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ON UNIVARIATE CARDINAL INTERPOLATION BY SHIFTED SPLINES

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1. Introduction. The object of this paper is to study cardinal interpolation of bounded data by integer translates of shifted B-splines. To set notation, M_n will denote the centered univariate B-spline of order n and, for any function $g(x)$ of the real variable x and a fixed real number $\alpha, g_\alpha(x)$ will stand for $g(x+\alpha)$; $\hat{g}$ will denote the Fourier transform of g. $I_{n,\alpha}f$ will represent the interpolant $\sum_{j\in\mathbf{Z}} a_j M_{n,\alpha}(\cdot - j)$ which agrees with a given function f on $\mathbf{Z}$ and $P_{n,\alpha}(x)$ will stand for the *characteristic polynomial*, viz.,

$$P_{n,\alpha}(x) = \sum_{j\in Z} M_{n,\alpha}(j)e^{-ijx}. \tag{1.1}$$

$I_{n,\alpha}f$ can also be written in the *Lagrange form*

$$I_{n,\alpha}f = \sum_{j\in Z} f(j)L_{n,\alpha}(\cdot - j), \tag{1.2}$$

where $L_{n,\alpha}$ is the *fundamental function* of interpolation.

An application of the Poisson summation formula to (1.1) yields the useful identity

$$\begin{aligned} P_{n,\alpha}(x) &= \sum_{j\in Z} \hat{M}_{n,\alpha}(x+2\pi j) \\ &= \sum_{j\in Z} \hat{M}_n(x+2\pi j)e^{i\alpha(x+2\pi j)}. \end{aligned} \tag{1.3}$$

It should also be recalled that the Fourier transforms of $L_{n,\alpha}$ and M_n are given by

$$\hat{L}_{n,\alpha}(x) = \frac{\hat{M}_{n,\alpha}(x)}{P_{n,\alpha}(x)} \tag{1.4}$$

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and

$$\hat{M}_n(x) = \left[\frac{\sin(x/2)}{(x/2)}\right]^n \tag{1.5}$$

respectively.

2. Correctness of the interpolation problem. This section focuses on the correctness of the cardinal interpolation problem by $M_{n,\alpha}$. To recall, cardinal interpolation with $M_{n,\alpha}$ is said to be *correct* if, given a bounded real valued function f defined on $\mathbf{R}$, there exists a unique bounded sequence $\{a_j; j \in \mathbf{Z}\}$ such that $\sum_{j\in Z} a_j N_{n,\alpha}(\cdot - j)$ agrees with f on $\mathbf{Z}$.

The following necessary and sufficient condition is well-known.

THEOREM 2.1. *Cardinal interpolation with $M_{n,\alpha}$ is correct if and only if $P_{n,\alpha}$ does not vanish in $[-\pi, \pi]$.*

With the aid of this theorem, it will be shown (Theorem 2.2) that the interpolation problem is correct for α in $(-1/2, 1/2)$. This result was proved by C.A. Micchelli (cf. [**2**]) and in a more general setting by T.N.T. Goodman (cf [**1**]) but the proof which will be given here is different. It relies on the identity (1.3) and supplies an expression for $P_{n,\alpha}(x)$ which will prove useful in §3 for the analysis of the convergence of $I_{n,\alpha}f$ as its order n tends to infinity.

The following lemma serves as a prelude.

LEMMA 2.1. *Let $0 \le b < a \le 1$ and $-\pi \le \theta \le \pi$. Then, for any positive integer j,*

$$\begin{aligned} h(\theta) :&= \left|\frac{1 + ae^{i(2j+1)\theta}}{1 + be^{i\theta}}\right| \\ &= \left[\frac{1 + a^2 + 2a\cos(2j+1)\theta}{1 + b^2 + 2b\cos\theta}\right]^{\frac{1}{2}} \le 4(2j+1). \end{aligned} \tag{2.1}$$

PROOF. Since $h(\theta)$is even, it may be assumed that θ belongs to $[0, \pi]$. The following cases will be considered.

Case (i). $0 \le \theta \le \pi/2$. Then

$$h^2(\theta) \le (1+a)^2 \le 4. \tag{2.2}$$

Case (ii). $\pi/2 < \theta \le \pi$. Setting $\eta = \pi - \theta$, it can be seen that

$$h^2(\theta) = \frac{1+a^2-2a\cos(2j+1)\eta}{1+b^2-2b\cos\eta}.$$

Since $1-\phi^2/2 \le \cos\phi \le 1-\phi^2/2+\phi^4/24$ for $\phi \ge 0$, it follows that

$$\begin{aligned} h^2(\theta) &\le \frac{1+a^2-2a[1-(2j+1)^2\eta^2/2]}{1+b^2-2b[1-\eta^2/2+\eta^4/24]} \\ &= \frac{(1-a)^2+a(2j+1)^2\eta^2}{(1-b)^2+b\eta^2[1-\eta^2/12]}. \end{aligned}$$

Observing that $0 \le \eta < \pi/2$ and $b < a \le 1$, it is clear that, when $0 \le b \le 1/2$,

$$h^2(\theta) \le \frac{(1-a)^2}{(1-b)^2} + \frac{(2j+1)^2\pi^2/4}{(1-b)^2} \le 1+\pi^2(2j+1)^2. \tag{2.3}$$

Similarly, if $1/2 \le n < a < 1$, one now obtains

$$h^2(\theta) \le \frac{(1-a)^2}{(1-b)^2} + \frac{(2j+1)^2}{b[1-\eta^2/12]} \le 1+\pi^2(2j+1)^2. \tag{2.4}$$

(2.2), (2.3) and (2.4) give (2.1) and the proof is complete. □

THEOREM 2.2.

(a) $P_{n,\alpha}(x) \ne 0$ *if* $-\pi \le x \le \pi$ *and* $-1/2 < \alpha < 1/2$ *or if* $-\pi < x < \pi$ *and* $\alpha = \pm 1/2$.

(b) $P_{n,1/2}$ *and* $P_{n,-1/2}$ *have simple zeroes at* $x = \pm\pi$ *respectively.*

PROOF. Firstly, it should be noted that the evenness of M_n guarantees that

$$P_{n,\alpha}(-x) = P_{n,-\alpha}(x), \quad x \in \mathbf{R}. \tag{2.5}$$

So it suffices to consider $x \in [0, \pi]$ whilst proving (a).

Setting $x = 2\pi u$, it then follows that $0 \le u \le 1/2$ and (1.3) reads

$$(2.6) \qquad e^{-i\alpha 2\pi u} P_{n,\alpha}(2\pi u) = \sum_{j \in Z} \hat{M}_n(2\pi u + 2\pi j) e^{i\alpha 2\pi j}.$$

Using(1.5) and the fact that $\hat{M}_n$ is even, (2.6) becomes

$$\begin{aligned} e^{-\alpha 2\pi u} P_{n,\alpha}(2\pi u) = \hat{M}_n(2\pi u)\Big[1 &+ \Big(\frac{u}{1-u}\Big)^n e^{-i\alpha 2\pi} \\ &+ \sum_{j=1}^{\infty} (-1)^{jn} \Big(\frac{u}{u+j}\Big)^n e^{i\alpha 2\pi j} \\ &+ \sum_{j=2}^{\infty} (-1)^{jn} \Big(\frac{u}{u-j}\Big)^n e^{-i\alpha 2\pi j}\Big] \end{aligned}$$

which, in turn, after some simplification, reduces to
(2.7)

$$\begin{aligned} e^{-i\alpha 2\pi u} P_{n,\alpha}(2\pi u) &= \hat{M}_n(2\pi u)\Big[1 + \Big(\frac{u}{1-u}\Big)^n e^{-i\alpha 2\pi}\Big] \\ &\times \Big[1 + \sum_{j=1}^{\infty} (-1)^{jn} e^{i\alpha 2\pi j} \Big(\frac{u}{u+j}\Big)^n A_{n,\alpha,j}(u)\Big] \end{aligned}$$

where, for brevity,

$$(2.8) \qquad A_{n,\alpha,j}(u) := \frac{1 + \Big(\frac{j+u}{j+1-u}\Big)^n e^{-i\alpha 2\pi(2j+1)}}{1 + \Big(\frac{u}{1-u}\Big)^n e^{-i\alpha 2\pi}}.$$

Since $\hat{M}_n(2\pi u)$ is non-zero for $0 \le u \le 1/2$, and it is clear that

$$1 + \Big(\frac{u}{1-u}\Big)^n e^{-\alpha 2\pi} = 0$$

if and only if $\alpha = \pm 1/2$ and $u = 1/2$, it suffices to prove that the remaining factor in (2.7) is non-zero for $0 \le u \le 1/2$ and

$-1/2 \leq \alpha \leq 1/2$ in order to establish the result. This would follow readily if it can be shown that

$$(2.9) \qquad \sum_{j=1}^{\infty} \left| \frac{u}{u+j} \right|^{n} |A_{n,\alpha,j}(u)| < 1 \quad \text{for } 0 \leq u \leq \frac{1}{2} \text{ and } -\frac{1}{2} \leq \alpha \leq \frac{1}{2}.$$

At the outset, an application of Lemma 2.1 to (2.8) yields the fact that $|A_{n,\alpha,j}(u)| \leq 4(2j+1)$ for $0 \leq u \leq 1/2$ and $-1/2 \leq \alpha \leq 1/2$. Since

$$\left| \frac{u}{u+j} \right| \leq \frac{1}{2j+1} \quad \text{for } 0 \leq u \leq 1/2,$$

it follows immediately that

$$(2.10) \qquad \sum_{j=1}^{\infty} \left| \frac{u}{u+j} \right|^{n} |A_{n,\alpha,j}(u)| \leq 4 \sum_{j=1}^{\infty} \left(\frac{1}{2j+1} \right)^{n-1} \leq C < 1 \quad \text{for } n \geq 3.$$

It should be noted that C is independent of u, n, and α; (2.9) is thus proved for $n \geq 3$.

The theorem can be checked directly for $n = 1, 2$ and is therefore proved in its entirety. □

3. Convergence of $\mathbf{I_{n,\alpha}f}$. This section deals with the problem of convergence of the interpolant $I_{n,\alpha}f$ as n approaches infinity; the object is to prove a convergence theorem of the Schoenberg type (cf. [**3, 4**]) for the class of shifted splines.

In what follows, $K(\alpha)$ will stand for a constant dependent only on α. It should be remarked, however, that its actual numerical value may differ at each appearance.

The following lemma is of consequence.

LEMMA 3.1. *Let* $-1/2 < \alpha < 1/2$ *and* $0 < \delta < \pi/2$. *Then the following hold:*

(a) $\lim_{n\to\infty} \hat{L}_{n,\alpha}(x) = 1$ *for* $-\pi < x < \pi$, *and the convergence is uniform on compact subintervals of* $(-\pi, \pi)$;

(b) $\lim_{n\to\infty} \hat{L}_{n,\alpha}(\pm\pi) = e^{\pi i\alpha\pi}/2(\cos\pi\alpha)$;

(c) *for* $x \in [-\pi - \delta, \pi + \delta], [\hat{L}_{n,\alpha}(x)| \leq K(\alpha)$; *and*

(d) *for* $\pi + \delta < |x| = 2\pi(u + j), u \in [-1/2, 1/2]$ *and* $j = 1, 2, 3, \ldots$,

$$|\hat{L}_{n,\alpha}(x)| \leq \begin{cases} K(\alpha)[(\pi - \delta)/(\pi + \delta)]^n, & \text{if } j = 1, \\ K(\alpha)(2j - 1)^{-n}, & \text{if } j = 1, 2, 3, \ldots . \end{cases}$$

PROOF. (a). By virtue of(1.4), (2.5) and the evenness of$\hat{M}_n$, it follows that

$$\hat{L}_{n,\alpha}(-x) = \hat{L}_{n,-\alpha}(x). \tag{3.1}$$

So x may be taken to belong to $[0, \pi)$. Now (a) follows easily from (1.4), (2.7), and (2.10).

(b). This is an easy consequence of (1.4), (2.7), (2.10), and (3.1).

(c). To begin with, let $x \in [0, \pi]$. Then $x = 2\pi u$ for $u \in [0, 1/2]$. For such u, it is not hard to see that

$$\left|1 + \left(\frac{u}{1-u}\right)^n e^{0i\alpha 2\pi}\right| \geq \begin{cases} 1, & \text{if } -1/4 \leq \alpha \leq 1/4, \\ |\sin 2\pi\alpha|, & \text{if } \pm\alpha \in (1/4, 1/2). \end{cases} \tag{3.2}$$

Inequality (3.2), taken in conjunction with (2.7) and (2.10), proves (c) for $x \in [0, \pi]$. The obvious symmetry of the lower bounds (w.r.t. α) in (3.2) coupled with (3.1) gives (c) for $x \in [-\pi, 0]$ as well. Now, for

$$|x| = 2\pi(u + j), \quad j = 1, 2, \ldots,$$

the periodicity of $P_{n,\alpha}$ permits the estimate

$$\begin{aligned} \left|\hat{L}_{n,\alpha}(|x|)\right| &= \left|\frac{\hat{M}_{n,\alpha}(2\pi u + 2\pi j)\hat{M}_{n,\alpha}(2\pi u)}{P_{n,\alpha}(2\pi u + 2\pi j)\hat{M}_{n,\alpha}(2\pi u)}\right| \\ &= |\hat{L}_{n,\alpha}(2\pi u)|\left|\frac{u}{u+j}\right|^n \\ &\leq \begin{cases} K(\alpha)[(\pi - \delta)/(\pi + \delta)]^n, & \text{if } j = 1, |x| > \pi + \delta; \\ K(\alpha)(2j - 1)^{-n}, & \text{if } j \geq 1. \end{cases} \end{aligned}$$

The remaining assertions of the lemma follow from this and (3.1). □

The convergence theorem can now be stated.

THEOREM 3.1. *Let*

$$f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ixt} d\mu(t)$$

for a bounded measure μ *on* $[-\pi, \pi)$ *and let* $-1/2 < \alpha < 1/2$.

(a) *If* μ *is absolutely continuous (w.r.t. Lebesgue measure),* $I_{n,\alpha} f$ *converges uniformly to* f.

(b) *If* $\mu = \delta_{-\pi}$, *then* $I_{n,\alpha} f(x)$ *converges uniformly to* $\cos\pi(x+\alpha)/(\cos\pi\alpha)$.

PROOF. (a) Since $\{f(-j) : j \in \mathbf{Z}\}$ are the Fourier series that

$$\begin{aligned} f(x) - I_{n,\alpha} f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ixt} d\mu(t) - \sum_{j\in Z} f(j) L_{n,\alpha}(x-j) \\ &= \frac{1}{2\pi}\Big[\int_{-\pi}^{\pi} e^{ixt} d\mu(t) \\ &\quad - \sum_{j\in Z} f(j)\int_{-\infty}^{\infty} e^{ijt}\hat{L}_{n,\alpha}(t) e^{ixt} dt\Big] \\ &= \frac{1}{2\pi}\Big[\int_{-\pi}^{\pi} e^{ixt} d\mu(t) - \int_{-\infty}^{\infty} \hat{L}_{n,\alpha}(t) e^{ixt} d\tilde{\mu}(t)\Big], \end{aligned} \tag{3.3}$$

where $\tilde{\mu}$ is the periodic extension of μ.

Let, for a given $\varepsilon > 0$, $N_{-\pi} := [-\pi-\delta, -\pi+\delta]$ and $N_{\pi} := [\pi-\delta, \pi+\delta]$ be chosen such that

$$\tilde{\mu}(N_{\pm\pi}) < \varepsilon \tag{3.4}$$

by the absolute continuity of μ.

From (3.3), it is clear that

$$
\begin{aligned}
|f(x) - I_{n,\alpha}f(x)| \le \frac{1}{2\pi}\Big[\int_{-\pi+\delta}^{\pi-\delta} &|1 - \hat{L}_{n,\alpha}(t)|d\mu \\
+ \int_{N_{-\pi}\cup N_{\pi}} &[1 + |\hat{L}_{n,\alpha}(t)|]d\tilde{\mu} \qquad (3.5)\\
+ \int_{-\infty}^{-\pi-\delta} &|\hat{L}_{n,\alpha}(t)|d\tilde{\mu} + \int_{\pi+\delta}^{\infty} |\hat{L}_{n,\alpha}(t)|d\tilde{\mu}\Big].
\end{aligned}
$$

Noticing that the sum of the last two integrals on the right hand side of (3.5) can be written as

$$
\begin{aligned}
&\int_{-3\pi}^{-\pi-\delta} |\hat{L}_{n,\alpha}(t)d\tilde{\mu} + \int_{\pi+\delta}^{3\pi} |\hat{L}_{n,\alpha}(t)|d\tilde{\mu} \\
&+ \sum_{j=2}^{\infty}\Big[\int_{(2j-1)\pi}^{(2j+1)\pi} |\hat{L}_{n,\alpha}(t)|d\tilde{\mu} + \int_{-(2j+1)\pi}^{(2j-1)\pi} |\hat{L}_{n,\alpha}(t)|d\tilde{\mu}\Big],
\end{aligned}
$$

and then using (a), (c), and (d) of Lemma 3.1 along with (3.4), it follows that

$$
\limsup_{n\to\infty} \|f - I_{n,\alpha}f\|_{\infty} \le \frac{1}{\pi}[1 + K(\alpha)]\varepsilon,
$$

from which the desired conclusion follows.

(b) When $\mu = \delta_{-\pi}$,

$$
\begin{aligned}
I_{n,\alpha}f(x) = \hat{L}_{n,\alpha}(-\pi)e^{-i\pi x} &+ \hat{L}_{n,\alpha}(\pi)e^{i\pi x} \\
&+ \sum_{j\in\mathbf{Z}\setminus\{0,1\}} \hat{L}_{n,\alpha}((2j-1)\pi)e^{i(2j-1)\pi x}
\end{aligned}
$$

which converges uniformly to

$$
\frac{e^{-i\pi(x+\alpha)} + e^{i\pi(x+\alpha)}}{2\cos\pi\alpha} = \frac{\cos\pi(x+\alpha)}{\cos\pi\alpha}
$$

by (b) and (d) of Lemma 3.1.

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