ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 18, Number 2, Spring 1988

AN INVARIANCE PRINCIPLE FOR A CLASS OF MONOTONE SYSTEMS AND APPLICATION TO DEGENERATE PARABOLIC EQUATIONS

N.D. ALIKAKOS AND P.W. BATES

1. Introduction. By extending the concept of Liapunov functional to define a Liapunov operator in an ordered Banach space we develop a method of proving stabilization of solutions to a class of monotone systems. This method is employed to prove that solutions to certain degenerate parabolic equations approach equilibria as t approaches $+\infty$.

Given a function p and a semi dynamical system we define the *upper Liapunov operator*, $\overline{V}(p)$, to be the smallest (in the order of the Banach space) super solution greater than or equal to p. The *lower Liapunov operator*, $\underline{V}(p)$, is defined in an analogous way. These operators are used to squeeze points on trajectories onto equilibria when the semi dynamical system has certain monotonicity and stability properties. This idea is inspired in part by work of C. Dafermos [**5**]. The general parabolic equation which can be treated in this way has the form

$$(1.1)\qquad \begin{aligned} &u_t = [a(x,u,\phi(u)_x) + b(x,u)]_x \text{ on } [0,1]\times(0,\infty)\\ &a(x,u,\phi(u)_x) + b(x,u) = 0 \text{ at } x = 0,1, \text{ for all } t > 0, \end{aligned}$$

where ϕ is increasing and $a(x,u,p)$ is increasing in p.

In this paper, for illustrative purposes, we restrict applications to equations of the form

$$(1.2)\qquad \begin{aligned} &u_t = ((u_x)^m + u^\alpha(u-1)V_x)_x \text{ on } [0,1]\times(0,\infty)\\ &(u_x)^m + u^\alpha(u-1)V_x = 0 \text{ at } x = 0,1, \text{ for all } t > 0, \end{aligned}$$

where V is a specified potential (see Figure 1). More varied and general applications will appear elsewhere.

Equation (1.2) is related to a model for the movement of two species with very different rates of mobility (e.g., cows and grass) and these rates are governed by biological pressure.

Received by the editors on April 9, 1986.

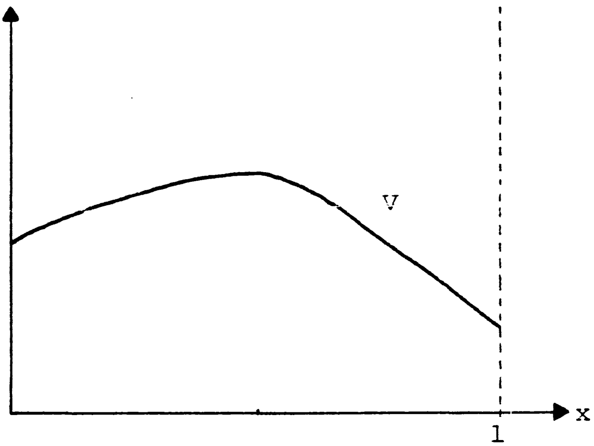

Figure 1. The potential.

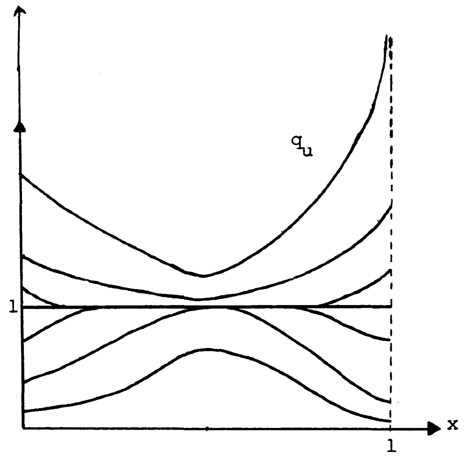

Figure 2. The equilibria.

Bertsch and Hilhorst [**4**] were the first to establish stabilization for a subclass of (1.1) using an idea due to Osher and Ralston [**10**]. For that approach to apply, however, it is necessary that the equation has no intertwining equilibria and this is not the case in (1.2), for example (see Figure 2).

Equation (1.1) is associated with a contraction semigroup on L^1 that is order preserving. Hirsch [**6**] (see also Matano [**9**]) establishes stabilization for *strongly* order preserving contraction semigroups on Hilbert space. There are easy counterexamples if strong monotonicity is relaxed to simple monotonicity. We note that (1.1) is not strongly order preserving, in general. We shall prove the following.

THEOREM. *Let* $u_0 \in L^\infty(0,1), u_0 \geq 0$.

(i) *If* $m \leq \alpha < 2m-1$, *then* $u(x,t)$ *converges to an equilibrium in the* $C^1([0,1])$ *norm, as* $t \to \infty$.

(ii) *If* $\alpha > 2m-1$ *and if* $\int_0^1 u_0 \geq \int_0^1 q_0$, *then* $u(x,t)$ *becomes unbounded in* L^∞ *as* $t \to \infty$.

Here q_0 is the minimal unbounded positive equilibrium (see Figure 2), and $u(x,t)$ is the solution to (1.2) with initial datum u_0. We remark that $\int_0^1 q_0 < +\infty$ when $\alpha > 2m-1$.

2. The Liapunov operator. Let X be a Banach space with a partial order defined by a cone K, that is a closed convex subset of X such that (i) $\alpha x \in K$ whenever $\alpha \geq 0$ and $x \in K$, and (ii) $K \cap (-K) = \{0\}$. We shall use some of the notation and results found in Krasnoselskii [**8**], (see also Amann [**2**]). The order is given by $y \geq x$ if and only if $y - x \in K$. A set, S, is said to be *order bounded* if and only if there exist elements $y, z \in X$ such that $y \geq x \geq z$ for all $x \in S$. Assume

(I) K is *regular*, that is, order bounded monotone sequences converge in X.

(II) K is *minihedral*, that is, given $x, y \in X$ there exists $z \in X$ such that $z \leq x, z \leq y$ and if $w \leq x, w \leq y$ for some $w \in X$ then $w \leq z$. We also assume that this element z is unique and write $z = \inf(x,y)$.

(III) There exists a functional, J on K such that if $x, y \in X$, then

(i) $y \geq x$ and $y \neq x$ implies $J(y) > J(x)$ (strict monotonicity),

(ii) $J(x+y) \geq J(x) + J(y)$ (super additivity).

REMARK. If X is separable then there exists a *continuous linear* functional, L such that $L(x) > 0$ for all $x \in K\backslash\{0\}$, so (III) holds.

REMARK. If X is a space of real valued functions and K is the cone of functions which are nonnegative then (II) is satisfied, the greatest lower bound of two functions being their pointwise minimum. If $X = L^p$ for some $p \geq 1$, then K is regular.

Let $S(t)$ be a semi dynamical system on X, that is, a one parameter family of maps from X into itself parametrized by $t \in \mathbf{R}^+$ and satisfying the axioms.

(a) $S(t) : X \to X$ is continuous for each $t \geq 0$;

(b) $S(t+\tau)f = S(t)S(\tau)f$ for $t \geq 0, \tau \geq 0$ and $f \in X$; and

(c) $S(\cdot)f : [0,\infty) \to X$ is continuous for each $f \in X$. We assume in addition that $S(t)$ is *order preserving*, that is, it satisfies

(d) $f \leq g \Rightarrow S(t)f \leq S(t)g$ for $t \geq 0$.

Following Matano **[9]** and Amann **[2]** we give the

DEFINITION. An element $f \in X$ is called a *super solution* if $S(t)f \leq f$ for all $t \geq 0$ and *subsolution* if $S(t)f \geq f$ for all $t \geq 0$.

DEFINITION. For $p \in X$ let $\Sigma_p = \{\rho \in X : \rho \geq p$ and ρ is a super solution $\}$ and $\sigma_p = \{\rho \in X : \rho \leq p$ and ρ is a subsolution $\}$.

We shall assume $\inf(x,y) \in \Sigma_p$ if $x,y \in \Sigma_p$ and $\sup(x,y) = -\inf(-x,-y) \in \sigma_p$ if $x,y \in \sigma_p$. This is not unreasonable if one considers the motivating examples.

Using the regularity of K and the existence of the strictly monotone functional, J, it is not difficult to prove

LEMMA 2.1. *If Σ_p is nonempty, then it contains a unique minimal element. If σ_p is nonempty, then it contains a unique maximal element.*

DEFINITION. The *upper Liapunov* operator, $\overline{V}$, is given by

$$\operatorname{dom}\overline{V} = \{p \in X : \Sigma_p \neq \phi\}$$
$$\overline{V}(p) = \text{minimal element of } \Sigma_p.$$

The *lower Liapunov* operator, $\underline{\mathrm{V}}$, is given by

$$\begin{aligned} \operatorname{dom} \underline{\mathrm{V}} &= \{p \in X : \sigma_p \neq \phi\} \\ \underline{\mathrm{V}}(p) &= \text{ maximal element of } \sigma_p. \end{aligned}$$

REMARK. Even in the simple case when $S(t)$ is the heat semigroup with homogeneous Dirichlet conditions, $\overline{V}$ is nontrivial. In that case if p has compact support in Ω it can be shown that $\overline{V}(p)$ is actually the solution to the obstacle problem (see [**7**]).

As may be anticipated, these Liapunov operators have certain *monotonicity along trajectories.*

LEMMA 2.2. *Let $u \in \operatorname{dom} \overline{V}$; then for $t \geq 0$, $S(t)u \in \operatorname{dom} \overline{V}$ and $\overline{V}(S(t)u) \leq \overline{V}(u)$. A similar statement holds for $\underline{V}$.*

The proof relies on the definitions of super solution, the minimality of $\overline{V}$ and the fact that $S(t)$ is order preserving.

DEFINITIONS.

(i) The trajectory $S(\cdot)u$ is called *Liapunov stable* if, for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $||u - v|| < \delta$ implies $||S(t)u - S(t)v|| < \varepsilon$ for all $t \geq 0$. Contraction semigroups clearly produce Liapunov stable trajectories.

(ii) The *orbit* of u is the set $\gamma(u) = \{S(t)u : t \geq 0\}$.

(iii) The *ω-limit set* of u is

$$\omega(u) = \{v \in X : v = \lim_{n \to \infty} S(t_n)u \text{ for some sequence } t_n \to \infty\}.$$

This brings us to our main abstract result.

THEOREM 2.1. (THE INVARIANCE PRINCIPLE). *Let $u \in \operatorname{dom} \overline{V}$. Suppose that $\gamma(u)$ is relatively compact and bounded below, and that the trajectory through any point of $\overline{\gamma(u)}$ is Liapunov stable. Then $\overline{V}$ takes the same value on $\omega(u)$ and that value is an equilibrium, i.e., it is fixed under $S(t)$ for all $t \geq 0$. A similar result holds for $u \in \operatorname{dom} \underline{\mathrm{V}}$.*

SKETCH OF PROOF. First, the relative compactness of the orbit and the Liapunov stability assumption imply that $\omega(u) \neq \emptyset$ and that, for any $v \in \omega(u)$, we have $\omega(v) = \omega(u)$, respectively.

Consider $v, w \in \omega(u)$ and choose an increasing sequence $\{t_n\}$ such that $S(t_n)v \to w$ in X. By monotonicity, $\{\overline{V}(S(t_n)v)\}$ is decreasing and this sequence is bounded below since $\{S(t_n)v\} \subset \gamma(u)$ is bounded below. Regularity implies $\lim \overline{V}(S(t_n)v) = q$ exists. Clearly, q is a supersolution by continuity of $S(t)$, and $\overline{V}(S(t_n)v) \geq S(t_n)v$ implies that $q \geq w$ so $q \geq \overline{V}(w)$. We have $\overline{V}(v) = \overline{V}(s(0)v) \geq \overline{V}(s(t_n)v) \geq q \geq \overline{V}(w)$. Symmetry shows that $\overline{V}$ takes only one value on $\omega(u)$, namely q. For $v \in \omega(u), v \leq q$ so $S(t)q$ and $q = \overline{V}(s(t)v) \leq \overline{V}(s(t)q) = S(t)q \leq q$ for all $t \geq 0$. This shows that q is an equilibrium.

REMARK. Under the hypotheses of Theorem 2.1, to establish stabilization it is sufficient to show that $\omega(u)$ consists of equilibria or even supersolutions since these remain fixed under $\overline{V}$.

3. An example. Consider the problem

$$u_t = ((u_x)^m + u^\alpha(u-1)V_x)_x \text{ on } (0,1) \times (0,\infty) \tag{3.1}$$

$$\begin{aligned} (u_x)^m + u^\alpha(u-1)V_x &= 0 \text{ for } x = 0, 1, t > 0, \\ u(x,0) = u_0(x) &\geq 0 \in L^\infty(0,1). \end{aligned} \tag{3.2}$$

Suppose $\alpha \geq m > 1$ and $V \in C^2(0,1)$ has the form given in Figure 1. By convention $z^m \equiv z|z|^{m-1}$. Let $X \equiv L^1(0,1)$ and let K be the cone of nonnegative functions in X. Then (I), (II) hold. One can show [**1**] that (3.1), (3.2) generates a *contraction semigroup* on X which is *order preserving* and *conserves the integral* and that, for $\alpha < 2m-1, \gamma(u_0)$ is relatively compact in $C^1([0,1])$ and bounded away from 0. Let q_0 be the minimal unbounded positive equilibrium of (3.1) and (3.2) (see Figure 2.)

THEOREM 3.1.

(i) *If $\alpha \leq 2m-1$, then $u(\cdot,t)$ converges as $t \to \infty$ in $C^1([0,1])$ to an equilibrium.*

(ii) *If $\alpha > 2m-1$ (in which case $\int_0^1 q_0(x)dx < \infty$) and if $\int_0^1 u_0(x)dx \geq \int_0^1 q_0(x)dx$, then $u(\cdot,t)$ becomes unbounded in $L^\infty(0,1)$ in finite or infinite time.*

REMARKS. (a) The case $\alpha = 2m - 1$ has been settled by C. Grant and will appear elsewhere.

(b) If $\alpha > 2m - 1$ and $\int_0^1 u_0(x)dx > \int_0^1 q_0(x)dx$ we have established in work in progress that the solution blows up in finite time.

PROOF OF PART (i). Suppose first that $u_0(x) < q_0(x)$ on $[0, 1]$. Then there exists an equilibrium q such that $u_0(x) \leq q(x) < q_0(x)$ on $[0, 1]$, and by definition of q_0 we know that $q \in L^\infty \subset X$. Hence, $u_0 \in \operatorname{dom} \overline{V}$. Also, since $u_0 \geq 0$ and 0 is an equilibrium, $u_0 \in \operatorname{dom} \underline{V}$. The above remarks and those preceding the statement of the theorem show that the hypotheses of Theorem 2.1 hold and so to establish stabilization it suffices to show that $\omega(u_0)$ consists of equilibria. Let $\xi \in \omega(u_0)$ and suppose that ξ is not an equilibrium. By Theorem 2.1 and using the invariance of $\omega(u_0)$ we note that there exist equilibria $\overline{q}$ and $\underline{q}$ in X with

$$\overline{V}(S(t)\xi) = \overline{q} \text{ and } \underline{V}(S(t)\xi) = \underline{q} \text{ for all } t \geq 0.$$

Noting the distribution of the equilibria, that is, functions satisfying (3.2) on the interval $[0, 1]$ (see Figure 2.), we know that either $\overline{q}(0) > \underline{q}(0)$ or $\overline{q}(1) > \underline{q}(1)$. We shall assume the latter and omit the argument for the other case since it is similar. Either $\xi(1) < \overline{q}(1)$ or $\xi(1) > \underline{q}(1)$. Again, we assume the latter and omit the proof in the other case. For $\varepsilon > 0$ and small, define

$$\rho(x) = \begin{cases} \underline{q}(x) & \text{on } [0, 1 - \varepsilon] \\ \text{linear} & \text{on } [1 - \varepsilon, 1] \end{cases}$$

such that

$$\rho \leq \xi \text{ and } (\rho')^m \geq -\rho^\alpha(\rho - 1)V_x \text{ on } [0, 1] \text{ (see Figure 3).}$$

Let
$$v_0(x) = \int_0^x \rho(y)dy \text{ and } v(x, t) = \int_0^x S(t)\rho(y)dy,$$

where S is the solution semigroup determined by (3.1), (3.2). Then v satisfies

$$v_t = (v_{xx})^m + v_x^\alpha(v_x - 1)V_x, \quad 0 < x < 1, \quad 0 < t \tag{3.3}$$

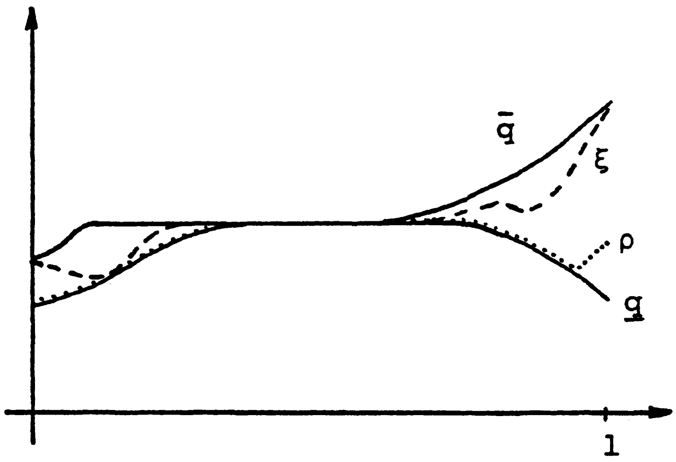

Figure 3.

$$v(0,t) = 0, \quad v(1,t) = \int_0^1 \rho(y)dy \text{ (by conservation), for } t \geq 0, \tag{3.4}$$

and $v(x,0) = v_0(x), 0 \leq x \leq 1$. Now $(v_{0xx})^m + v_{0x}^\alpha(v_{0x} - 1)V_x = (\rho')^m + \rho^\alpha(\rho - 1)V_x \geq 0$ on $[0,1]$. Using an argument similar to those found in Sattinger [**11**] on can show that $v(x,t)$ is nondecreasing in t. Since $v(x,t) \leq \int_0^x \bar{q}(y)dy$ we see $v(x,t)$ converges to some functions $v_\infty(x)$ as t tends to $+\infty$. By the compactness of $\gamma(\rho)$ we know that $\omega(\rho) \neq \emptyset$. The uniqueness of v_∞ implies that $\omega(\rho)$ must be a single point and, hence an equilibrium, $\tilde{q}$. But now we have $\underline{q} = \underline{V}(\omega(\xi)) \geq \underline{V}(\omega(\rho)) = \tilde{q} \geq \underline{q}$, so $\tilde{q} = \underline{q}$. The inequalities result from $\xi \geq \rho \geq \underline{q}$. We have reached a contradiction since $\int_0^1 \tilde{q} = \int_0^1 \rho > \int_0^1 \underline{q}$. Now we relax the condition $u_0 \leq q_0$.

Since u_0 is in L^∞ and $\alpha < 2m - 1$, $|u(\cdot,t)|_{L^\infty} < C$ (see [**1**].) Let $f(u) = u^\alpha(u-1)$ for $u \leq C + 1$ and extend this function in a positive smooth and bounded fashion, $\tilde{f}$, on $\mathbf{R}^+$. Note that the equilibria of the new equation do not blow up for finite x, and so, by the above argument, $u(x,t)$ stabilizes to some equilibrium q,

$$(q_x)^m + \tilde{f}(q)V_x = 0.$$

Hence $q \leq C$, and so q is an equilibrium solution to (3.1) and (3.2).

PROOF OF PART (ii). Note that, near the blow-up point $x = 1, q_0$ satisfies essentially an equation of the form

$$\dot{q}_0 = q_0^{\frac{\alpha+1}{m}}.$$

Therefore

$$q_0(x) \sim (1-x)^{\frac{1}{1-\lambda}}, \quad \lambda = \frac{\alpha+1}{m},$$

and so if $\alpha \leq 2m-1$ the mass of q_0 is infinite. A consequence of this is that, in this range, there are bounded equilibria of arbitrarily large mass. On the other hand, if $\alpha > 2m-1$, then q_0 has finite mass. If $|u(\cdot,t)|_{L^\infty}$ is uniformly bounded, by the proof of (i) above, the solution has to converge to a bounded equilibrium of (3.1), an impossibility by conservation since all the bounded equilibria have mass less than that of q_0 due to the ordering.

REMARKS.

(1) From Figure 2 one can see that the integral does not uniquely determine the equilibrium.

(2) The principal part of the operator in (3.3) is known as the dual of porous medium operator.

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MATHEMATICS DEPARTMENT, UNIVERSITY OF TENNESSEE, KNOXVILLE, TN 37996

MATHEMATICS DEPARTMENT, BRIGHAM YOUNG UNIVERSITY, PROVO, UT 94602