. Define

$$x(t) = \begin{cases} 1 & t < A, \\ \dfrac{\sum_{i=1}^{n} p_i(t) \exp\Big(- \int_{\tau_i(t)}^{t} \xi_{\mu(s)}\Big(- \lambda \sum_{i=1}^{n} p_i(s)\Big) x(s)\Delta s\Big)}{\lambda \sum_{i=1}^{n} p_i(t)} & t > A. \end{cases} \tag{3.2}$$

Then $0 < x(t) \leq 1$.

Let

$$z(t) = 1 - \sum_{i=1}^{n} p_i(t) \exp\Big(- \int_{\tau_i(t)}^{t} \xi_{\mu(s)}\Big(- \lambda \sum_{i=1}^{n} p_i(s)\Big) x(s)\Delta s\Big). \tag{3.3}$$

Then

$$z(t) = 1 - \lambda\Big(\sum_{i=1}^{n} p_i(t)\Big) x(t). \tag{3.4}$$

Since $x(t) \in (0, 1]$, $\lambda \in E$, we have $z(t) > 0$.

Define

$$y(t) = \begin{cases} 1 & t < A, \\ \exp\Big(\int_{A}^{t} \xi_{\mu(s)}(z(s) - 1)\Delta s\Big) & t \geq A. \end{cases} \tag{3.5}$$

Then $y(t) > 0$,

$$y(t) = \exp\Big(\int_{A}^{t} \xi_{\mu(s)}\Big(- \lambda \sum_{i=1}^{n} p_i(s)\Big) x(s)\Big) \Delta s. \tag{3.6}$$

From the definition

$$y^{\Delta}(t) = (z(t)-1)y(t), \quad t \geq A. \tag{3.7}$$

From (3.3) and (3.7), we obtain

$$\begin{aligned} 1+\frac{y^{\Delta}(t)}{y(t)} &= z(t) \\ &= 1-\sum_{i=1}^{n} p_i(t)\exp\left(\left(-\int_{\tau_i(t)}^{t}\xi_{\mu(s)}\left(-\lambda\sum_{i=1}^{n}p_i(s)\right)x(s)\right)\Delta s\right) \\ &= 1-\sum_{i=1}^{n}p_i(t)\left(\frac{y(\tau_i(t))}{y(t)}\right), \end{aligned}$$

which implies that

$$y^{\Delta}(t)+\sum_{i=1}^{n}p_i(t)y(\tau_i(t)) = 0.$$

Thus, we obtain a positive solution of equation (1.1). The proof is complete. □

In the following theorem we get a necessary condition for the existence of positive solutions of equation (1.1).

Theorem 3.2. *Assume that equation* (1.1) *has an eventually positive solution and* H_1, H_2 *hold. Then the following inequality holds*:

$$\limsup_{t_0\to\infty}\sup_{t>t_0}\sup_{\lambda\in E}\left\{\frac{\lambda\sum_{i=1}^{n}p_i(t)}{\sum_{i=1}^{n}p_i(t)\exp\left(-\int_{\tau_i(t)}^{t}\xi_{\mu(s)}\left(-\lambda\sum_{i=1}^{n}p_i(s)\right)\Delta s\right)}\right\} \geq 1. \tag{3.8}$$

Proof. Let $y(t)$ be an eventually positive solution of equation (1.1). Then

$$y^{\Delta}(t) = -\sum_{i=1}^{n}p_i(t)y(\tau_i(t)) < 0, \tag{3.9}$$

and consequently $y(t) \leq y(\tau_i(t))$, $i = 1, 2, \dots, n$, eventually hold. Hence,

$$y^{\Delta}(t) + \Big(\sum_{i=1}^{n} p_i(t)\Big) y(t) \leq 0. \tag{3.10}$$

Define

$$\Lambda = \Big\{\lambda > 0 : \lambda^{\Delta}(t) + \lambda\Big(\sum_{i=1}^{n} p_i(t)\Big) y(t) \leq 0\Big\}. \tag{3.11}$$

Clearly, $1 \in \Lambda$. Because

$$\begin{aligned} 0 &\geq \int_t^{\sigma(t)} y^{\Delta}(s)\Delta s + \lambda \int_t^{\sigma(t)} \Big(\sum_{i=1}^{n} p_i(s)\Big) y(s)\Delta s \\ &= y(\sigma(t)) - y(t) + \lambda\Big(\sum_{i=1}^{n} p_i(t)\Big) y(t)\mu(t) \\ &> -y(t) + \lambda\Big(\sum_{i=1}^{n} p_i(t)\Big) y(t)\mu(t), \end{aligned}$$

which implies that $\lambda \in E$. Therefore, $\Lambda \subset E$. So E is nonempty. Define

$$f(t,\lambda) = \frac{\lambda \sum_{i=1}^{n} p_i(t)}{\sum_{i=1}^{n} p_i(t) \exp\Big(- \int_{\tau_i(t)}^{t} \xi_{\mu(s)}\Big(- \lambda \sum_{i=1}^{n} p_i(s)\Big)\Delta s\Big)}. \tag{3.12}$$

Assume that (3.8) is not true, which implies that

$$\lim_{t_0 \to \infty} \sup_{t > t_0} \sup_{\lambda \in E} f(t,\lambda) < 1. \tag{3.13}$$

Then there exist t_0 and a positive constant $c > 1$ such that $f(t,\lambda) < 1/c$ for all $t > t_0$. For any $\lambda \in \Lambda$, by Lemma 1 there is t_1 such that

$$y(t) \leq y(\tau_i(t)) \exp\Big(\int_{\tau_i(t)}^{t} \xi_{\mu(s)}\Big(- \lambda \sum_{i=1}^{n} p_i(s)\Big)\Delta s\Big) \tag{3.14}$$

i.e.,

$$\frac{y(\tau_i(t))}{y(t)} \geq \exp\left(-\int_{\tau_i(t)}^{t} \xi_{\mu(s)}\Big(-\lambda \sum_{i=1}^{n} p_i(s)\Big)\Delta s\right) \tag{3.15}$$

and then

$$\begin{aligned}\frac{\sum_{i=1}^{n} p_i(t)y(\tau_i(t))}{y(t)} &\geq \sum_{i=1}^{n} p_i(t)\exp\left(-\int_{\tau_i(t)}^{t} \xi_{\mu(s)}\Big(-\lambda \sum_{i=1}^{n} p_i(s)\Big)\Delta s\right) \\ &= \frac{\lambda\sum_{i=1}^{n} p_i(t)}{f(t,\lambda)} \geq \lambda c\Big(\sum_{i=1}^{n} p_i(t)\Big),\end{aligned}$$

which implies that

$$\sum_{i=1}^{n} p_i(t)y(\tau_i(t)) \geq \lambda c\Big(\sum_{i=1}^{n} p_i(t)\Big)y(t), \tag{3.16}$$

and then

$$0 = y^{\Delta}(t) + \sum_{i=1}^{n} p_i(t)y(\tau_i(t)) \geq y^{\Delta}(t) + \lambda c\Big(\sum_{i=1}^{n} p_i(t)\Big)y(t),$$

i.e.,

$$y^{\Delta}(t) + \lambda c\Big(\sum_{i=1}^{n} p_i(t)\Big)y(t) \leq 0,$$

and consequently $\lambda c \in \Lambda$.

Following the above analysis, there is t_1 such that for any $t > t_1$,

$$y(t) \leq y(\tau_i(t))\exp\left(\int_{\tau_i(t)}^{t} \xi_{\mu(s)}\Big(-\lambda c \sum_{i=1}^{n} p_i(s)\Big)\Delta s\right),$$

i.e.,

$$\frac{y(\tau_i(t))}{y(t)} \geq \exp\left(-\int_{\tau_i(t)}^{t} \xi_{\mu(s)}\Big(-\lambda c \sum_{i=1}^{n} p_i(s)\Big)\Delta s\right)$$

and then

$$\frac{\sum_{i=1}^{n} p_i(t)y(\tau_i(t))}{y(t)} \geq \sum_{i=1}^{n} p_i(t)\exp\Big(-\int_{\tau_i(t)}^{t} \xi_{\mu(s)}\Big(-\lambda c\sum_{i=1}^{n} p_i(s)\Big)\Delta s\Big)$$
$$\geq \lambda c^2\Big(\sum_{i=1}^{n} p_i(t)\Big).$$

Repeating the above procedure, we have, for any positive k, $\lambda c^{k-1} \in \Lambda$ and there exists t_k such that for any $t > t_k$,

$$y(t) \leq y(\tau_i(t))\exp\Big(\int_{\tau_i(t)}^{t} \xi_{\mu(s)}\Big(-\lambda c^{k-1}\sum_{i=1}^{n} p_i(s)\Big)\Delta s\Big)$$

and

$$\frac{\sum_{i=1}^{n} p_i(t)y(\tau_i(t))}{y(t)} \geq \lambda c^k\Big(\sum_{i=1}^{n} p_i(t)\Big). \tag{3.17}$$

Therefore,

$$\frac{\sum_{i=1}^{n} p_i(t)y(\tau_i(t))}{y(t)} \underset{t\to\infty}{\longrightarrow} \infty. \tag{3.18}$$

On the other hand, from Lemma 2 we have
(3.19)

$$1-\sum_{i=1}^{n}\int_{\tau_i(t)}^{t} p_i(s)\Delta s \leq \frac{\sum_{i=1}^{n} p_i(t)}{\sum_{i=1}^{n} p_i(t)\exp\Big(-\int_{\tau_i(t)}^{t} \xi_{\mu(s)}\Big(-\lambda\sum_{i=1}^{n} p_i(s)\Big)\Delta s\Big)}.$$

Since $f(t,1) < 1/c$, then

$$\frac{\sum_{i=1}^{n} p_i(t)}{\sum_{i=1}^{n} p_i(t)\exp\Big(-\int_{\tau_i(t)}^{t} \xi_{\mu(s)}\Big(-\lambda\sum_{i=1}^{n} p_i(s)\Big)\Delta s\Big)} < \frac{1}{c}, \tag{3.20}$$

and consequently,

$$\sum_{i=1}^{n}\int_{\tau_i(t)}^{t} p_i(s)\Delta s \geq 1-\frac{1}{c} > 0. \tag{3.21}$$

Let $1-1/c=\delta$ and $\sum_{i=1}^{n}\int_{\tau_i(t)}^{t} p_i(s)\Delta s = w(t)$. Define

$$g(r)\equiv \sum_{i=1}^{n}\int_{\tau_i(t)}^{r} p_i(s)\Delta s - \frac{1}{2}w(t), \quad t\geq r>\tau_i(t), \quad i=1,2,\dots,n,$$

$$A_1=\Big\{r\in\bigcup_{i=1}^{n}[\tau_i(t),t]\cap\mathbf{T}: g(r)\leq 0\Big\},$$
$$A_2=\Big\{r\in\bigcup_{i=1}^{n}[\tau_i(t),t]\cap\mathbf{T}: g(r)\geq 0\}.$$

It is easy to see that:

(i) $g(\cdot)$ is nondecreasing,

(ii) $\tau_i(t)\in A_1$, $t\in A_2$, $A_1\cup A_2=\cup_{i=1}^{n}[\tau_i(t),t]\cap\mathbf{T}$,

(iii) for any $r_1\in A_1$, $r_2\in A_2$, we have $r_1\leq r_2$.

Consequently, we have that there exists $t^*\in\cup_{i=1}^{n}[\tau_i(t),t]\cap\mathbf{T}$ such that

$$g(t^*)g(\sigma(t^*))\leq 0.$$

From (ii), we obtain that $t^*\in A_1$, $\sigma(t^*)\in A_2$ and for any $r_1\in A_1$, $r_2\in A_2$, we have $r_1\leq t^*$, $\sigma(t^*)\leq r_2$.

If

$$\sum_{i=1}^{n}\int_{t^*}^{\sigma(t^*)} p_i(s)\Delta s = \mu(t^*)\left(\sum_{i=1}^{n} p_i(t^*)\right)\geq \frac{\delta}{3}, \tag{3.22}$$

then integrating equation (1.1) from t^* to $\sigma(t^*)$, we have

$$y(\sigma(t^*))-y(t^*)+\mu(t^*)\sum_{i=1}^{n} p_i(t^*)y(\tau_i(t^*))=0.$$

Because $y(\sigma(t^*)) > 0$, we have

$$\mu(t^*)\sum_{i=1}^{n} p_i(t^*)(\tau_i(t^*)) < y(t^*),$$

i.e.,

$$\frac{\sum_{i=1}^{n} p_i(t^*)y(\tau_i(t^*))}{y(t^*)} < \frac{1}{\mu(t^*)}, \tag{3.23}$$

and consequently, from (3.17) and (3.23), we obtain

$$\lambda c^k \le \frac{\sum_{i=1}^{n} p_i(t^*)y(\tau_i(t^*))}{y(t^*)\left(\sum_{i=1}^{n} p_i(t^*)\right)} < \frac{1}{\mu(t^*)\left(\sum_{i=1}^{n} p_i(t^*)\right)} < \frac{3}{\delta},$$

which is a contradiction, because $c > 1$.

Now, we consider the case that

$$\sum_{i=1}^{n}\int_{t^*}^{\sigma(t^*)} p_i(s)\Delta s < \frac{\delta}{3} \le \frac{w(t)}{3}.$$

If $\mu(t^*) = 0$, then

$$\sum_{i=1}^{n}\int_{t^*}^{\sigma(t^*)} p_i(s)\Delta s = \mu(t^*)\left(\sum_{i=1}^{n} p_i(t^*)\right) = 0.$$

Then we have

$$\begin{aligned} &\sum_{i=1}^{n}\int_{\tau_i(t)}^{t^*} p_i(s)\Delta s = \frac{w(t)}{2} > \frac{w(t)}{3}, \\ &\sum_{i=1}^{n}\int_{\tau_i(t)}^{t^*} p_i(s)\Delta s = \frac{w(t)}{2} > \frac{w(t)}{3}, \\ &t^* \in \bigcup_{i=1}^{n} [\tau_i(t), t] \cap \mathbf{T}. \end{aligned} \tag{3.24}$$

If $\mu(t^*) > 0$, we can also obtain (3.24); otherwise,

$$\sum_{i=1}^{n} \int_{\tau_i(t)}^{t^*} p_i(s)\Delta s < \frac{w(t)}{3}$$

or

$$\sum_{i=1}^{n} \int_{\tau_i(t)}^{t^*} p_i(s)\Delta s < \frac{w(t)}{3}, \quad t^* \in \bigcup_{i=1}^{n} [\tau_i(t), t] \cap \mathbf{T}.$$

If

$$\sum_{i=1}^{n} \int_{\tau_i(t)}^{t^*} p_i(s)\Delta s < \frac{w(t)}{3},$$

then for any $r_1 \in A_1$,

$$\sum_{i=1}^{n} \int_{\tau_1(t)}^{r_1} p_i(s)\Delta s < \frac{w(t)}{3}.$$

If

$$\sum_{i=1}^{n} \int_{\tau_i(t)}^{t^*} p_i(s)\Delta s < \frac{w(t)}{3},$$

then for any $r_2 \in A_2$,

$$\sum_{i=1}^{n} \int_{\tau_1(t)}^{r_2} p_i(s)\Delta s > \frac{2w(t)}{3}.$$

Therefore,

$$\left| \sum_{i=1}^{n} \int_{\tau_1(t)}^{r_2} p_i(s)\Delta s - \sum_{i=1}^{n} \int_{\tau_i(t)}^{r_1} p_i(s)\Delta s \right| = \sum_{i=1}^{n} \int_{r_1}^{r_2} p_i(s)\Delta s > \frac{w(t)}{6},$$

so we have

$$\sum_{i=1}^{n} \int_{t^*}^{\sigma(t^*)} p_i(s)\Delta s > \frac{w(t)}{6},$$

which is a contradiction. Therefore, (3.24) holds.

Integrating (1.1) from $\tau^*(t)$ to t^{**}, where

$$t^{**} \in \bigcap_{i=1}^{n} [\tau_i(t), t] \cap \mathbf{T} = [\tau^*(t), t],$$

we obtain

$$y(t^{**}) - y(\tau^*(t)) + \sum_{i=1}^{n} \int_{\tau^*(t)}^{t^{**}} p_i(s) y(\tau_i(s)) \Delta s = 0,$$

which implies that

$$y(\tau^*(t)) \geq y(\tau^*(t^{**})) \int_{\tau^*(t)}^{t^{**}} \sum_{i=1}^{n} p_i(s) \Delta s \geq \frac{\delta}{3} y(\tau^*(t^{**})). \tag{3.25}$$

Integrating (1.1) from t^{**} to t, we get

$$y(t) - y(t^{**}) + \sum_{i=1}^{n} \int_{t^{**}}^{t} p_i(s) y(\tau_i(s)) \Delta s = 0,$$

which implies that

$$y(t^{**}) \geq y(\tau^*(t)) \int_{t^{**}}^{t} \sum_{i=1}^{n} p_i(s) \Delta s \geq \frac{\delta}{3} y(\tau^*(t)). \tag{3.26}$$

From (3.25) and (3.26), we obtain

$$y(t^{**}) \geq \left(\frac{\delta}{3}\right)^2 y(\tau^*(t^{**})). \tag{3.27}$$

From (3.17) we have

$$\lambda c^k \leq \frac{\sum_{i=1}^{n} p_i(t) y(\tau_i(t))}{y(t)\left(\sum_{i=1}^{n} p_i(t)\right)} \leq \frac{y(\tau^*(t^{**}))}{y(t^{**})}; \tag{3.28}$$

hence, from (3.27) and (3.28) we get

$$\lambda c^k \leq \frac{y(\tau^*(t^{**}))}{y(t^{**})} \leq \left(\frac{3}{\delta}\right)^2,$$

which is a contradiction, because $c > 1$ and k can be taken to be a sufficiently large integer. Therefore,

$$\lim_{t_0\to\infty} \sup_{t>t_0} \sup_{\lambda\in E} f(t,\lambda) \geq 1.$$

The proof is complete. □

Corollary 3.1. *If*

(3.29)

$$\lim_{t_0\to\infty} \sup_{t>t_0} \sup_{\lambda\in E} \left\{ \frac{\lambda \sum_{i=1}^{n} p_i(t)}{\sum_{i=1}^{n} p_i(t) \exp\left(-\int_{\tau_i(t)}^{t} \xi_{\mu(s)}\left(-\lambda \sum_{i=1}^{n} p_i(s)\right) \Delta s\right)} \right\} < 1,$$

then all solutions of equation (1.1) *are oscillatory.*

Remark 3.1. We note that, if $\mathbf{T} = \mathbf{R}$, equation (1.1) becomes the first order delay differential equation (1.2) with several delays. From this, we have

$$\lim_{t_0\to\infty} \sup_{t>t_0} \sup_{\lambda\in E} \left\{ \frac{\lambda \sum_{i=1}^{n} p_i(t)}{\sum_{i=1}^{n} p_i(t) \exp\left(\int_{\tau_i(t)}^{t} \lambda \sum_{i=1}^{n} p_i(s)\right) ds} \right\} < 1$$

where $E = \mathbf{R}^+$. Let

$$g(\lambda,t) = \frac{\lambda \sum_{i=1}^{n} p_i(t)}{\sum_{i=1}^{n} p_i(t) \exp\left(\int_{\tau_i(t)}^{t} \lambda \sum_{i=1}^{n} p_i(s)\right) ds}.$$

Then

$$\sup_{\lambda\in E} g(\lambda,t)=\frac{\sum_{i=1}^{n}p_i(t)}{e\sum_{i=1}^{n}p_i(t)\int_{\tau_i(t)}^{t}\sum_{i=1}^{n}p_i(s)\,ds}<1;$$

therefore, if

$$\liminf_{t\to\infty}\frac{\sum_{i=1}^{n}p_i(t)\int_{\tau_i(t)}^{t}\sum_{i=1}^{n}p_i(s)\,ds}{\sum_{i=1}^{n}p_i(t)}>\frac{1}{e}, \tag{3.30}$$

then all solutions of equation (1.2) are oscillatory. Condition (3.30) is a new criteria for oscillation of equation (1.2).

In the following we give an example to show the importance of the above result.

Example. Consider the delay differential equation

$$y'(t)+\frac{1}{3e}(1+\cos t)y(t-\pi)+\frac{1}{15e}(1+\sin t)y(t-2\pi)=0,\quad t\geq 0, \tag{3.31}$$

i.e.,

$$p_1(t)=\frac{1}{3e}(1+\cos t),\quad \tau_1=\pi,$$
$$p_2(t)=\frac{1}{15e}(1+\sin t),\quad \tau_2=2\pi.$$

We have

(1)

$$\liminf_{t\to\infty}\left(\int_t^{t+\pi}\frac{1}{3e}(1+\cos\zeta)\,d\zeta+\int_t^{t+2\pi}\frac{1}{15e}(1+\sin\zeta)\,d\zeta\right)=\frac{1}{15e}(7\pi-10)<\frac{1}{e}.$$

This shows that the well-known sufficient condition

$$\liminf_{t\to\infty}\sum_{i=1}^{n}\int_t^{t+\tau_i}p_i(s)\,ds>\frac{1}{e}$$

does not hold.

(2)

$$\liminf_{t\to\infty}\left(\left[\frac{1}{3e}(1+\cos t)\right]\left[\frac{1}{15e}(1+\sin t)\right]\right)^{1/2}(3\pi)<\frac{1}{e}.$$

This shows that the well-known sufficient condition

$$\liminf_{t\to\infty}\left(\prod_{i=1}^{n}p_i(t)\right)^{1/n}\sum_{i=1}^{n}\tau_i>\frac{1}{e}$$

does not hold.

(3)

$$\limsup_{t\to\infty}\int_{t-\pi}^{t}\left\{\frac{1}{3e}(1+\cos\zeta)+\frac{1}{15e}(1+\sin\zeta)\right\}d\zeta$$
$$=\frac{1}{15e}(6\pi-\sqrt{26})<1.$$

This shows that the well-known sufficient condition

$$\limsup_{t\to\infty}\int_{t-\tau_{\min}}^{t}\sum_{i=1}^{n}p_i(s)\,ds>1$$

does not hold.

(4)

$$\liminf_{t\to\infty}\left(\frac{\pi}{3e}(1+\cos t)+\frac{2\pi}{15e}(1+\sin t)\right)=\frac{\pi}{15e}(7-\sqrt{29})<\frac{1}{e}.$$

This shows that the well-known sufficient condition

$$\liminf_{t\to\infty}\sum_{i=1}^{n}\tau_i p_i(t)>\frac{1}{e}$$

does not hold. But, according to (3.30), one can write

$$p_1(t)+p_2(t)=\frac{1}{15e}(6+5\cos t+\sin t)$$

$$\liminf_{t\to\infty}\left(\frac{p_1(t)\int_{t-\pi}^{t}(p_1(s)+p_2(s))\,ds+p_2(t)\int_{t-2\pi}^{t}(p_1(s)+p_2(s))\,ds}{p_1(t)+p_2(t)}\right)$$
$$=4.8021>\frac{1}{e}.$$

This shows that every solution of (3.31) oscillates.

Remark 3.2. Equation (3.31) is also oscillatory, since the well-known sufficient condition

$$\liminf_{t\to\infty}\int_{t-\tau_{\max}}^{t}\sum_{i=1}^{n}p_i(s)\,ds>\frac{1}{e}$$

holds. Since

$$\liminf_{t\to\infty}\int_{t-2\pi}^{t}(p_1(s)+p_2(s))\,ds=\frac{4\pi}{5e}>\frac{1}{e}.$$

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