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A REMARK ON REGULAR OPERATORS

PIOTR MIKUSIŃSKI

ABSTRACT. A simple definition of regular Mikusiński operators is given and equivalence with the original one is proved.

Regular operators were introduced and studied by T.K. Boehme in [**2**]. Regular operators form a subalgebra of Mikusiński operators and have local properties. For example each regular operator has a well-defined support. We will recall the definition of Mikusiński operators and regular operators briefly.

Let $C(\mathbf{R}^N)$ be the space of continuous functions on $\mathbf{R}^N$. The support of $f \in C(\mathbf{R}^N)$, denoted by $\operatorname{supp} f$, is the closure of the set on which f is not zero. For $a = (a_1, \dots, a_N) \in \mathbf{R}^N$ and $x = (x_1, \dots, x_N) \in \mathbf{R}^N$, let $\mathbf{R}_a^N = \{x \in \mathbf{R}^N : x_i \geq a_i \text{ for } i = 1, \dots, N\}$. Put

$$C^+(\mathbf{R}^N) = \{f \in C(\mathbf{R}^N) : \operatorname{supp} f \subset \mathbf{R}_a^N \text{ for some } a \in \mathbf{R}^N\}.$$

The space $C^+(\mathbf{R}^N)$ is an algebra with respect to the *convolution*

$$(f * g)(x) = \int_{\mathbf{R}^N} f(x-u)f(u)du \ (f, g \in C^+(\mathbf{R}^N)).$$

Moreover $C^+(\mathbf{R}^N)$ has no divisors of zero. The field of operators $M(\mathbf{R}^N)$ is the quotient field of $C^+(\mathbf{R}^N)$. Thus an operator $A \in M(\mathbf{R}^N)$ can be represented as f/g with $f, g \in C^+(\mathbf{R}^N)$.

By $S(\varepsilon), \varepsilon > 0$, we mean the ε-ball about the origin in $\mathbf{R}^N$:

$$S(\varepsilon) = \{x \in \mathbf{R}^N : x_1^2 + \cdots + x_N^2 < \varepsilon^2\}.$$

By a *delta sequence* (or approximate identity) we shall mean a sequence $\varphi_n, n = 1, 2, \dots$, such that all the following are satisfied:

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(1) $\varphi_n \in C^+(\mathbf{R}^N), n = 1, 2, \ldots;$

(2) for each $\varepsilon > 0$ there is an n_0 such that $n > n_0$ implies $\operatorname{supp} \varphi_n \subset S(\varepsilon)$;

(3) $\varphi_n(x) \geq 0$ for all $x \in \mathbf{R}^N$ and for all n;

(4) $\int_{\mathbf{R}^N} \varphi_N(x)dx = 1$ for all n.

For a function $f \in C(\mathbf{R}^N)$ with compact support put

$$s(f) = \inf\{\varepsilon > 0 : \operatorname{supp} f \subset S(\varepsilon)\}.$$

An operator $A \in M(\mathbf{R}^N)$ is called *regular* if there exists a delta sequence $\varphi_1, \varphi_2, \ldots$ and a sequence $f_n \in C^+(\mathbf{R}^N)$ $(n = 1, 2, \ldots)$ such that

$$A = f_1/\varphi_1 = \cdots = f_n/\varphi_n = \cdots .$$

The simplest way to construct a delta sequence is the following. Take a continuous function f with compact support in $\mathbf{R}^N$ such that $\int_{\mathbf{R}^N} f(x)dx = 1$. Define, for $n = 1, 2, \ldots$, $\varphi_n(x) = n^N f(nx)$. It is easy to check that the sequence φ_n is a delta sequence. Following [**3**], a sequence obtained in this way will be called a *model sequence.*

It is clear that there are delta sequences that are not model sequences. On the other hand, in the definition of regular operators delta sequences can be replaced by model sequences, which follows from

THEOREM. *For every regular operator A there exists a model sequence $\varphi_n (n = 1, 2, \ldots)$ and $g_n \in C^+(\mathbf{R}^N)$ $(n = 1, 2, \ldots)$ such that*

$$A = g_1/\varphi_1 = \cdots = g_n/\varphi_n = \cdots .$$

To prove the theorem we will use infinite convolutions. The method of infinite convolutions was first used by T.K. Boehme in [**1**].

LEMMA. *Let $\psi_1, \psi_2, \ldots$ be a sequence of continuous functions with compact supports in $\mathbf{R}^N$. If $\sum_1^\infty s(\psi_n) < \infty$, then the sequence*

$$\gamma_n = \psi_1 * \cdots * \psi_n$$

converges uniformly to a continuous function γ. *The function* γ *has compact support and* $s(\gamma) \leq \sum_1^\infty s(\psi_n)$.

The lemma is a slight modification of a theorem proved in [**1**], (see also [**4**]).

PROOF OF THEOREM. Let A be a regular operator and let $\delta_1, \delta_2, \dots,$ be a delta sequence such that $A = f_n/\delta_n$ $(n = 1, 2, \dots)$ for some $f_n \in C^+(\mathbf{R}^N)$. Since $s(\delta_n) \to 0$ as $n \to \infty$, by (2), there exists an increasing sequence of positive integers $p_1, p_2, \dots,$ such that $s(\delta_{p_n}) < n^{-3}$ for all n. Define

$$\psi_n(x) = n^{-N}\delta_{p_n}(x/n), \quad n = 1, 2, \dots . \tag{5}$$

Then $s(\psi_n) \leq n^{-2}$ and thus the limit $\varphi = \lim \psi_1 * \cdots * \psi_n$ exists and φ is a continuous function with compact support. Moreover $\int_{\mathbf{R}^N} \varphi(x)dx = 1$. We will prove that the model sequence $\varphi_n(x) = n^N\varphi(nx)$ $(n = 1, 2, \dots)$ has the required property.

For clarity of the proof we will use the notation

$$m^N\eta(mx) = \eta^m(x).$$

First note that

$$\eta_1^n * \cdots * \eta_k^m = (\eta_1 * \cdots * \eta_k)^m, \tag{6}$$

which can be easily proved by induction with respect to k.

Since $\varphi = \lim \psi_1 * \cdots * \psi_n$ we have, by (6),

$$\begin{aligned}\varphi^m &= \lim_{n\to\infty} (\psi_1 * \cdots * \psi_n)^m = \lim_{n\to\infty} \psi_1^m * \cdots * \psi_n^m \\ &= \psi_m^m * (\lim_{n\to\infty} \psi_1^m * \cdots * \psi_{m-1}^m * \psi_{m+1}^m * \cdots * \psi_n^m).\end{aligned}$$

By the lemma, the limit $\lim_{n\to\infty} \psi_1^m * \cdots * \psi_{m-1}^m * \psi_{m+1}^m * \cdots * \psi_n^m = \gamma_m$ exists for all m. By (5), we have $\psi_m^m = \delta_{p_m}$ and hence $\varphi_n = \delta_{p_n} * \gamma_n$. Therefore

$$A = (f_{p_n} * \gamma_n)/(\delta_{p_n} * \gamma_n) = (f_{p_n} * \gamma_n)/\varphi_n$$

for all n. Since φ_n is a model sequence, the proof is complete.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CENTRAL FLORIDA, ORLANDO, FL 32816