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AN EXAMPLE OF A CHEBYSHEV SET, THE COMPLEX CASE

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ABSTRACT. Recently, the authors were asked by Professor Klaus Donner whether the set of matrices, the determinant of which in its absolute value is less than or equal to one, is Chebyshevian within the space of $n \times n$ matrices endowed with the l^2-operator norm. The answer is yes. Although our proof is elementary we think that the result and its proof are of general interest. In addition, there is one further nice example of a Chebyshev set.

We denote by $\mathbf{C}^{n\times m}$ the vector space of complex $n \times m$ matrices over $\mathbf{C}, n, m \in \mathbf{N}$, with elements $A, B, \ldots$ For $A \in \mathbf{C}^{n\times p}$ and $B \in \mathbf{C}^{p\times m}, n, m, p \in \mathbf{N}, A^*$ denotes the adjoint of A in $\mathbf{C}^{p\times n}$ and AB the matrix product of A and B in $\mathbf{C}^{n\times m}$. Instead of $\mathbf{C}^{n\times 1}$ we write $\mathbf{C}^n$ the vector space of complex column vectors, we also write $z, u, \ldots$ to denote its elements.

By l_n^2 we denote $\mathbf{C}^n$ endowed with the Euclidean norm $|\cdot|_2$, and by $\mathcal{L}(l_n^2)$ the vector space of linear transformations of l_n^2 into itself. $\mathcal{L}(l_n^2)$ can be identified with $\mathbf{C}^{n\times n}$ endowed with the l^2-operator norm $||\cdot||_2$ where $A \in \mathbf{C}^{n\times n}$ acts on $z \in \mathbf{C}^n$ via the matrix product Az. Since it will not lead to any ambiguity, in the following we shall write $\mathbf{C}^n$ and $\mathbf{C}^{n\times n}$ instead of l_n^2 and $\mathcal{L}(l_n^2)$, respectively.

For an $A \in \mathbf{C}^{n\times n}$, we denote its singular value decomposition (SVD) by $U\Sigma V^*, U$ and V are unitary matrices in $\mathbf{C}^{n\times n}$ and $\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n), \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$. Σ is uniquely determined by A, its elements are the so-called singular values of A. If A is non-singular, the UV^* is also uniquely defined.

THEOREM.

$$\mathcal{S} = \{S \in \mathbf{C}^{n\times n} : |\det S| \leq 1\}$$

is a Chebyshev set in $\mathbf{C}^{n\times n}$.

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Moreover, if $A \in \mathbf{C}^{n\times n}\backslash\mathcal{S}$ and if $U\Sigma V^*$ is the SVD of A, then $A - dUV^*$ is the approximant of A in $\mathcal{S}$, where

$$0 < d = \min\{t > 0 : (\sigma_1 - t)\dots(\sigma_n - t) = 1\} < \sigma_n$$

is the distance of A from $\mathcal{S}$.

The proof of the theorem rests upon the following lemmata.

LEMMA 1. *Let $B, C \in \mathbf{C}^{n\times n}$ be Hermitian. If B is positive definite, then*

$$\det B < |\det(B + iC)|$$

unless C is the zero matrix.

LEMMA 2. *Let $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \dots, \sigma_n) \in \mathbf{C}^{n\times n}, \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n > 0$. For each $S \in \mathbf{C}^{n\times n}, S \neq \Sigma$ and such that $|\det S| \leq \det \Sigma$,*

$$\min_{|u|_2=1} \mathrm{Re}\, u^*(S - \Sigma)u < 0.$$

The first Lemma is due to A.M. Ostrowski and O. Taussky [**3**]. To make the paper self-contained, we sketch their proof.

We have to show that

$$1 \leq |\det(I + iB^{-1}C)|.$$

This follows from the observation that the eigenvalues of $B^{-1}C$ are real. If $B^{-1}Cz = \lambda z$ for some $\lambda \in \mathbf{C}$ and $z \in \mathbf{C}^n, |z|_2 = 1$, then $Cz = \lambda Bz$ and consequently

$$z^*Cz = \lambda z^*Bz.$$

Since C and B are Hermitian, the quadratic forms z^*Cz and z^*Bz are real. The latter one is even strictly positive, because of B being positive definite.

To prove Lemma 2, let us assume there exists an $S \in \mathbf{C}^{n\times n}$ such that $|\det S| \leq \det \Sigma$ and

$$0 \leq \mathrm{Re}\, u^*(S - \Sigma)u \quad \forall u \in \mathbf{C}^n,\ |u|_2 = 1.$$

Setting $S = B + iC, B$ and C Hermitian, we obtain

$$0 \le u^*(B - \Sigma)u \quad \forall u \in \mathbf{C}^n, |u|_2 = 1.$$

Since

$$0 < \sigma_n \le u^*\Sigma u \le u^* B u \quad \forall u \in \mathbf{C}^n, |u|_2 = 1,$$

B is positive definite and, by Lemma 1,

$$0 < \det B \le |\det(B + iC)| \le \det \Sigma.$$

We use Cayley's expansion of the determinant by diagonal elements for $B = (B - \Sigma) + \Sigma$, see, e.g., A.C. Aitken [**1**, p. 87], and obtain

$$\begin{aligned}
\det B = \det \Sigma & \\
& + (b_{11} - \sigma_1)\sigma_2 \cdots \sigma_n + (b_{22} - \sigma_2)\sigma_1\sigma_3 \cdots \sigma_n \\
& + \cdots + (b_{nn} - \sigma_n)\sigma_1 \cdots \sigma_{n-1} \\
& + \det \begin{pmatrix} b_{11} - \sigma_1 & b_{12} \\ b_{21} & b_{22} - \sigma_2 \end{pmatrix} \sigma_3 \cdots \sigma_n \\
& + \det \begin{pmatrix} b_{11} - \sigma_1 & b_{13} \\ b_{31} & b_{33} - \sigma_3 \end{pmatrix} \sigma_2\sigma_4 \cdots \sigma_n \\
& + \cdots + \det \begin{pmatrix} b_{n-1n-1} - \sigma_{n-1} & b_{n-1n} \\ b_{nn-1} & b_{nn} - \sigma_n \end{pmatrix} \sigma_1 \cdots \sigma_{n-2} + \cdots \\
& + \det \begin{pmatrix} b_{22} - \sigma_2 & \cdots & b_{2n} \\ \vdots & \ddots & \vdots \\ b_{n2} & \cdots & b_{nn} - \sigma_n \end{pmatrix} \sigma_1 \\
& + \cdots + \det \begin{pmatrix} b_{11} - \sigma_1 & \cdots & b_{1n-1} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn-1} - \sigma_{n-1} \end{pmatrix} \sigma_n \\
& + \det(B - \Sigma).
\end{aligned}$$

By assumption $B - \Sigma$ is positive semi-definite. Hence all its minors are non-negative, and it follows from the inequality and the expansion that its minors vanish, giving B equal to Σ. Applying Lemma 1 once more, we obtain, for $C \neq 0$,

$$\det \Sigma < |\det(\Sigma + iC)| \le \det \Sigma,$$

a contradiction. Thus $S = B + iC = \Sigma$, which proves Lemma 2.

PROOF OF THE THEOREM. Let $A \in \mathbf{C}^{n\times n}\backslash\mathcal{S}$, and let $U\Sigma V^*$ be its SVD. Since the mapping

$$\mathbf{C}^{n\times n} \ni B \to \Theta(B) = U^*BV$$

defines an isometric isomorphism on $\mathbf{C}^{n\times n}$ and since $\mathcal{S}$ is invariant under Θ, the theorem is proved, if we can show that $\Sigma_1 = \Sigma - dI$ is the unique approximant of Σ in $\mathcal{S}$. Indeed, by Lemma 2,

$$(*) \qquad \forall S \in \mathcal{S}, S \neq \Sigma_1 \quad 0 < \max_{|u|_2=1} \operatorname{Re} u^*(\Sigma_1 - S)u$$

or replacing Σ_1 by $\Sigma - dI$ in the quadratic form

$$\forall S \in \mathcal{S},\ S \neq \Sigma_1,\ ||\Sigma - \Sigma_1||_2 = d < \max_{|u|_2=1} \operatorname{Re} u^*(\Sigma - S)u \leq ||\Sigma - S||_2,$$

proving the claim.

REMARK. It was not difficult to make a reasonable guess for Σ_1 to be the approximant of Σ in S. To get a better understanding of the proof, we would like to point out that condition $(*)$ can be interpreted as a strict Kolmogorov condition. Indeed, let $\langle\cdot,\cdot\rangle_s$ denote the semi -inner product on $\mathbf{C}^{n\times n}$, defined by

$$\forall A, B \in \mathbf{C}^{n\times n}\ \langle B, A\rangle_s = \lim_{t\to 0+} \frac{||A + tB||_2^2 - ||A||_2^2}{2t}.$$

Then it is not too hard to verify that

$$d \max_{|u|_2=1} \operatorname{Re} u^*(\Sigma_1 - S)u = \langle \Sigma_1 - S, \Sigma - \Sigma_1\rangle_s,$$

see, e.g., H. Berens and U. Westphal [2] for detailed information on semi-inner products and the Kolmogorov condition.

For each $r \geq 0$, let us define $\mathcal{S}_r = \{S \in C^{n\times n} : |\det S| \leq r\}$. Thus $\mathcal{S}_1$ is equal to $\mathcal{S}$ while $\mathcal{S}_0$ defines the set of singular matrices in $\mathbf{C}^{n\times n}$. It follows from above that, for each $r > 0$, $\mathcal{S}_r$ is a Chebyshev set and

$$\mathbf{C}^{n\times n}\backslash\mathcal{S}_r \ni A \to \Pi_r(A) = A - d_r UV^*$$

is the approximant of A in $\mathcal{S}_r$, where $U\Sigma V^*$ is the SVD of A and

$$0 < d_r = \min\{t > 0 : (\sigma_1 - t)\cdots(\sigma_n - t) = r\} < \sigma_n$$

is the distance of A from $\mathcal{S}_r$. Moreover, we have the

THEOREM. *$\mathcal{S}_0$ is a sun in $\mathbf{C}^{n\times n}$ (in the sense of Vlasov) and*

$$\mathbf{C}^{n\times n} \ni A \to \Pi_0(A) = A - \sigma_n UV^*$$

defines a continuous ray selection of the metric projection of $\mathbf{C}^{n\times n}$ onto $\mathcal{S}_0$, i.e., $\Pi_0 : \mathbf{C}^{n\times n} \to \mathcal{S}_0$ is a continuous selection of the metric projection of $\mathbf{C}^{n\times n}$ onto $\mathcal{S}_0$ and such that, for each $A \in \mathbf{C}^{n\times n}, \Pi_0(A)$ is the image under Π_0 of each element of the ray from $\Pi_0(A)$ through A.

REMARK. It is well-known that, for a given $A \in \mathbf{C}^{n\times n}/\mathcal{S}_0$, each approximant of A in $\mathcal{S}_0$, where $\mathbf{C}^{n\times n}$ is endowed with the Frobenius norm, is also an approximant of A in $\mathcal{S}_0$, where $C^{n\times n}$ is endowed with the l^2-operator norm. The metric projection with respect to the Frobenius norm, however, does not admit a continuous selection.

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Added in proof. This is the complex version of our paper in Approximation Theory and Applications **3** (1987), 37-41. We hope that the complex extension and its modified proof justify the note.

REFERENCES

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