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ON THE PIERCE-BIRKHOFF CONJECTURE OVER ORDERED FIELDS

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1. Introduction. Let R be a real closed field and let K be a subfield of R with the inherited order. Endow R with the usual order topology (generated by the open intervals), and R^n with the usual product topology. Let $X := (X_1, \ldots, X_n)$ be n indeterminates, and let $x = (x_1, \ldots x_n) \in R^n$. For $m \geq 1$ let $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_m) \in \{\langle\ ,\ \rangle, =\}^m$ be a sequence of m relations chosen from $\langle\ ,\ \rangle$, and $=$. For any set $A \subseteq R^n$, a *basic semi-algebraic* (s.a.) *subset* of A is a set of the form $\cap_{i=1}^m \{x \in A \mid a_i(x)\varepsilon_i 0\}$, for some ε, where $m \geq 1$ and $a_i \in K[X]$. An *s.a. subset* of A is a finite union of basic s.a. subsets of A. An *s.a. set* is an s.a. subset of R^n. We call a function $h : A \to R$ *piecewise-polynomial* (p.p.) over K if there exist $g_1, \ldots, g_l \in K[X]$ such that the subsets $A_i = \{x \in A \mid h(x) = g_i(x)\}$ are s.a. and cover A, i.e., $A = \cup_i A_i$. If we call a function h simply "p.p.," then h is understood to be "p.p. over K." Write $\mathrm{PWP}(A)$ for the set of continuous p.p. functions on A; $\mathrm{PWP}(A)$ is clearly closed under sums and products, and so is a ring; it is also closed under pointwise suprema and infima.

CONJECTURE 1.1. *If $h : R^n \to R$ is continuous and piecewise-polynomial (i.e., if $h \in \mathrm{PWP}(R^n)$), then h is a* sup *of* infs *of finitely many polynomial functions (i.e.,*

$$h = \sup_j \inf_k f_{jk}, \tag{1.1.1}$$

for some finite number of $f_{jk} \in K[X]$.) (The converse is easy.)

A simple example is $h(X_1) = |X_1| = \sup\{X_1, -X_1\}$.

Writing $\mathrm{SIPD}(A)$ for the set of "sup-inf-polynomially-definable" functions $h : A \to R$ defined as in (1.1.1), we see that Conjecture 1.1 asserts that the obvious inclusion $\mathrm{SIPD}(R^n) \subseteq \mathrm{PWP}(R^n)$ can be reversed.

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Note that, for $A \subseteq B \subseteq R^n$, any function defined sup-inf-polynomially on A extends sup-inf-polynomially to B. This is not true of PWP(A) and PWP(B) if $n \geq 2$ and A and B are properly chosen:

EXAMPLE 1.2. [**12**] Let

$$A = \{(x_1, x_2) \in R^2 \mid x_1 < 0 \text{ or } x_2 < 0 \text{ or } x_2 > x_1^2\},$$

$B = R^2$, and let $g : A \to R$ be defined by

$$g(x_1, x_2) = \begin{cases} x_1 & \text{if } x_1 > 0 \text{ and } x_2 > x_1^2, \text{ and} \\ 0 & \text{elsewhere.} \end{cases}$$

Then $g \in$ PWP(A), but g does not extend to a function $\overline{g} \in$ PWP(R^2), since such a $\overline{g}$ must have unbounded partial derivative $\partial\overline{g}/\partial X_2$ as $(x_1, x_2) \in R^2 \backslash A$ approaches $(0,0)$. As a consequence, the obvious inclusion SIPD(A) $\subseteq$ PWP(A) can*not* be reversed for all A. J. Madden has suggested extending Conjecture 1.1 by asking whether PWP(A) = SIPD(A) under the hypothesis that A be a convex and/or compact regular algebraic subset of R^n (the above set A is neither).

Conjecture 1.1 was proposed by G. Birkhoff and R.S. Pierce in [**2**]; however, their formulation was slightly incorrect, or at least vague. Apparently it was Isbell who clarified and formulated it as stated above, at least in the case $K = R = \mathbf{R}$. Although the conjecture arose out of lattice theory, its proof or disproof would probably use little lattice theory, but mainly semi-algebraic geometry instead.

The Pierce-Birkhoff conjecture has received active interest in recent years. Henriksen and Isbell [**7**] showed that SIPD(A) is closed under addition and multiplication (see Lemma 2.1), and so is a ring. Mahé [**12**] and, independently, Efroymson (unpublished) proved Conjecture 1.1 for $n \leq 2$, but only in the case where K (and not just R) is real closed (specifically, $K = R = \mathbf{R}$). Madden [**9**] has announced a proof that the rings SIPD(R^n) and PWP(R^n) have homeomorphic "Brumfiel spectra" (= prime convex ideal spectra). He also gave two conjectures [**9**, **10**] equivalent to the Pierce-Birkhoff conjecture, in terms of total orders on, and ideals in, $R[X]$. In [**11**], he used these to give a new proof of the Pierce-Birkhoff conjecture for $n \leq 2$, but again only for real closed K.

In this note we shall extend Mahé's 2-variable methods to arbitrary (not just real closed) subfields K of R:

THEOREM 1.3. Conjecture 1.1 *is true for* $n \leq 2$.

One way to view the Pierce-Birkhoff conjecture is to say that it asserts that if a p.p. function h is continuous (a property which may be hard to check if h is presented by giving many A_i and g_i), then it can be represented in a form which makes this continuity *obvious*, since a sup of infs of finitely many continuous functions is obviously continuous. Viewed this way, the Pierce-Birkhoff conjecture is in the same spirit as some other results in real algebraic geometry, such as: (1) The various "Stellensätze," which say that if a polynomial $f \in K[X]$ is (a) zero, (b) positive semidefinite (psd), or (c) positive definite, respectively, on a basic closed s.a. subset $W \cap V$ of a real algebraic set V (properties which may be hard to check if f is presented in the usual way as a sum of monomials), then f can be represented in a way which makes (a), (b), or (c), respectively, obvious (see, for example, [**8**, §7]). (A simple case of (b) is $W = V = R^n$; then the "Nichtnegativstellensatz" is Artin's solution to Hilbert's 17^{th} problem, which represents f in the form $\sum_j c_j r_j^2$, where $0 \leq c_j \in K$ and $r_j \in K(X)$, making clear that f is psd.) (2) Another representation theorem in this spirit is the finiteness theorem for open s.a. sets, which says that if an s.a. set A is open (a property which may be hard to check if A is presented in the usual way as a finite union of basic—not—necessarily open–s.a. sets), then A can be represented in a way which makes this openness obvious, namely as a finite union of basic s.a. sets in each of which the only relations ε_i which occur are either $<$ or $>$, and not $=$ (see, for example, [**4**]).

There are three reasons why, in Conjecture 1.1 and Theorem 1.3, we have added the condition that the coefficients of the f_{jk} come from the same subfield K of R from which the coefficients of the polynomials defining h come: (1) The new method of proof may help prove Conjecture 1.1 for $n > 2$. (2) This refinement has also been made to other results in real algebraic geometry, such as Artin's solution to Hilbert's 17^{th} problem, the Tarski-Seidenberg theorem, and the finiteness theorem for open semi-algebraic sets. (3) The third reason is that it might help improve our continuous solution [**5**] to Hilbert's 17^{th} problem, which we now review. Here we can let $K = \mathbf{Q}$. A function from one s.a. set to another is called *s.a.* if its graph, in the product space, is s.a. Let $f \in \mathbf{Z}[C;X]$ be the general polynomial of even degree d in X with coefficients $C := (C_k)$, $1 \le k \le \binom{n+d}{n}$, and let

$$P_{nd} := \{c \in R^{\binom{n+d}{n}} \mid f(c;X) \text{ is psd over } R \text{ in } X\}.$$

This is a closed, convex, s.a. set. Then we constructed ((5.1) of [**5**]) $s \in \mathbf{N}$ and finitely many continuous s.a. functions $q_j : P_{nd} \to R$ ($q_j \ge 0$) and $a_{ij} : P_{nd} \to R^{m_i}$ such that $\forall c \in P_{nd}$,

$$f(c;X) = \frac{\sum_j q_j(c) f_1(a_{1,j}(c);X)^2}{f(c;X)^{2s} + \sum_j q_j(c) f_2(a_{2,j}(c);X)^2}.$$

Here $m_i = \binom{n+e_i}{n}$ (where $i = 1,2$, $e_1 = ds + d/2$, and $e_2 = ds$) and f_i is the general polynomial of degree e_i in X. The main shortcoming of this result was that if the components of some fixed c lie in a subfield $K \subset R$ which is not real closed, then the $q_j(c)$ and $a_{ij}(c)$ may lie in some proper (real algebraic) extension of K, since the q_j and a_{ij} are only s.a. While we showed that the q_j and a_{ij} may not be chosen to be (even discontinuous) *rational* functions $\in K(C)$ (unless $d \le 2$, where they may be chosen to be polynomial functions $\in K[C]$), we have conjectured [**5**] that they may be chosen to be continuous *piecewise*-rational, or perhaps even p.p., functions defined on P_{nd}. If this is true, and if Conjecture 1.1 (or Conjecture 2.2 below) could be proven also for functions defined on convex domains as Madden suggested in Example 1.2, then we could represent the q_j and a_{ij} as sups of infs of finitely many polynomial (or rational) functions of C with coefficients in $\mathbf{Q}$. Then both the positive semidefiniteness of $f(c;X)$ in X *and* the continuity in C of the q_j and a_{ij} would be obvious; this would be the best possible outcome. Of course, most of this is speculative; but we hope it helps motivate our refinement of the Pierce-Birkhoff conjecture to arbitrary ordered, not necessarily real closed, subfields $K \subseteq R$.

In §2 we shall present those lattice-ordered ring identities of Henriksen and Isbell which show that SIPD(A) is a ring; we shall also consider what can happen when we replace the word "(piecewise-)polynomial" with "(piecewise-)rational". In §3 we shall review Mahé's cylindrical algebraic decomposition of R^n, which represents each continuous p.p. function as a sup-inf-polynomially-definable function, but only within 1 cylinder at a time. Finally, in §4, we shall complete the proof of Theorem 1.3.

I am grateful to James Madden for calling my attention to this problem, and to the referee for several suggestions and corrections.

2. Piecewise-rational functions.

LEMMA 2.1. *For A an arbitrary subset of R^n, $\mathrm{SIPD}(A)$ is a ring.*

PROOF. We must show that $\mathrm{SIPD}(A)$ is closed under differences and products. Closure under $-$ is easy, using identities such as

$$-\sup\{a,b\} = \inf\{-a,-b\},$$
$$a+\sup\{b,c\} = \sup\{a+b,a+c\},$$

and

$$a+\inf\{b,c\} = \inf\{a+b,a+c\}.$$

To show closure under $\cdot$, write $a^+ = \sup\{a,0\}$ and $a^- = \inf\{a,0\}$ for every $a \in \mathrm{SIPD}(A)$, and note that the identities

$$a \sup\{b,c\} = a(c+(b-c)^+),$$
$$a \inf\{b,c\} = a(c+(b-c)^-),$$

and

$$a^- = -(-a)^+$$

reduce our task to showing that $ab^+ \in \mathrm{SIPD}(A)$, for $a,b \in \mathrm{SIPD}(A)$. This follows by successive use of the identities

$$ab^+ = \sup\{\inf\{ab, a^2b+b\}, \inf\{0,-a^2b-b\}\}$$

and

$$\inf\{a,\sup\{b,c\}\} = \sup\{\inf\{a,b\},\inf\{a,c\}\}.$$

Actually, these identities hold in arbitrary "f-rings," and not just in $\mathrm{SIPD}(A)$; see [**7**, 3.2-3.4]. □

For $A \subseteq R^n$, call a function $h : A \to R$ *piecewise-rational* (p.r.) (over K) if there exist rational functions $g_1, \dots g_l \in K(X)$, all defined throughout A, such that the subsets $A_i := \{x \in A \mid h(x) = g_i(x)\}$ are s.a. and cover A, i.e., $A = \cup_i A_i$. Write PWR(A) for the set of continuous p.r. functions; like PWP(A), PWR(A) is a ring.

Let SIRD(A) be the set of "sup-inf-rationally-definable" functions $h : A \to R$ expressible in the form (1.1.1), with each f_{jk} a *rational* function contained in $K(X)$ defined throughout A. Like SIPD(A), SIRD(A) is a ring. Unlike functions in SIPD(A), however, not every function SIRD(A) can be extended to a function in SIRD(B), where $A \subseteq B \subseteq R^n$, already for $n = 1$: take $A = [1, \infty), B = R^1$, and $h = 1/X_1$.

We now investigate conditions under which the obvious inclusion SIRD(A) $\subseteq$ PWR(A) can be reversed. For any subset $I \subseteq K[X]$ write $V(I) = \{x \in R^n \mid \forall f \in I,\ f(x) = 0\}$. For any subset $D \subseteq R^n$ write $I(D) = \{f \in K[X] \mid \forall x \in D,\ f(x) = 0\}$. $I(D)$ is an ideal, and is, of course, finitely generated: $I(D) = (f_1, \ldots, f_s)$, for some $f_i \in I(D)$. Write $(D)_{zc} = V(I(D))$, the Zariski closure of D. Then $(D)_{zc} = V((f_1, \ldots, f_s)) = V((a))$, where $a = f_1^2 + \cdots + f_s^2$. As usual, $\overline{D} \subseteq (D)_{zc}$, where $\overline{D} := \{x \in R^n \mid$ every open n-ball about x meets $D\}$ is the closure of D in the usual topology of R^n. Finally, write $D^\circ = \{x \in R^n \mid \exists$ an open n-ball about x in $D\}$, the interior of D in R^n, and write $\partial D := \{x \in R^n \mid$ every open n-ball about x meets both D and $R^n \backslash D\} = \overline{D} \backslash D^\circ$ for the boundary of D. If D is s.a., then both ∂D and $(\partial D)_{zc}$ are nowhere dense in R^n (by, say, the triangulability of s.a. sets).

CONJECTURE 2.2. *Suppose $A \subseteq R^n$ is s.a. and $(\partial A)_{zc} \cap A = \emptyset$ (so that A is open in R^n). Then* SIRD(A) = PWR(A); *that is, each continuous p.r. function $h : A \to R$ is expressible in the form* (1.1.1), *for finitely many $f_{jk} \in K(X)$, all defined throughout A.* (The inclusion $\subseteq$ is easy.)

It is easy to show, for each fixed n, that Conjecture 1.1 $\Rightarrow$ Conjecture 2.2: Choose $a \in K[X]$ so that $(\partial A)_{zc} = V((a))$, and let $d \in K[X]$ be a least common denominator of the $g_i \in K(X)$ defining h. Then $V((a)) \cap A = \emptyset = V((d)) \cap A$, and $a^2 d^2 h \in$ PWP(A) vanishes on $(\partial A)_{zc} \supseteq \partial A$. It therefore extends continuously, by 0, to a function $\overline{h} \in$ PWP(R^n) (i.e., define $\overline{h}(x) = 0\ \forall x \in R^n \backslash A$). Apply Conjecture 1.1 to $\overline{h}$ and then divide both sides of the resulting (1.1.1) by $a^2 d^2$, which is positive definite on A, so that the order relations between the f_{jk} are unaffected by this division.

EXAMPLE 2.3. Our first illustration of Conjecture 2.2 is simple, and was suggested by the referee: let $n = 1, A = R\backslash\{0\}$, and

$$h(x_1) = \begin{cases} 1 & \text{if } x_1 > 0 \text{ and} \\ -1 & \text{if } x_1 < 0; \end{cases}$$

note that $\partial A = \{0\} = V((X_1))$, so that $(\partial A)_{zc} \cap A = \emptyset$, as hypothesized in Conjecture 2.2. We have $h \in \mathrm{PWP}(A) \subset \mathrm{PWR}(A)$. Since h is not continuously extendible to $R^1, h \notin \mathrm{SIPD}(A)$; but, by Conjecture 2.2 (which is true at least for $n \leq 2$), $h \in \mathrm{SIRD}(A)$: specifically, the Henriksen-Isbell identities in the proof of (2.1) lead from $h(X_1) = |X_1|/X_1$ to

$$h(X_1) = \sup\{\inf\{1, X_1 + 1/X_1\}, \inf\{-1, -X_1 - 1/X_1\}\}.$$

Thus we see that the obvious inclusion $\mathrm{SIRD}(A) \subseteq \mathrm{PWR}(A)$ appears to be reversible, while $\mathrm{SIPD}(A) \subseteq \mathrm{SIRD}(A) \cap \mathrm{PWP}(A)$ is not.

EXAMPLE 2.4. Our second illustration of Conjecture 2.2 is based on Mahé's Example 1.2. Recall that the $h : A \longrightarrow R$ given there belonged to $\mathrm{PWP}(A)$ but not $\mathrm{SIPD}(A)$. Write $a = X_2 - X_1^2$. Note that $(\partial A)_{zc} = V((X_2 a))$, which meets A; thus Conjecture 2.2 does not necessarily apply to this A. However, if we restrict h to $A' := A\backslash V((X_2 a))$, then it does apply. Specifically, we see that $a^2 h$ agrees on A with $\overline{h}(X_1, X_2) = X_1^+(a^+)^2$. By Lemma 2.1, $\overline{h} \in \mathrm{SIPD}(R^2)$; explicitly,

$$\begin{aligned}\overline{h}(X_1, X_2) = \sup\{&\inf\{X_1 a^2, X_1(a^2+1), a(a^2+1)(X_1^2+1)\},\\ &\inf\{0, -X_1(a^2+1), -a(a^2+1)(X_1^2+1)\}\}.\end{aligned}$$

Therefore

$$\begin{aligned} h(X_1, X_2) = \sup\Big\{&\inf\Big\{X_1, \ \frac{X_1(a^2+1)}{a^2}, \ \frac{(a^2+1)(X_1^2+1)}{a}\Big\},\\ &\inf\Big\{0, \ -\frac{X_1(a^2+1)}{a^2}, \ -\frac{(a^2+1)(X_1^2+1)}{a}\Big\}\Big\}\\ &\in \mathrm{SIRD}(A''),\end{aligned}$$

where $A'' := A\backslash V((a))$ is a dense subset of A (containing A').

More generally, note that, given any s.a. set $A \subseteq R^n$ satisfying $\overline{A^\circ} = \overline{A}$, we can find a *dense* subset $A' \subseteq A$ satisfying $(\partial A')_{zc} \cap A' = \emptyset$; namely, let A' be $A\backslash(\partial A)_{zc}$. A' is dense in A because $\overline{A^\circ} = \overline{A}$ and $(\partial A)_{zc}$ is nowhere dense in R^n. To check $(\partial A')_{zc} \cap A' = \emptyset$, note that $(\partial A')_{zc} = (\partial A)_{zc}$.

The referee wondered whether Conjecture 1.1 implies a stronger version of Conjecture 2.2, obtained by weakening the hypothesis $(\partial A)_{zc} \cap A = \emptyset$ to $\partial A \cap A = \emptyset$ (i.e., A is open). Despite Theorem 1.3, this stronger form of Conjecture 2.2 is false already for $n = 1$ if K is not real closed, and for $n = 2$ if K is real closed:

EXAMPLES 2.5. Let $K = \mathbf{Q}$, $A = R\backslash\{\sqrt{2}\}$, and $h(x_1) = 1$ for $x_1 > \sqrt{2}$ and $h(x_1) = 0$ otherwise. A is open and $h \in \mathrm{PWP}(A) \subset \mathrm{PWR}(A)$, but $h \neq \mathrm{SIRD}(A)$, for otherwise one of the $f_{jk} \in K(X_1)$ appearing in (1.1.1) would have to be undefined at $\sqrt{2}$, and hence also at $-\sqrt{2} \in A$. Even if K is real closed, the hypothesis cannot be weakened if $n \geq 2$: let $n = 2$, let A be the plane R^2 slit along the nonnegative X_1-axis, and let

$$h(x_1, x_2) = \begin{cases} x_1 & \text{if } x_1 > 0 \text{ and } x_2 > 0, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Again A is open and $h \in \mathrm{PWP}(A) \subset \mathrm{PWR}(A)$, but $h \notin \mathrm{SIRD}(A)$, for otherwise one of the $f_{jk} \in K(X)$ appearing in (1.1.1) would have to be undefined along a Zariski dense subset of the nonnegative X_1-axis, and hence also along the negative X_1-axis, which lies in A.

But Example 2.5 is not really satisfying, because the h's given there were not continuously extendible to $\overline{A}$. Recall that, in Example 2.4, however, h *was* continuously extendible to $\overline{A}$.

EXAMPLE 2.4. (improved) Using the identity

$$\frac{X_1X_2(X_2+1)}{X_1^2+X_2^2} - X_1 = \frac{X_1(X_2 - X_1^2)}{X_1^2+X_2^2},$$

we see that

$$h(X_1, X_2) = \inf\left\{\frac{X_1^+X_2^+(X_2^+ + 1)}{X_1^2+X_2^2},\ X_1^+\right\}.$$

Thus h is sup-inf-rationally-definable over $R^2\backslash\{(0,0)\} \supset A$, and not just over $A'' \supset A'$.

However, upon adding another dimension to Example 2.4, we shall see that Conjecture 2.2 becomes false if the hypothesis is weakened by replacing $(\partial A)_{zc}$ with ∂A, even if it is at the same time strengthened by requiring that h be continuously extendible to $\overline{A}$.

EXAMPLE 2.6. Let $n = 3$, let $A = \{x \in R^3 \mid x_1 < 0 \text{ or } x_2 < 0 \text{ or } x_3 < 0 \text{ or } x_2 > x_1^2\}$, and let

$$h(x) = \begin{cases} x_1 & \text{when } x_2 > x_1^2, x_1 > 0, \text{ and } x_3 > 1, \\ x_1 x_3 & \text{when } x_2 > x_1^2, x_1 > 0, \text{ and } 0 < x_3 \leq 1, \text{ and} \\ 0 & \text{elsewhere.} \end{cases}$$

Then $h \in \mathrm{PWP}(A) \subset \mathrm{PWR}(A)$, and h extends continuously to $\overline{A}$. But $h \notin \mathrm{SIRD}(A)$, for otherwise the argument in Example 1.2 would show that one of the $f_{jk} \in K(X)$ appearing in (1.1.1) would have to be undefined along a Zariski dense subset of the half-line $\{(0,0,x_3) \mid x_3 \geq 1\}$, and hence also along even the negative X_3-axis, which lies in A.

Therefore, we leave Conjecture 2.2 as it is. In the positive direction, we proved [**6**] in 1987 that, for every $n \in \mathbf{N}$, for every—not necessarily open—s.a. set $A \subseteq R^n$, and for every piecewise-rational function $h : A \to R$ (not necessarily continuous), h can be represented "almost everywhere" in A as in (1.1.1) with rational functions $f_{jk} \in K(X)$ which are *not* necessarily defined throughout A (of course, the set where all the f_{jk} are defined is dense in R^n, though not necessarily dense in A, unless $\overline{A^\circ} = \overline{A}$). After learning of [**6**], Madden found a new proof and a generalization of this result, see [**11**]. This result is surely evidence in favor of Conjectures 1.1 and 2.2; on the other hand, this evidence is weakened by the fact that it has nothing to do with continuity, while Conjecture 1.1 does. Anyway, if Conjecture 1.1 is false for some $n > 2$, then we would have a situation similar to the situation for Hilbert's 17$^{\text{th}}$ problem: for $n < 2$, every positive semidefinite polynomial $f \in R[X]$ can be represented as a sum of squares of polynomials in $R[X]$, while, for $n \geq 2$, such f can, in general, be represented only almost everywhere, as sums of squares of *rational* functions in $R(X)$.

3. Mahé's cylindrical algebraic decomposition. To prove Theorem 1.3 we shall need to review Mahé's cyclindrical algebraic decomposition (Lemma 3.2 and Proposition 3.3 below) of R^n, which applies even to $n > 2$, and not just $n \leq 2$. The reader will notice that we don't need to change significantly Mahé's proof of Lemma 3.2 and Proposition 3.3 in order to generalize it to the case where the subfield $K \subseteq R$ is no longer real closed. Rather, it is later, in §4, where the rest of Mahé's proof of Theorem 1.3 for real closed K needs to be modified significantly to achieve the full generality of Theorem 1.3 for all $K \subseteq R$.

For $p \in \omega := \{0, 1, \ldots\}$ let $K[X]^p$ be the set of sequences $\mathcal{A} := (a_1, \ldots, a_p)$ of length p of polynomials $a_i \in K[X]$ ($K[X]^0 = \{\emptyset\}$, the set consisting of the empty sequence). Let $K[X]^{\overset{\omega}{\smile}} := \cup_{p \in \omega} K[X]^p$ be the set of all such finite sequences. Let $\rho := \{\langle\ ,\ \rangle, =\}$ consist of the three binary relations $\langle\ ,\ \rangle$, and $=$ on R. For $p \in \omega$ let ρ^p be the set of sequences $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_p)$, of length p, of relations $\varepsilon_i \in \rho$. For $\mathcal{A} \in K[X]^p$ and $\varepsilon \in \rho^p$, we define $\mathcal{A}(\varepsilon) = \cap_{i=1}^p \{x \in R^n \mid a_i(x)\varepsilon_i 0\}$. Then $R^n = \cup_\varepsilon \mathcal{A}(\varepsilon)$ and the $\mathcal{A}(\varepsilon)$ are pairwise-disjoint. For those ε for which $\mathcal{A}(\varepsilon)$ has non-empty interior in R^n, and for those i such that $a_i \not\equiv 0, \varepsilon_i$ is either $<$ or $>$; therefore such $\mathcal{A}(\varepsilon)$ are open. Let $\pi(\mathcal{A})$ be the set of non-empty open $\mathcal{A}(\varepsilon)$ so obtained; thus $1 \leq |\pi(\mathcal{A})| \leq 2^m$. Then $\cup\pi(\mathcal{A})$, the union of the $\mathcal{A}(\varepsilon) \in \pi(\mathcal{A})$, is dense in R^n. If $\mathcal{A}, \mathcal{B} \in K[X]^{\overset{\omega}{\smile}}$ and $\mathcal{A}$ is a subsequence of $\mathcal{B}$, then every $B \in \pi(\mathcal{B})$ is a subset of some $A \in \pi(\mathcal{A})$.

For $1 \leq m \leq n$, write $\hat{X}_m = (X_1, \ldots, X_{m-1}, X_{m+1}, \ldots, X_n)$ and $\hat{x}_m = (x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_n) \in R^{n-1}$. Define $\mathrm{proj}_m : R^n \to R^{n-1}$ by $\mathrm{proj}_m(x) := \hat{x}_m$.

LEMMA 3.1. *Let $1 \leq m \leq n$. There is a function $\Pi_m : K[X]^{\overset{\omega}{\smile}} \to K[\hat{X}_m]^{\overset{\omega}{\smile}}$ such that, for $0 \not\equiv a \in \mathcal{A} \in K[X]^{\overset{\omega}{\smile}}$, and for each cylinder $A \in \pi(\Pi_m(\mathcal{A}))$, the zeros of a which lie in A are the graphs of continuous s.a. functions $x_m = \xi_{a,i}(\hat{x}_m)$, $i = 1, 2, \ldots s$ (where $s := s(a, A, m)$ satisfies $0 \leq s \leq \deg_{X_m} a$), with $\xi_{a,1} < \cdots < \xi_{a,s}$ on $\mathrm{proj}_m(A)$. Moreover, $\forall a_1, a_2 \in \mathcal{A}\backslash\{0\}$, $\forall i_1 \leq s(a_1, A, m)$, $\forall i_2 \leq s(a_2, A, m)$, throughout $\mathrm{proj}_m(A)$ only one of the three relations $\xi_{a_1,i_1} < \xi_{a_2,i_2}$, $\xi_{a_1,i_1} = \xi_{a_2,i_2}$, or $\xi_{a_1,i_1} > \xi_{a_2,i_2}$ holds.* (This is basically Corollary 2.4 of [**3**].)

Now set $\xi_{a,0}(\hat{x}_m) = -\infty$ and $\xi_{a,s+1}(\hat{x}_m) = +\infty\ \forall x \in A$, where

$s = s(a, A, m)$ as in Lemma 2.1.

Let $\mathcal{A} \in K[X]^{\overset{\omega}{\smile}}$. Define the "$m$-skeleton" $\Gamma_m(\mathcal{A})$ of $\mathcal{A}$ to be the smallest subset of $K[X]$ (arranged in some sequential order) containing $\mathcal{A}$ and closed under the following two operations, for each nonconstant $a \in \mathcal{A} : a \to a'$ and $a \to r := r_{a,m}$, where a' will denote $\frac{\partial a}{\partial X_m}$ and $r(X) = a(X) - \frac{X_m}{d} a'(X)$ (here, $d = \deg_{X_m} a > 0$). Since, for nonconstant a, $\deg_{X_m} a' < d$ and $\deg_{X_m} r < d$, $\Gamma_m(\mathcal{A})$ is finite, and so it is in $K[X]^{\overset{\omega}{\smile}}$.

LEMMA 3.2. (MAHÉ 1983). *Suppose* $\mathcal{A} \in K[X]^{\overset{\omega}{\smile}}$, $1 \leq m \leq n$, and $0 \neq a \in \Gamma_m(\mathcal{A})$. *Then, for each cylinder* $A \in \pi(\Pi_m(\Gamma_m(\mathcal{A})))$, *and* $\forall i$ *such that* $0 \leq i \leq 1 + s(a, A, m)$, $\exists c := c(a,i) := c_{A,m}(a,i) \in \mathrm{SIPD}(R^n)$ *such that* $\forall\, x \in A$,

$$c(x) = \begin{cases} a(x) & \textit{if } x_m > \xi_{a,i}(\hat{x}_m), \textit{ and} \\ 0 & \textit{otherwise.} \end{cases}$$

Concerning the proof, Mahé's synopsis [**12**] states only that it may be proved by induction on $d := \deg_{X_m} a$. Since Mahé's full proof has been presented only in his thesis (1983, unpublished), we take this opportunity to present it here, with minor alterations.

PROOF. Obviously, $c(a, 0) = a$ and $c(a, 1 + s) = 0$ (handling the case $d = 0$). So, for $1 \leq i \leq s(a, A, m)$, we may assume, using induction on i, that $c(a, i - 1)$ has already been constructed. For $d \geq 1$ we may suppose, by induction on d, that $c(a', j)$ and $c(r, k)$ have been constructed, for all suitable j and k.

Throughout this paragraph, it will be understood that $x \in A$. Let j be the smallest index such that $\xi_{a,i} \leq \xi_{a',j}$ (then $1 \leq j \leq 1 + s(a', A, m)$). Let k be the smallest index such that $\xi_{a',j} \leq \xi_{r,k}$ (then $1 \leq k \leq 1 + s(r, A, m)$). Then

$$e(x) := \frac{x_m}{d} c(a', j)(x) + c(r, k)(x) = \begin{cases} 0 & \text{if } x_m \leq \xi_{a',j}, \\ a(x) - r(x) & \text{if } \xi_{a',j} \leq x_m \leq \xi_{r,k}, \\ & \text{and} \\ a(x) & \text{if } \xi_{r,k} \leq x_m. \end{cases}$$

If $a(x) = 0$ for $x_m = \xi_{a',j}$, then $\xi_{a',j} = \xi_{r,k}$, and so we may take $c(a,i) = e$. Otherwise we may assume, by symmetry, that, for $x_m = \xi_{a',j}$, $a(x) > 0$. Then (1) for $\xi_{a',j} \leq x_m < \xi_{r,k}$, $r > 0$ (hence $e < a$) and (2) for $\xi_{a,i-1} < x_m < \xi_{a,i}$, $a(x) < 0$. By (1),

$$\sup\{a,e\} = \begin{cases} a & \text{if } x_m \geq \xi_{a,i}, \text{ and} \\ a^+ & \text{if } x_m \leq \zeta_{a,i}. \end{cases}$$

Therefore, by (2), we may take $c(a,i) = \inf\{c(a,i-1)^+, \sup\{a,e\}\}$. □

Now let $h \in \mathrm{PWP}(R^n)$ be as in §1, i.e., $h = g_i$ on A_i, where the $g_i \in K[X]$ and the A_i are s.a. and cover R^n. For the rest of this paper let $\mathcal{A} = \{g_i - g_j \mid 1 \leq i \leq j \leq l\} \in K[X]^{\overset{\omega}{\smile}}$.

PROPOSITION 3.3. **[12]** *Let h and $\mathcal{A}$ be as above. For $1 \leq m \leq n$, and for each cylinder $A \in \pi(\Pi_m(\Gamma_m(\mathcal{A})))$, there is a function $q := q_{A,m} \in \mathrm{SIPD}(R^n)$ which coincides with h on A.*

PROOF. The graphs of the functions $x_m = \xi_{a,m}(\hat{x}_m)$, for all $a \in \Gamma_m(\mathcal{A})\backslash\{0\}$ and for all i with $1 \leq i \leq s(a, A, m)$, separate A into disjoint connected open s.a. subsets ("sausages") $D_1, \ldots, D_t$ whose union is dense in A. We may suppose that the D's are listed in order of increasing x_m-coordinates—precisely, $\forall\, \hat{x}_m \in \mathrm{proj}_m(A)$, if $(x_1, \ldots, d_k, \ldots x_n) \in D_k$, $(1 \leq k \leq t)$, then $d_1 < \cdots < d_t$ (this is similar to what Arnon, Collins, and McCallum **[1]** call a "stack"). For each $k = 1, \ldots, t$ there exists a unique $\mu := \mu(k, A)$ such that $D_k \subseteq A_\mu$ (hence $h = g_\mu$ on D_k), since h is continuous.

If $t = 1$ we may define $q := g_{\mu(1,A)} \in \mathrm{SIPD}(R^n)$; if $t > 1$, then we shall define q as follows. For $k = 1, \ldots, t-1$, let $v_k := q_{\mu(k+1,A)} - g_{\mu(k,A)}$. We have $v_k = 0$ on $\overline{D_k} \cap \overline{D_{k+1}}$ since h is continuous. For $i = 1, 2, \ldots$, define the function $c := c(0,i) := c_{A,m}(0,i)$ by $c(x) = 0 \;\; \forall x \in R^n$. If $v_k \neq 0$, then there exists a unique $i := i(k)$ such that $1 \leq i \leq s(v_k, A, m)$ and the graph of $x_m = \xi_{v_k,i}(\hat{x}_m)$ over $\mathrm{proj}_m(A)$ separates D_k from D_{k+1}. By Lemma 3.2 we may take

$$q = g_{\mu(1,A)} + \sum_{k=1}^{t-1} c(v_k, i(k)) \in \mathrm{SIPD}(R^n).$$

□

(Mahé's proof of Proposition 3.3 [**4**] used the transversal zeros theorem; there appears to be no need for this theorem here.)

4. Proof of the Pierce-Birkhoff Conjecture for n $\leq$ 2. For $n = 1$, Theorem 1.3 reduces to Proposition 3.3, for since $\Pi_1(\Gamma_1(\mathcal{A})) \subset K$, there is only one cylinder A, which must be all of R^1.

To establish Theorem 1.3 for $n = 2$ we shall need four lemmas. Let T be a single indeterminate and $t \in R$. For any function $b : R \to R$ and any $\delta \in R$, the notation $\lim_{t\to\delta^+} b(t)$ will mean the right-hand limit of b at δ, and not the 2-sided limit of b at $\sup\{\delta, 0\}$, despite the notation introduced in the proof of Lemma 2.1.

Let $\overline{K} \subseteq R$ denote the real closure of K. Note that if a function $c : R \to R$ of one variable t is p.p. and $c(\delta) = \lim_{t\to\delta^+} c(t)$ for some $\delta \in R$, then c has a right-hand derivative $c'_+(t) := \lim_{\varepsilon\to 0^+}(c(t+\varepsilon) - c(t))/\epsilon$ at δ (this holds even if c is not continuous at δ).

LEMMA 4.1. *For all* $\delta \in \overline{K}$ *we can construct a function* $c_\delta \in$ SIPD(R^1) *such that* $c_\delta(t) > 0$ *for all* $t > \delta$, $c_\delta(t) = 0$ *for* $t \leq \delta$, *and* $c'_{\delta_+}(\delta) > 0$.

PROOF. Let $c \in K[T]$ be the minimal polynomial of δ over K. Then $c'(\delta) \neq 0$. If c has a real root $> \delta$, then, by Rolle's theorem, c' has a real root $> \delta$; let η be the smallest such root. Using induction on $\deg_K \delta := \deg c \geq 1$, we can construct $c_\eta \in$ SIPD(R^1) such that $c_\eta(t) > 0$ for $t > \eta$ and $c_\eta(t) = 0$ for $t \leq \eta$ (we can also arrange for $c'_{\eta_+}(\eta) > 0$, but we do not need this here). In this case define $e = c_\eta$; if c has no real root $> \delta$ (in particular, if $\deg_K \delta = 1$), define $e = 0$. Then we may define

$$c_\delta(t) := \begin{cases} \sup\{|c(t)|, e(t)\} & \text{if } t \geq \delta, \text{ and} \\ 0 & \text{if } t \leq \delta, \end{cases}$$

and apply Theorem 1.3 to conclude that $c_\delta \in$ SIPD(R^1). □

LEMMA 4.2. *Suppose* $\delta \leq \xi \in \overline{K}$ *and* $b : R \to R$ *is p.p. over* $\overline{K}$; *in case* $\delta = \xi$, *suppose also that* $b(\xi) = 0 = \lim_{t\to\xi^+} b(t)$. *Then we* *can construct a function* $u := u_{b,\delta,\xi} \in \mathrm{SIPD}(R^1)$ *(over K) such that* $u(t) \geq b(t)$ *for* $t \geq \xi$ *and* $u(t) = 0$ *for* $t \leq \delta$.

REMARK. In Lemma 4.2, if we had allowed u to be defined using coefficients in $\overline{K}$, then (4.2) would reduce to the one-variable case of the Pierce-Birkhoff conjecture (over $\overline{K}$), which was already established.

PROOF. Use Lemma 4.1 to construct $c_\delta \in \mathrm{SIPD}(R^1)$ with the properties listed there. In case $\delta = \xi$, choose $v \in K$ larger than $b'_+(\xi)/c'_{\delta+}(\xi) \in \overline{K}$; then $vc_\delta(t) > b(t)$ for $\xi < t < \xi + \epsilon$, some $\epsilon > 0$. And if $\delta < \xi$, so that $c_\delta(\xi) > 0$, it is even easier to see that we may choose $v \in K$ such that $vc_\delta(t) \geq b(t)$ for $\xi < t < \xi + \varepsilon$, some $\varepsilon > 0$. Either way, if $vc_\delta(t) \geq b(t)$ for all $t \geq \xi$, then we may set $u = vc_\delta$. Otherwise, set $\zeta = \inf\{t \in (\xi, \infty) \,|\, vc_\delta(t) < b(t)\} \in \overline{K}$ $(\xi + \varepsilon \leq \zeta < \infty)$.

Let $c_\delta^\infty \in K[T]$ (respectively $b^\infty \in \overline{K}[X]$) be the polynomial which coincides with c_δ (respectively b) for large t. Set $d := \max\{0, \deg b^\infty - \deg c_\delta^\infty\}$. Then $(1 + |t|)^{d+1} c_\delta(t) \geq b(t)$ for all t larger than some $\eta \in \overline{K}$ (we may assume $\eta > \zeta$). Choose $w \in K$ greater than $\sup_{t \in [\zeta, \eta]} b(t)/c_\delta(t) \leq (\sup b(t))/\min c_\delta(t) \in \overline{K}$. Then we may take $u(t) = (1 + |t|)^{d+1} c_\delta(t) \max\{1, v, w\}$. □

For $1 \leq m \leq n$ and $\mathcal{B} \in K[X]^{\breve{\omega}}$ let $\Delta_m(\mathcal{B}) \in K[X]^{\breve{\omega}}$ be the smallest set, arranged in some sequential order, containing $\mathcal{B}$ and closed under partial differentiation with respect to X_m. Thom's lemma (cf. [**3**, Proposition 3.1]) says that if $n = 1$ and $\mathcal{B} = \Delta_1(\mathcal{B}) \in K[X]^{\breve{\omega}}$, then each $B \in \pi(\mathcal{B})$ is connected (and hence an open interval). From this one can see that if $1 \leq m \leq n$ and $\mathcal{B} = \Delta_m(\mathcal{B}) \in K[X]^{\breve{\omega}}$, then each $B \in \pi(\mathcal{B})$ is connected *provided* that each cylinder $C \in \pi(\Pi_m(\mathcal{B}))$ is connected.

PROOF OF THEOREM 1.3 For $n = 2$. As in Proposition 3.3, let $\mathcal{A} = \{g_i - g_j \,|\, 1 \leq i \leq j \leq l\}$. Let $\mathcal{B} = \Gamma_1(\mathcal{A})$; then $\Delta_1(\mathcal{B}) = \mathcal{B}$. Let $\mathcal{C} = \Gamma_2(\Pi_1(\mathcal{B})) \in K[X_2]^{\breve{\omega}}$. Then $\mathcal{D} := \mathcal{B} \cup \mathcal{C}$ (arranged in some sequential order) satisfies $\mathcal{D} = \Delta_1(\mathcal{D})$ and $\Pi_1(\mathcal{D}) = \mathcal{C}$. Since $\Delta_2(\mathcal{C}) = \mathcal{C}$, each cylinder $C \in \pi(\mathcal{C})$ is connected. Then, by the preceding paragraph, each $D \in \pi(\mathcal{D})$ is connected.

Since h is continuous, the A_i are closed, hence $\overline{A_i^\circ} \subseteq A_i$. We may assume that the g_i are distinct; thus $A_i^\circ \cap A_j^\circ = \emptyset$ for $i \neq j$.

LEMMA 4.3. $\cup_{i=1}^{l} A_i^\circ = R^n \setminus \cup_{1 \le i < j \le l} (\overline{A_i^\circ} \cap \overline{A_j^\circ})$.

PROOF. $\subseteq$. Let $x \in A_i^\circ$ and suppose $j \neq i$. It is enough to show that $x \notin \overline{A_j^\circ}$. There exists an open ball in A_i about x. In fact, this ball is in A_i°, and hence is disjoint from A_j°. Therefore $x \notin \overline{A_j^\circ}$.

$\supseteq$. Suppose $x \in R^n \setminus \cup_i A_i^\circ$. Since the A_i are s.a., the ∂A_i are nowhere dense; therefore $\cup_i A_i^\circ$ is dense in R^n. Thus every ball about x meets at least 2 different A_i°. Since there are only finitely many A_i, there exist at least 2 indices $i < j$ such that every ball about x meets A_i° and A_j°. Therefore $x \in \overline{A_i^\circ} \cap \overline{A_j^\circ}$. □

LEMMA 4.4. *Suppose $\mathcal{D} \in K[X]^{\breve{\omega}}$ is as above and $\mathcal{D}$ is a subsequence of $\mathcal{E} \in K[X]^{\breve{\omega}}$. Then there is a function $\nu = \nu_{\mathcal{E}} : \pi(\mathcal{E}) \to \{1, \dots, l\}$ such that $\forall\, E \in \pi(\mathcal{E})$, $h = g_{\nu(E)}$ on E (i.e., $E \subseteq A_{\nu(E)}$).*

PROOF. It suffices to prove the lemma in the case $\mathcal{E} = \mathcal{D}$. Fix $D \in \pi(\mathcal{D})$. Since $\mathcal{A}$ is a subsequence of $\mathcal{D}$, $g_i - g_j$ has constant sign, $+$, $-$, or 0, on D. For $i \neq j$, this sign is actually either $+$ or $-$, since the g_i are distinct and D is open (and nonempty). Since h is continuous, each $g_i - g_j = 0$ on $A_i \cap A_j$, and so, for $i \neq j$, $D \cap (\overline{A_i^\circ} \cap \overline{A_j^\circ}) \subseteq D \cap (A_i \cap A_j) = \emptyset$. By Lemma 4.3, $D \subseteq \cup_i A_i^\circ$. But D is connected, as discussed before Lemma 4.3, so $\exists \nu(D)$ such that $D \subseteq A_{\nu(D)}^\circ$. □

Lemmas 4.3 and 4.4 hold even for $n > 2$.

REMARK. The only purpose of Lemmas 4.3 and 4.4 was to construct ν as in Lemma 4.4—we shall need ν in the proof of Theorem 1.3. However, we could have constructed ν rather trivially by adding to $\mathcal{E}$ any finite set of polynomials in $K[X]$ defining all the A_i as s.a. sets, for then each $E \in \pi(\mathcal{E})$ would be a subset of some A_i. Our purpose in constructing ν without using any set of polynomials defining the A_i (besides $\mathcal{D}$) was to show that h (and hence the A_i) can be recovered from the set $\{g_1, \ldots g_l\}$ together with the function $\nu_{\mathcal{D}}$—it is not necessary to know the A_i in advance. We do not need this fact for the proof of the Pierce-Birkhoff conjecture – it is just an interesting fact. One consequence of this fact is that we did not need to assume that the sets A_i were s.a. when we defined "h is p.p.," at least not if h is also continuous: for the A_i are definable by polynomials in $\mathcal{D}$, and hence are automatically s.a. (for $n > 2$ we must iterate the operations Γ_i and Π_i also for $i > 2$ to get a suitable $\mathcal{D}$; this is no problem).

Returning to the proof of Theorem 1.3, let $\mathcal{G} := \Gamma_2(\mathcal{A})$ and $\mathcal{E} := \mathcal{D} \cup \mathcal{G} \cup \Delta_1(\Pi_2(\mathcal{G}))$, arranged in some sequential order. Construct $\nu_{\mathcal{E}}$ as in Lemma 4.4. As in [**12**], the idea now is to construct, $\forall\, E, F \in \pi(\mathcal{E})$, functions $e_{EF} \in \mathrm{SIPD}(R^2)$ such that $e_{EF} \leq g_{\nu(E)}$ on E and $e_{EF} \geq g_{\nu(F)}$ on F. Then we shall be done, since the function $e_F := \inf_E\{e_{EF}, g_{\nu(F)}\} \in \mathrm{SIPD}(R^2)$ will satisfy $e_F = g_{\nu(F)}$ on F and $e_F \leq g_{\nu(E)}$ on each E; then $h = \sup_F e_F$.

So suppose $E, F \in \pi(\mathcal{E})$. If E and F are both subsets of the same "horizontal" cylinder $C \in \pi(\mathcal{C})$ or "vertical" cylinder $G \in \pi(\Delta_1(\Pi_2(\mathcal{G})))$, then we may use Proposition 3.3 and take e_{EF} to be either $q_{C,1}$ or $q_{G,2}$, respectively.

The difficult case is when E and F do not lie in a common cylinder (in either the X_1 - or the X_2-direction). We may assume, without loss of generality, that E is below and to the left of F (i.e, that points in E have X_1- and X_2-coordinates less than the X_1- and X_2-coordinates, respectively, of points in F); the other three possibilities could be handled similarly.

E lies in a unique cylinder $C \in \pi(\mathcal{C})$ in the X_1-direction, and in another unique cylinder $G \in \pi(\Delta_1(\Pi_2(\mathcal{G})))$ in the X_2-direction. Let ξ_1 (respectively ξ_2) $\in \overline{K}$ be the right endpoint of the interval $\mathrm{proj}_2(G)$ (respectively $\mathrm{proj}_1(C)$) $\subset R$. For $t \in R$ let $L_t := \{(x_1, x_2) \in R^2 \,|\, x_1 + x_2 = t$ and $(x_1 - \xi_1)(x_2 - \xi_2) \geq 0\}$ (see Figure). Define $I(t) := \{i \,|\, 1 \leq i \leq l$ and $A_i \cap L_t \neq \emptyset\}$ ($\neq \emptyset \quad \forall t$). Let $p(t) = \max_{(x_1,x_2)\in L_t}(h - g_{\nu(E)})(x_1, x_2)$. Then

$$p(t) \leq \max_{\substack{i \in I(t) \\ (x_1,x_2)\in L_t}} (g_i - g_{\nu(E)})(x_1, x_2).$$

We shift the point (ξ_1, ξ_2) to the origin and rotate the X_1- and X_2-axes

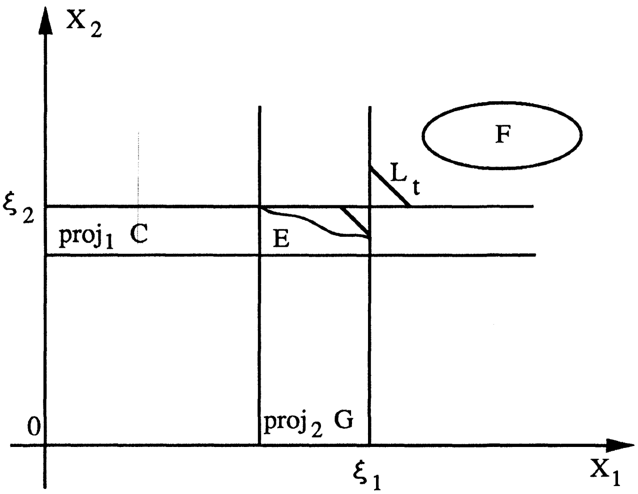

FIGURE 1

by $\pi/4$ radians, using the following $\overline{K}$-linear change of coordinates: $Y_1 = (X_1 - \xi_1) + (X_2 - \xi_2)$ and $Y_2 = (X_1 - \xi_1) - (X_2 - \xi_2)$. For each i expand $g_i - g_{\nu(E)}$ in powers of Y_1 and Y_2:

$$\begin{aligned}(g_i - g_{\nu(E)})(X_1, X_2) = \gamma_{i00} + \gamma_{i10}Y_1 + \gamma_{i01}Y_2 + \gamma_{i20}Y_1^2 \\ + \gamma_{i11}Y_1Y_2 + \gamma_{i02}Y_2^2 + \cdots ,\end{aligned}$$

for finitely many $\gamma_{ijk} \in \overline{K}$. Since, for $(x_1, x_2) \in L_t$, $|(x_1 - \xi_1) \pm (x_2 - \xi_2)| \leq |t - \xi_1 - \xi_2|$,

$$\begin{aligned}p(t) \leq \max_{i \in I(t)} [|\gamma_{i00}| + (|\gamma_{i10}| + |\gamma_{i01}|) \\ |t - \xi_1 - \xi_2| + (|\gamma_{i20}| + |\delta_{i11}| + |\gamma_{i02}|)\, |t - \xi_1 - \xi_2|^2 + \cdots];\end{aligned}$$

denote the righthand side by $b(t)$. Choose the smallest value of $\delta \in \overline{K}$ such that $\delta < t < \xi_1 + \xi_2 \Rightarrow L_t \cap E = \emptyset$; then $\delta \leq \xi_1 + \xi_2$. If $\delta = \xi_1 + \xi_2$, then $b(\delta) = 0 = \lim_{t \to \delta^+} b(t)$, since this implies $i \in I(\delta) \Rightarrow \gamma_{i00} = 0$. By Lemma 4.2 we can construct a function $u := u_{b,\delta\xi_1+\xi_2} \in \mathrm{SIPD}(R^1)$ such that $u(t) \geq b(t) \geq p(t)$ for $t \geq \xi_1 + \xi_2$ and $u(t) = 0$ for $t \leq \delta$. Then we may set $e_{EF}(X_1, X_2) := g_{\nu(E)}(X_1, X_2) + u(X_1 + X_2)$. Then (1) $e_{EF}(x_1, x_2) \geq g_{\nu(F)}(x_1, x_2)$ for $(x_1, x_2) \in F$ since, in F, $x_1 \geq \xi_1$,

$x_2 \geq \xi_2$, and $h = g_{\nu(F)}$; and (2) $e(x_1, x_2) = g_{\nu(E)}(x_1, x_2)$ for $(x_1, x_2) \in E$ (where $x_1 + x_2 \leq \delta$). □

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