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BASIC QUASI-PROXIMITIES, GRILLS AND COMPACTIFICATIONS

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ABSTRACT. In this paper we make a study of nonsymmetric basic proximity structures defined by ignoring the symmetry axiom from the definition of basic proximities given in Čech [1]. They have been used to construct a type of compactification of D_0-closure spaces.

1. Introduction. Pervin [14] introduced the concept of quasi-proximities in 1963. He defined quasi-proximities by ignoring the symmetry axiom from the definition of classical proximities (EF-proximities) defined by Efremovič [4]. Steiner [15] has proved that there is a quasi-proximity as defined by Pervin, compatible with each topological space. Gastl [6] has also investigated quasi-proximities as defined by Pervin [14].

Mattson [11] has investigated nonsymmetric proximities, which are defined by including distributive properties in addition to what has been defined by Pervin [14]. This type of nonsymmetric proximities have also been studied by E.P. Lane [8], Singal and Sunder Lal [17] among others.

After the introduction of Efremovič proximities, extensive investigations to generalize the theory of proximities in different ways have been made by Leader [9], Lodato [10], Harris [7], Gagrat and Naimpally [5], Sharma and Naimpally [16] Thron and Warren [22] and Mozzochi, Gagrat and Naimpally [12].

Basic proximities were introduced by Čech [1]. It has been shown that the closure operator induced by a basic proximity satisfies a symmetry axiom. Thus one can not expect a basic proximity compatible with an arbitrary closure space.

In this paper, an attempt, parallel to what has been done by Pervin to subsume all topological spaces under proximity-like structures defined by modifying suitably the definition of EF-proximities, has been made to subsume all closure spaces under basic proximity-like structures defined by modifying the definition of basic proximities of Čech. We have introduced the concept of basic quasi-proximities and used the theory of grills to develop the theory in line with what has been done by Thron [19] for

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the theory of proximities. We can show that many of the results of **[19]** remain valid for quasi-proximities.

Finally, we have used the theory of quasi-proximities to construct D_0-extensions (in particular compactifications) of D_0-closure spaces. But, as it stands right now, we have failed to characterize what compactifications of D_0-closure spaces could be achieved by our method. We have concluded by making a remark about the Riesz' problem in the context of quasi-proximities.

2. Preliminaries. In this section we fix our notation, collect several definitions, and state some results without proof.

In what follows there is always an underlying nonempty set X. It will be convenient to denote the elements of X by $x, y, \ldots$ and its subsets by $A, B, \ldots$. Families of subsets will be denoted by $\mathscr{A}, \mathscr{B}, \ldots$. In particular, $\mathscr{F}$ will be used for filters, $\mathscr{U}, \mathscr{V}$ for ultrafilters and $\mathscr{G}$ for grills. The collection of filters, ultrafilters and grills will be denoted by $\Phi(X), \Omega(X)$ and $\Gamma(X)$ respectively. Though, an element $x \in X$ and the set $\{x\}$ containing the single element x are conceptually different, they are not distinguished.

We begin by recalling the definition of a filter, ultrafilter and grill. Basic results on grills are given in Thron **[19]**.

DEFINITIONS 2.1. A stack $\mathscr{S}$ on a set X is a subfamily of the power set $\mathscr{P}(X)$ satisfying the condition

$$A \supset B \in \mathscr{S} \Rightarrow A \in \mathscr{S}.$$

We denote, by $\sum(X)$, the set of all stacks on the set X. Note that $\sum(X)$ is closed under arbitrary unions and intersections.

A filter $\mathscr{F}$ on a set X is a nonempty stack satisfying the condition $A, B \in \mathscr{F} \Rightarrow A \cap B \in \mathscr{F}$. A filter is said to be a proper filter if it does not contain the empty set.

An ultrafilter $\mathscr{U}$ on X is a maximal proper filter on X. In particular, if $\mathscr{U}$ contains a singleton $\{x\}$, then $\mathscr{U}$ is called a principal ultrafilter and is denoted by $\mathscr{U}(x)$.

A grill $\mathscr{G}$ on X is a stack which satisfies the additional conditions

$$\phi \notin \mathscr{G} \text{ and } A \cup B \in \mathscr{G} \Rightarrow A \in \mathscr{G} \text{ or } B \in \mathscr{G}.$$

A nonempty grill is called a proper grill. Observe that $\Phi(X)$ is closed under arbitrary intersections and $\Gamma(X)$ is closed under arbitrary unions.

The words "filter, grill" will be used to mean 'proper filter, proper grill' unless stated otherwise.

DEFINITIONS 2.2. A function $c: \mathscr{P}(X) \rightarrow \mathscr{P}(X)$ is called a closure operator on the set X if it satisfies the following three conditions:

C_1: $c(\emptyset) = \emptyset$;
C_2: $c(A \cup B) = c(A) \cup c(B)$;
C_3: $c(A) \supset A$.

The pair (X, c), where c is a closure operator on the set X, is called a closure space. These concepts are generalizations of the more familiar concepts of Kuratowski clousre operators and topological spaces, respectively. If, in addition, a closure operator c satisfies

$$C_4\colon c(c(A)) \subset c(A),$$

then c is called a Kuratowski closure operator and (X, c) is called a topological space.

Closure spaces were introduced by Čech [1]. They form the basic spatial structures in [1]. One advantage of this approach is that it provides a convenient basis for the study of general proximities and uniform structures.

DEFINITIONS 2.3. A binary relation π on the power set $\mathscr{P}(X)$ is said to be a basic quasi-proximity on X if the following conditions hold:

QP_1: (i) $(A, B \cup C) \in \pi \Leftrightarrow (A, B) \in \pi$ or $(A, C) \in \pi$;
(ii) $(A \cup B, C) \in \pi \Leftrightarrow (A, C) \in \pi$ or $(B, C) \in \pi$;
QP_2: $A \cap B \neq \phi \Rightarrow (A, B) \in \pi$; and
QP_3: $(A, B) \in \pi \Rightarrow (A \neq \phi \neq B)$.

The pair (X, π) is called a basic quasi-proximity space. If, in addition, the basic quasi-proximity π satisfies

QP_4: $(A, B) \in \pi \Rightarrow (B, A) \in \pi$,

then π is called a basic proximity on X and (X, π) is called a basic proximity space. The basic proximity spaces were introduced and studied by Cech [1]. For each basic quasi-proximity π, we define

$$c_\pi(A) = \{x \in X\colon (A, x) \in \pi\}, \qquad \text{for all } A \subset X.$$

One can easily verify that $c_\pi\colon \mathscr{P}(X) \to \mathscr{P}(X)$ is a closure operator; it is called the closure operator induced by π. By π^{-1} we mean $\pi^{-1} = \{(A, B)\colon (B, A) \in \pi\}$. It can be verified that π^{-1} is also a basic quasi-proximity and it is called the conjugate of π.

We define

$$\pi(A) = \{B \subset X\colon (A, B) \in \pi\}, \qquad \text{for all } A \subset X.$$

Henceforth we shall drop the prefix 'basic' and just talk of quasi-proximities.

DEFINITION 2.4. A duality function $d\colon \sum(X) \to \sum(X)$ is defined by

$$d(\mathscr{S}) = \{B \subset X\colon B \cap A \neq \phi, A \in \mathscr{S}\}.$$

The duality function satisfies the following results:

(i) $d(\mathcal{S}) = \{B \subset X: X - B \notin \mathcal{S}\}$;
(ii) $d(d(\mathcal{S})) = \mathcal{S}$;
(iii) d is a bijection map from $\Phi(X)$ onto $\Gamma(X)$ and from $\Gamma(X)$ onto $\Phi(X)$;
(iv) For any element $\mathcal{U} \in \Omega(X)$, $d(\mathcal{U}) = \mathcal{U}$;
(v) $d(\bigcup \mathcal{S}_i) = \bigcap d(\mathcal{S}_i)$ and $d(\bigcap \mathcal{S}_i) = \bigcup d(\mathcal{S}_i)$; and
(iv) $\mathcal{S}_1, \mathcal{S}_2 \in \sum(X)$, $\mathcal{S}_1 \supset \mathcal{S}_2 \Rightarrow d(\mathcal{S}_1) \subset d(\mathcal{S}_2)$.

Below, we state without proof some results from Thron **[19]**.

THEOREM 2.5. *Let $\mathcal{G} \in \Gamma(X)$ be any element. Then*

$$\mathcal{G} = \bigcup \{\mathcal{U} \in \Omega(X): \mathcal{U} \subset \mathcal{G}\}.$$

THEOREM 2.6. *Let $\mathcal{F} \in \Phi(X)$ and $\mathcal{G} \in \Gamma(X)$. Then $\mathcal{F} \subset \mathcal{G}$ holds if and only if there exists an element $\mathcal{U} \in \Omega(X)$ such that $\mathcal{F} \subset \mathcal{U} \subset \mathcal{G}$.*

THEOREM 2.7. *Let $\mathcal{F} \in \Phi(X)$ and $\mathcal{G}_1, \mathcal{G}_2 \in \Gamma(X)$ be such that $\mathcal{F} \subset \mathcal{G}_1 \bigcup \mathcal{G}_2$. Then $\mathcal{F} \subset \mathcal{G}_1$ cr $\mathcal{F} \subset \mathcal{G}_2$.*

From Theorem 2.7 one can easily deduce the following corollary.

COROLLARY 2.8. *Let $\mathcal{G} \in \Gamma(X)$ and $\mathcal{F}_1, \mathcal{F}_2 \in \Phi(X)$ be such that $\mathcal{F}_1 \bigcap \mathcal{F}_2 \subset \mathcal{G}$. Then $\mathcal{F}_1 \subset \mathcal{G}$ or $\mathcal{F}_2 \subset \mathcal{G}$.*

THEOREM 2.9. *A binary relation π on $\mathcal{P}(X)$ is a quasi-proximity on X if,* for each $A \subset X$,
(i) *$\pi(A)$ and $\pi^{-1}(A)$ are grills*; and
(ii) $\pi(A) \supset \bigcup \{\mathcal{U} \in \Omega(X): A \in \mathcal{U}\} = \{B \subset X: A \bigcap B \neq \phi\}$.

PROOF. It is easy to verify that the condition (i) is equivalent to QP_1 and QP_3, and the condition (ii) is equivalent to QP_2.

THEOREM 2.10. *Let π be a binary relation on $\mathcal{P}(X)$. Then π is a quasi-proximity on X if and only if*
(i) *$\pi(A)$ is a grill for each $A \subset X$*;
(ii) *$\pi(A \bigcup B) = \pi(A) \bigcup \pi(B)$ for all $A, B \subset X$*;
(iii) $\pi(A) \supset \bigcup\{\mathcal{U} \in \Omega(X): A \in \mathcal{U}\}$ *for all $A \subset X$*; and
(iv) $\pi(\phi) = \phi$.

PROOF. Conditions (i), (iii) and (iv) are equivalent to conditions QP_1 and QP_3. Condition QP_2 is equivalent to condition (iii).

THEOREM 2.11. *Let π_1, π_2 be any two quasi-proximities on X and let A, B be subsets of X. Then*
i) *$\pi_1 \bigcup \pi_2$ is a quasi-proximity on X*;
ii) *if $A \supset B$, then $\pi_1(A) \supset \pi_1(B)$*;
iii) $(\pi_1 \bigcup \pi_2)(A) = \pi_1(A) \bigcup \pi_2(A)$; and
iv) $c_{\pi_1 \bigcup \pi_2}(A) = c_{\pi_1}(A) \bigcup c_{\pi_2}(A)$.

PROOF. The proof is straightforward and hence we omit it.

COROLLARY 2.12. *For any quasi-proximity π on X, $\pi \bigcup \pi^{-1}$ is a proximity on X.*

PROOF. The proof follows from the fact that $(\pi \bigcup \pi^{-1})^{-1} = \pi \bigcap \pi^{-1}$.

THEOREM 2.13. *Let π be a quasi-proximity on X. Then $A \in \pi(B)$ implies there is a $\mathscr{U} \in \Omega(X)$ such that $\mathscr{U} \subset \pi(A) \bigcap \pi(B)$.*

PROOF. The proof follows from the fact that all grills are unions of ultrafilters, and if $C \in \mathscr{U}$, $\mathscr{U} \in \Omega(X)$, then $\mathscr{U} \subset \pi(C)$.

THEOREM 2.14. *Let π_1 and π_2 be any two quasi-proximities on X. Then $c_{\pi_1} = c_{\pi_2}$ if and only if $\pi_1(x) = \pi_2^{-1}(x)$, for all $x \in X$.*

PROOF. The proof follows from the following observation. Let π be any quasi-proximity on X and A be any subset of X. Then

$$\begin{aligned} c_\pi(A) = \{x \in X: (A, x) \in \pi\} &= \{x \in X: (x, A) \in \pi^{-1}\} \\ &= \{x \in X: A \in \pi^{-1}(x)\}. \end{aligned}$$

DEFINITION 2.15. For any quasi-proximity π on X and for any filter $\mathscr{F}$ on X, we define

$$\pi(\mathscr{F}) = \bigcap \{\pi(F): F \in \mathscr{F}\}.$$

THEOREM 2.16. *For any quasi-proximity π on X,*
(a) *$\pi(\mathscr{F})$ is a grill on X, for each $\mathscr{F} \in \Phi(X)$;*
(b) *$\pi(\mathscr{F}_1 \bigcap \mathscr{F}_2) = \pi(\mathscr{F}_1) \bigcup \pi(\mathscr{F}_2)$, for any two $\mathscr{F}_1, \mathscr{F}_2 \in \Phi(X)$;*
(c) *$\pi(A) = \bigcup \{\pi(\mathscr{U}): A \in \mathscr{U}, \mathscr{U} \in \Omega(X)\}$;*
(d) *$\pi(\mathscr{F}) \supset \mathscr{F}$, for each $\mathscr{F} \in \Phi(X)$;*
(e) *$\mathscr{F}_1 \subset \pi(\mathscr{F}_2) \Leftrightarrow \mathscr{F}_2 \subset \pi^{-1}(\mathscr{F}_1)$, where $\mathscr{F}_1, \mathscr{F}_2 \in \Phi(X)$;*
(f) *$\mathscr{V} \subset \pi(A)$, $\exists$ a $\mathscr{U} \in \Omega(X)$ such that $A \in \mathscr{U}$ and $\mathscr{V} \subset \pi(\mathscr{U})$;* and
(g) *$\pi(\mathscr{U}(x)) = \pi(x)$, for all $x \in X$.*

PROOF. (a). Clearly, $\pi(\mathscr{F})$ is a stack on X not containing the empty set ϕ. For any two subsets A, B of X, $A \bigcup B \in \pi(\mathscr{F})$ implies $\mathscr{F} \subset \pi^{-1}(A \bigcup B) = \pi^{-1}(A) \bigcup \pi^{-1}(B)$. In view of Theorem 2.7, we conclude that

$$\mathscr{F} \subset \pi^{-1}(A) \quad \text{or} \quad \mathscr{F} \subset \pi^{-1}(B).$$

Hence $A \in \pi(\mathscr{F})$ or $B \in \pi(\mathscr{F})$. Thus $\pi(\mathscr{F})$ is a grill.

(b). We have $A \in \pi(\mathscr{F}_1 \bigcap \mathscr{F}_2)$ if and only if $\mathscr{F}_1 \bigcap \mathscr{F}_2 \subset \pi^{-1}(A)$ if and only if $\mathscr{F}_1 \subset \pi^{-1}(A)$ or $\mathscr{F}_2 \subset \pi^{-1}(A)$ (see corollary 2.8) if and only if $A \in \pi(\mathscr{F}_1)$ or $A \in \pi(\mathscr{F}_2)$ if and only if $A \in \pi(\mathscr{F}_1) \bigcup \pi(\mathscr{F}_2)$.

(c). $B \in \pi(A)$ if and only if $A \in \pi^{-1}(B)$ if and only if $\exists$ a $\mathscr{U} \in \Omega(X)$ such that $A \in \mathscr{U}$ and $\mathscr{U} \subset \pi^{-1}(B)$ if and only if $\exists$ a $\mathscr{U} \in \Omega(X)$ such that $A \in \mathscr{U}$ and $B \in \pi(\mathscr{U})$.

Thus $\pi(A) = \bigcup \{\pi(\mathscr{U}): \mathscr{U} \in \Omega(X), A \in \mathscr{U}\}$.

(d). The result is immediate from QP_2.

(e). $\mathscr{F}_1 \subset \pi(\mathscr{F}_2)$ if and only if $\mathscr{F}_1 \subset \pi(F)$, for all $F \in \mathscr{F}_2$, if and only if $F \in \pi^{-1}(\mathscr{F}_1)$, for all $F \in \mathscr{F}_2$, if and only if $\mathscr{F}_2 \subset \pi^{-1}(\mathscr{F}_1)$.

(f). $\mathscr{V} \subset \pi(A) \Rightarrow A \in \pi^{-1}(\mathscr{V}) \Rightarrow \exists$ a $\mathscr{U} \in \Omega(X)$ such that $A \in \mathscr{U} \subset \pi^{-1}(\mathscr{V}) \Rightarrow \mathscr{V} \subset \pi(\mathscr{U})$.

(g). $\pi(\mathscr{U}(x)) = \bigcap \{\pi(A): A \in \mathscr{U}(x)\} = \pi(x)$ (See Theorem 2.11 (ii)).

DEFINITIONS 2.17. A closure space (X, c) is said to satisfy the R_0-axiom if, given $x, y \in X$ such that $x \in c(y)$, then $y \in c(x)$.

Now we prove an interesting result in this section. Let (X, c) be a closure space. A quasi-proximity π on X is said to be compatible with c if $c_\pi = c$. In what follows we shall see that, for any closure space, there exists a compatible quasi-proximity. This is not true in the case of basic proximities. Since the basic proximity is symmetric, to get a compatible proximity, the given closure space is required to satisfy the R_0-separation axiom.

THEOREM 2.18. *Let (X, c) be any closure space. Define*

$$\pi_c^0 = \{(A, B): c(A) \cap B \neq \phi\}.$$

Then π_c^0 is a quasi-proximity on X compatible with c. Also, π_c^0 is the smallest of all quasi-proximities compatible c.

PROOF. Verification of the fact that π_c^0 is a quasi-proximity is straightforward and hence we omit it. Also, $x \in c(A)$ if and only if $(A, x) \in \pi_c^0$ if and only if $x \in c_{\pi_c^0}(A)$. Thus π_c^0 is compatible with c. Finally, let π be any quasi-proximity on X compatible with c. Then $c(A) \cap B \neq \phi \Rightarrow c_\pi(A) \cap B \neq \phi \Rightarrow (A, x) \in \pi$, for some $x \in B \Rightarrow (A, B) \in \pi$. This completes the proof.

REMARK 2.19. Now we give an example of a closure space for which there exists no compatible basic proximity.

EXAMPLE 2.20. Let X be the set of all natural numbers. Define an operator $c: \mathscr{P}(X) \to \mathscr{P}(X)$ by the rule

$$c(A) = \{x \in X: x \geqq a, \text{ for some } a \in A\}.$$

It can be easily verified that c is a closure operator. By the above theorem, we can get a quasi-proximity π_c^0 compatible with c. Since, by the definition of c, we have $2 \in c(1)$ but $1 \notin c(2)$, it follows that c does not satisfy the R_0-axiom and hence there does not exist a basic proximity on X compatible with c.

THEOREM 2.21. *Let (X, c) be a closure space. We define a relation π_c^1 on $\mathscr{P}(X)$ by*

$$\pi_c^1 = \pi_c^0 \cup \{(A, B): A \neq \phi \textit{ and } |B| \geqq \aleph_0\}.$$

Then π_c^1 *is the largest quasi-proximity on* X *compatible with* c, *where* π_c^0 *is defined in Theorem* 2.18.

($\aleph_0$ denotes the cardinal number of the set of natural numbers and $|B|$ the cardinal number of the set B.)

PROOF. We denote, by Δ, the set $\{(A, B): A \neq \emptyset, |B| \geqq \aleph_0\}$. Clearly Δ satisfies QP_1 and QP_3 of Definition 2.3. Since π_c^0 is a quasi-proximity it follows that $\pi_c^1 = \pi_c^0 \cup \Delta$ is a quasi-proximity on X.

Now, from the construction of π_c^1, it follows that $(A, x) \in \pi_c^1$ if and only if $(A, x) \in \pi_c^0$ if and only if $x \in c_{\pi_c^0}(A) = c(A)$. Hence it follows that π_c^1 is a quasi-proximity compatible with c.

Finally we shall show that π_c^1 is the largest quasi-proximity on X compatible with c. Let π be any quasi-proximity on X compatible with c. Let $(A, B) \in \pi$. Clearly $A \neq \phi \neq B$. If $|B| \geqq \aleph_0$, then $(A, B) \in \pi_c^1$. So suppose that B is finite. Then there exists $x \in B$ such that $(A, x) \in \pi$ and hence $x \in c_\pi(A) = c(A) = c_{\pi_c^1}(A)$. That is, $(A, x) \in \pi_c^1$. Consequently $(A, B) \in \pi_c^1$. This completes the proof.

3. Proximity Neighbourhoods. In this section we give an alternative approach to introduce quasi-proximities by axiomatizing certain properties of neighbourhood filters.

DEFINITION 3.1. Let (X, π) be a quasi-proximity space. A subset B of X is called a proximal neighbourhood of a subset A with respect to π if $(X - B) \notin \pi(A)$ (notation $A \Subset_\pi B$).

Now it is clear that $A \Subset_\pi B$ if and only if $(X - B) \Subset_{\pi^{-1}} (X - A)$. If no confusion is likely to arise about the proximity, then we simply write $A \Subset B$. We denote by $\mathfrak{N}(\pi, A)$ the set of all proximal neighbourhoods of A with respect to π, i.e., $\mathfrak{N}(\pi, A) = \{B \subset X: X - B \notin \pi(A)\}$.

THEOREM 3.2. *Let* π *be a quasi-proximity on* X. *Then, for any* $A \subset X$:

i) $\mathfrak{N}(\pi, A) = d(\pi(A))$, *where d is the duality function*;

ii) $\mathfrak{N}(\pi, A) = \bigcap \{\mathscr{U} \in \Omega(X): \mathscr{U} \subset \pi(A)\} \subset \{B \subset X: A \subset B\}$; *and*

iii) $\mathfrak{N}(\pi, A) \subset \bigcap \{\mathscr{U} \in \Omega(X): A \in \mathscr{U}\}$.

PROOF. i). $B \in \mathfrak{N}(\pi, A)$ if and only if $X - B \notin \pi(A)$ if and only if $B \in d(\pi(A))$.

ii). Since $\pi(A)$ is a grill, by (i), we have

$$\begin{aligned}\mathfrak{N}(\pi, A) = d(\pi(A)) &= d(\bigcup\{\mathscr{U}\in\Omega(X): \mathscr{U} \subset \pi(A)\}\\ &= \bigcap \{d(\mathscr{U}): \mathscr{U} \subset \pi(A)\}\\ &= \bigcap \{\mathscr{U}: \mathscr{U} \subset \pi(A)\}.\end{aligned}$$

Also $B \in \mathfrak{N}(\pi, A) \Rightarrow X - B \notin \pi(A) \Rightarrow (X - B) \cap A = \phi \Rightarrow A \subset B$.

iii) Since $\pi(A) \supset \bigcup \{\mathscr{U} \in \Omega(X): A \in \mathscr{U}\}$, it follows that

$$\begin{aligned}\mathfrak{N}(\pi, A) = d(\pi(A)) &\subset d(\bigcup\{\mathscr{U} \in \Omega(X): A \in \mathscr{U}\} \\ &= \bigcap\{d(\mathscr{U}): A \in \mathscr{U}\} \\ &= \bigcap\{\mathscr{U} \in \Omega(X): A \in \mathscr{U}\}.\end{aligned}$$

COROLLARY 3.3. *For any quasi-proximity π on X and for any $A \subset X$, $\mathfrak{N}(\pi, A)$ is a filter on X.*

PROOF. Since, for each $A \subset X$, $\pi(A)$ is a grill, $d(\pi(A))$ is a filter on X and hence by the above theorem, $\mathfrak{N}(\pi, A)$ is a filter on X.

COROLLARY 3.4. *Let π be a quesi-proximity on X. Then $B \in \pi(A)$ if and only if $B \cap N_A \neq \varnothing$, for all $N_A \in \mathfrak{N}(\pi, A)$.*

PROOF. By the above Theorem,

$$\pi(A) = d(d(\pi(A))) = d(\mathfrak{N}(\pi, A)) = \{B \subset X: B \cap N_A \neq \varnothing, N_A \in \mathfrak{N}(\pi, A)\}.$$

This completes the proof.

COROLLARY 3.5. *Let π be a quasi-proximity on X. Then for any subset A of X,*

$$c_\pi(A) = \bigcap\{N_A: N_A \in \mathfrak{N}(\pi, A)\}.$$

PROOF. From the above results, $x \in c_\pi(A)$ if and only if $(A, x) \in \pi$ if and only if $x \in N_A$, for all $N_A \in \mathfrak{N}(\pi, A)$, if and only if $x \in \bigcap\{N_A: N_A \in \mathfrak{N}(\pi, A)\}$.

THEOREM 3.6. *Let π_1 and π_2 be any two quasi-proximities on X and let A, B be any two subsets of X. Then*

(i) $\mathfrak{N}(\pi_1 \cup \pi_2, A) = \mathfrak{N}(\pi_1, A) \cap \mathfrak{N}(\pi_2, A)$; and

(ii) $\mathfrak{N}(\pi_1, A \cup B) = \mathfrak{N}(\pi_1, A) \cap \mathfrak{N}(\pi_1, B)$.

PROOF. (i). $\mathfrak{N}(\pi_1 \cup \pi_2, A) = d((\pi_1 \cup \pi_2)(A)) = d(\pi_1(A) \cup \pi_2(A)) = d(\pi_1(A)) \cap d(\pi_2(A)) = \mathfrak{N}(\pi_1, A) \cap \mathfrak{N}(\pi_2, A)$.

(ii). $\mathfrak{N}(\pi_1, A \cup B) = d(\pi_1(A \cup B)) = d(\pi_1(A) \cup \pi_1(B)) = d(\pi_1(A)) \cap d(\pi_1(B)) = \mathfrak{N}(\pi_1, A) \cap \mathfrak{N}(\pi_1, B)$.

COROLLARY 3.7. *Let π be a quasi-proximity on X let A, B be subsets of X. Then*

$$\mathfrak{N}(\pi, A \cup B) = \{N_A \cup N_B: N_A \in \mathfrak{N}(\pi, A), N_B \in \mathfrak{N}(\pi, B)\}.$$

PROOF. Note that $N_A \in \mathfrak{N}(\pi, A)$ and $N_B \in \mathfrak{N}(\pi, B)$ implies that $N_A \cup N_B \in \mathfrak{N}(\pi, A) \cap \mathfrak{N}(\pi, B)$ and, hence, by the above results,

$$N_A \cup N_B \in \mathfrak{N}(\pi, A \cup B).$$

Also, if $E \in \mathfrak{N}(\pi, A \cup B)$, then $E \in \mathfrak{N}(\pi, A)$ and $E \in \mathfrak{N}(\pi, B)$. Thus $\mathfrak{N}(\pi, A \cup B) = \{N_A \cup N_B: N_A \in \mathfrak{N}(\pi, A), N_B \in \mathfrak{N}(\pi, B)\}$.

LEMMA 3.8. *Let π be a quasi-proximity on X and let A, B be subsets of X. Then*
(i) $A \subset B \Rightarrow \mathfrak{N}(\pi, A) \supset \mathfrak{N}(\pi, B)$;
(ii) $\mathfrak{N}(\pi, X) = \{X\}$; and
(iii) $\mathfrak{N}(\pi, A) = \mathscr{P}(X)$ *if and only if* $A = \varnothing$.

PROOF. (i). We know that $A \subset B \Rightarrow \pi(A) \subset \pi(B)$. Hence $A \subset B \Rightarrow d(\pi(A)) \supset d(\pi(B))$. Thus we have $A \subset B \Rightarrow \mathfrak{N}(\pi, A) \supset \mathfrak{N}(\pi, B)$.

(ii). Note that $(\mathscr{P}(X) - \{\varnothing\})$ is a stack. Thus $\mathfrak{N}(\pi, X) = d(\pi(X)) = d(\mathscr{P}(X) - \{\varnothing\}) = \{X\}$.

(iii). Finally, $\mathfrak{N}(\pi, A) = \mathscr{P}(X)$ if and only if $d(\mathfrak{N}(\pi, A)) = d(\mathscr{P}(X))$ if and only if $\pi(A) = \varnothing$ if and only if $A = \varnothing$.

COROLLARY 3.9. *Let π be a quasi-proximity on X and let A_i ($i = 1, 2, \ldots m$), B_j ($j = 1, 2, \ldots n$) be subsets of X. If, for any pair (i, j), $i = 1, 2, \ldots m$, $j = 1, 2, \ldots n$, $B_j \in \mathfrak{N}(\pi, A_1)$, then*
(i) $\bigcap_{j=1}^n B_j \in \mathfrak{N}(\pi, \bigcup_{i=1}^m A_i)$;
(ii) $\bigcup_{j=1}^n B_j \in \mathfrak{N}(\pi, \bigcup_{i=1}^m A_i)$; and
(iii) $\bigcap_{j=1}^n B_j \in \mathfrak{N}(\pi, \bigcap_{i=1}^m A_i)$.

PROOF. (i). Since, for each i, $\mathfrak{N}(\pi, A_i)$ is a filter on X, it follows that $B_j \in \mathfrak{N}(\pi, A_i) \Rightarrow \bigcap_{j=1}^n B_j \in \mathfrak{N}(\pi, A_i)$, for all $i = 1, 2, \ldots m$. Hence, by Theorem 3.6 (ii), $\bigcap_{j=1}^n B_j \in \bigcap_{i=1}^m \mathfrak{N}(\pi, A_i) = \mathfrak{N}(\pi, \bigcup_{i=1}^m A_i)$.

(ii). Since $\mathfrak{N}(\pi, \bigcup_{i=1}^m A_i)$ is a filter, it follows from above that $\bigcup_{j=1}^n B_j \in \mathfrak{N}(\pi, \bigcup_{i=1}^m A_i)$.

(iii). From above, we get $\bigcap_{j=1}^n B_j \in \mathfrak{N}(\pi, A_i)$, for all $i = 1, 2, \ldots m$. Now, since $\bigcap_{i=1}^m A_i \subset A_i$, by Lemma 3.8 (i), $\mathfrak{N}(\pi, A_i) \subset \mathfrak{N}(\pi, \bigcap_{i=1}^m A_i)$. Hence we get

$$\bigcap_{j=1}^{n} B_j \in \mathfrak{N}(\pi, \bigcap_{i=1}^{m} A_i).$$

REMARK 3.10. Observe that, in the above theorem, some of the neighbourhood filters may not be proper. This, however does not affect the proof of the theorem.

REMARK 3.11. From the above results, it is clear that if we are given a quasi-proximity space (X, π), then a unique filter $\mathfrak{N}(A)$ is assigned to each $A \subset X$, defined by $\mathfrak{N}(A) = \mathfrak{N}(\pi, A)$, such that the following conditions hold:
(N_1) $\mathfrak{N}(A \cup B) = \mathfrak{N}(A) \cap \mathfrak{N}(B)$;
(N_2) $\mathfrak{N}(A) \subset \bigcap \{\mathscr{U} \in \Omega(X): A \in \mathscr{U}\}$; and

(N_3) $\mathfrak{N}(\varnothing) = \mathscr{P}(X)$.

THEOREM 3.12. *Let X be a nonempty set and $\mathfrak{N}(A)$ be a unique filter on X for each $A \subset X$ such that the conditions N_1, N_2, N_3 of Remark* 3.11 *are satisfied. Then the binary relation π on $\mathscr{P}(X)$, defined by*

$$\pi(A) = d(\mathfrak{N}(A)) \text{ for each } A \subset X,$$

is a quasi-proximity on X such that $\mathfrak{N}(\pi, A) = \mathfrak{N}(A)$, for each $A \subset X$.

PROOF. Since $\mathfrak{N}(A)$ is a filter, it follows that $\pi(A)$ is a grill, for each $A \subset X$. Since $\mathfrak{N}(A) \subset \bigcap \{\mathscr{U} \in \Omega(X) : A \in \mathscr{U}\}$, it follows that

$$\begin{aligned} \pi(A) = d(\mathfrak{N}(A)) &\supset \bigcup \{d(\mathscr{U}) : A \in \mathscr{U}\} \qquad (\text{by } N_2) \\ &= \bigcup \{\mathscr{U} : A \in \mathscr{U}\}. \end{aligned}$$

Also

$$\begin{aligned} \pi(A \cup B) &= d(\mathfrak{N}(A \cup B)) = d(\mathfrak{N}(A) \cap \mathfrak{N}(B)) \qquad (\text{by } N_1) \\ &= d(\mathfrak{N}(A)) \cup d(\mathfrak{N}(B)) = \pi(A) \cup \pi(B). \end{aligned}$$

Finally,

$$\pi(\varnothing) = d(\mathfrak{N}(\varnothing)) = d(\mathscr{P}(X)) = \varnothing \qquad (\text{By } N_3).$$

Thus, by Theorem 2.10, it follows that π is a quasi-proximity on X. Also, $\mathfrak{N}(\pi, A) = d(\pi(A)) = d(d(\mathfrak{N}(A))) = \mathfrak{N}(A)$ (by Theorem 3.2 (*i*)). This completes the proof.

4. Extension of D_0-closure spaces.

DEFINITION 4.1. Let (X, c) be a closure space. A grill $\mathscr{G}$ on X is called a c-grill if

$$A \in \mathscr{G} \text{ if } c(A) \in \mathscr{G}.$$

Dual to the concept of a neighborhood filter of a point x is that of an adherence grill of the point x. By this, we mean the grill

$$\mathscr{G}_c(x) = \{A \subset X : x \in c(A)\}.$$

LEMMA 4.2. *Let (X, π) be a quasi-proximity space. Then, for each $x \in X$, $\pi^{-1}(x) = \mathscr{G}_{c_\pi}(x)$.*

PROOF. The result follows from the appropriate definitions.

DEFINITION 4.3. A closure space (X, c) is said to satisfy the G_0-axiom if $\mathscr{G}_c(x_1) = \mathscr{G}_c(x_2) \Rightarrow x_1 = x_2$.

A closure space (X, c) is said to satisfy the D_0-axiom if $x_1 \in c(x_2)$ and $x_2 \in c(x_1) \Rightarrow x_1 = x_2$. (It is immediate that the D_0-axiom implies the G_0-axiom. But the converse need not be true in general.)

For any grill $\mathscr{G}$ on X, we define

$$\mathscr{G}^+ = \{\mathscr{U} \in \Omega(X) : \mathscr{U} \subset \mathscr{G}\}.$$

DEFINITION 4.4. A quasi-proximity space (X, π) is said to be separated if, for each pair of points x, y in X,

$$(x, y) \in \pi \cap \pi^{-1} \Rightarrow x = y.$$

THEOREM 4.5. *A quasi-proximity space (X, π) is separated if and only if (X, c_π) isa D_0-closure space.*

PROOF. The result follows from the fact that, for x, y in X, $(x, y) \in \pi \cap \pi^{-1}$ if and only if $(x, y) \in \pi$ and $(y\ x) \in \pi$ if and only if $y \in e_\pi(x)$ and $x \in c_\pi(y)$.

DEFINITION 4.6. Let (X, c) and (Y, k) be two closure spaces and let $\Psi: (X, c) \to (Y, k)$ be an injection map. Then $E = (\Psi, (Y, k))$ is called an extension of (X, c) if

(i) $\Psi(c(A)) = k(\Psi(A)) \cap \Psi(X)$, for all $A \subset X$; and

(ii) $k(\Psi(X)) = Y$.

(Since Ψ is an injection map, (i) insures that Ψ is a homeomorphism from (X, c) onto $(\Psi(X), k')$ where k' is the relativised closure operator on $\Psi(X)$ induced by k. Condition (ii) insures that $\Psi(X)$ is dense in (Y, k).)

Two extensions $E_1 = (\Psi_1, (Y_1, k_1))$ and $E_2 = (\Psi_2, (Y_2, k_2))$ of the space (X, c) are called equivalent if there exists a homeomorphism θ from (Y_1, k_1) onto (Y_2, k_2) such that, on X, $\theta \circ \Psi_1 = \Psi_2$,

The extension E_1 is said to be greater than the extension E_2 if there exists a continuous function θ from (Y_1, k_1) onto (Y_2, k_2) such that, on X, $\theta \circ \Psi_1 = \Psi_2$.

Associated to each extension $E = (\Psi, (Y, k))$ of (X, c) is the trace system

$$X^\pi = X^\pi(E) = \{\tau(y) : y \in Y\},$$

where

$$\tau(y) = \tau(y, E) = \{A \subset X : y \in k\ (\Psi(A))\}.$$

(Extension theory of G_0-closure spaces has been studied in [**2**].)

DEFINITION 4.7. Let π be a quasi-proximity on X and $\mathscr{G}$ be a grill on X. Then $\mathscr{G}$ is said to be a *π-family* if $(c_\pi(A), c_\pi(B)) \in \pi \cap \pi^{-1}$ for all $A, B \in \mathscr{G}$.

LEMMA 4.8. *For each $x \in X$, $\pi^{-1}(x)$ is a π-family.*

PROOF. Since A, $B \in \pi^{-1}(x) \Rightarrow x \in c_\pi(A)$ and $x \in c_\pi(B) \Rightarrow x \in c_\pi(A) \cap c_\pi(B) \Rightarrow (c_\pi(A), c_\pi(B)) \in \pi \cap \pi^{-1}$, it follows that $\pi^{-1}(x)$ is a π-family.

LEMMA 4.9. *Each π-family is contained in a maximal π-family.*

PROOF. The result follows by Zorn's lemma.

THEOREM 4.10. *Let π be a quasi-proximity on X and let $\mathscr{A}$ be a family of subsets of X. If each finite subfamily of $\mathscr{A}$ is contained in a π-family, then $\mathscr{A}$ is contained in a maximal π-family.*

PROOF. By Zorn's Lemma, there exists a subfamily $\mathscr{B} \subset \mathscr{P}(X)$ such that $\mathscr{A} \subset \mathscr{B}$ and each finite subfamily of $\mathscr{B}$ is contained in a π-family and such that it is maximal with respect to this property. Since each finite subfamily of $\mathscr{B}$ is contained in a π-family, to complete the proof, one needs to check that $\mathscr{B}$ is a grill. Since each finite subfamily of $\mathscr{B}$ is contained in a π-family, it follows that $\emptyset \notin \mathscr{B}$.

Now let $B \supset A \in \mathscr{B}$. It is clear that the subfamily $\{B\} \cup \mathscr{B} \subset \mathscr{P}(X)$ contains $\mathscr{A}$ and each finite subfamily of its contained in a π-family. By maximality of $\mathscr{B}$, it follows that $B \in \mathscr{B}$.

Next let $A \notin \mathscr{B}$ and $B \notin \mathscr{B}$. Then there exist two finite subfamilies $\mathscr{B}_1$ and $\mathscr{B}_2$ of $\mathscr{B}$ such that none of $\{A\} \cup \mathscr{B}_1$ and $\{B\} \cup \mathscr{B}_2$ is contained in a π-family. Consequently, $\{A \cup B\} \cup \mathscr{B}_1 \cup \mathscr{B}_2$ is not contained in a π-family. Since $\mathscr{B}_1 \cup \mathscr{B}_2$ is a finite sub-family of $\mathscr{B}$, it follows that $A \cup B \notin \mathscr{B}$. Thus $\mathscr{B}$ is a grill on X.

Finally, since any π-family containing $\mathscr{B}$ has the property with respect to which $\mathscr{B}$ is maximal, $\mathscr{B}$ is a maximal π-family containing $\mathscr{A}$.

Now we are in a position to construct extensions of D_0-closure spaces with the help of the theory developed here for quasi-proximities.

THEOREM 4.11. *Let (X, π) be a separated quasi-proximity space and let X^π be the collection of all maximal π-families and $\pi^{-1}(x)$, for all $x \in X$. Define, for each $A \subset X$,*

$$A^\pi = \{\mathscr{G} \in X^\pi : A \in \mathscr{G}\}$$

and a mapping φ from X into X^π by the rule

$$\varphi(x) = \pi^{-1}(x), \quad \text{for all } x \in X.$$

Define, an operator $h_r: \mathscr{P}(X^\pi) \to \mathscr{P}(X^\pi)$ by the rule $h_r(\alpha) = (\varphi^{-1}(\alpha))^\pi \cup r(\alpha - \varphi(X))$ for all $\alpha \subset X^\pi$, where $r: \mathscr{P}(X^\pi - \varphi(X)) \to \mathscr{P}(X^\pi)$ is an arbitrary function satisfying $r(\phi) = \phi$, $\alpha \subset r(\alpha)$ and $r(\alpha \cup \beta) = r(\alpha) \cup r(\beta)$, for all $\alpha, \beta \subset X^\pi - \varphi(X)$. Then $(\varphi, (X^\pi, h_r))$ is a G_0-extension of (X, c_π), for each such r, with trace system X^π such that $\tau(\mathscr{G}) = \mathscr{G}$, for all $\mathscr{G} \in X^\pi$. Moreover, all extensions of (X, c_π) on X^π with traces $\tau(\mathscr{G}) = \mathscr{G}$, for all $\mathscr{G} \in X^\pi$, can be obtained by a suitable choice of r.

PROOF. In Lemma 4.2 we have seen that $\pi^{-1}(x) = \mathscr{G}_{c_\pi}(x)$ for all $x \in X$. Hence X^π is a collection of grills on X containing all adherence grills of (X, c_π). Now, it is clear that the proof of the above theorem may be omitted, for it can be completed in a way similar to what has been done by Chattopadhyay and Thron in [**2**].

4.12. Special choices of r and topological extensions.

A permissible choice of r in h_r is $r(\beta) = \beta$, for all $\beta \subset X^\pi - \varphi(X)$. The identity mapping from (X, c) into (X, c') is continuous if $c(A) \subset c'(A)$, for all $A \subset X$. So, if we define $h_1(\alpha) = (\varphi^{-1}(\alpha))^\pi \cup (\alpha - \varphi(X))$, for all $\alpha \subset X^\pi$, then, for any r, it is clear that $h_1(\alpha) \subset h_r(\alpha)$. Consequently, it follows that $(\varphi, (X^\pi, h_1))$ is the largest among all the extensions $(\varphi, (X^\pi, h_r))$ of (X, c_π). Now, if we define, for all $\alpha \subset X^\pi$,

$$h_0(\alpha) = (\varphi^{-1}(\alpha))^\pi \cup r_0(\alpha - \varphi(X)),$$

where $r_0(\phi) = \phi$ and $r_0(\beta) = X^\pi$ if β is a nonempty subset of $X^\pi - \varphi(X)$, then, as above, it can be verified that $(\varphi, (X^\pi, h_0))$ is the smallest among all the extensions $(\varphi, (X^\pi, h_r))$ of (X, c_π).

Thus we get the result

Theorem 4.13. *Let (X, π) be a separated quasi-proximity space. Then $(\varphi, (X^\pi, h_1))$ and $(\varphi, (X^\pi, h_0))$ are the laregest and the smallest among all extensions $(\varphi, (X^\pi, h_r)$ of (X, c_π).*

It can be easily verified that $\{A^\pi: A \subset X\}$ forms a base for the closed sets of some topology on X^π. Hence the closure operator s defined on X^π by $s(\alpha) = \bigcap \{A^\pi: \alpha \subset A^\pi\}$, for all $\alpha \subset X^\pi$, is a Kuratowski closure operator. Now for each $\alpha \subset X^\pi$, define

$$h_s(\alpha) = (\varphi^{-1}(\alpha))^\pi \cup s(\alpha - \varphi(X)).$$

Since $s(\varphi(A)) \subset A^\pi$, it can be easily verified that $h_s(\alpha) = (\varphi^{-1}(\alpha)^\pi \cup s(\alpha)$, for all $\alpha \subset X^\pi$ (see [**2**]). However, if (X, π) is a topological space, then all $\pi^{-1}(x)$ are c_π-grills and if, in addition, all maximal π-families are also c_π-grills, then $A^\pi = (c_\pi(A))^\pi$ and, hence, $\varphi(A) \subset B^\pi$ if $A^\pi \subset B^\pi$. Consequently, we get $s(\varphi(A)) = A^\pi$. Therefore,

$$\begin{aligned} h_s(\alpha) &= (\varphi^{-1}(\alpha))^\pi \cup s(\alpha - \varphi(X)) \\ &= s(\varphi(\varphi^{-1}(\alpha)) \cup s(\alpha - \varphi(X)) \\ &= s(\alpha). \end{aligned}$$

Since the restriction of s to $X^\pi - \varphi(X)$ is a choice of r in Theorem 4.11 and s is a Kuratowski closure operator on X^π, it follows that $(\varphi, (X^\pi, s))$ is a topological extension of (X, c_π) with the trace $\tau(\mathscr{G}) = \mathscr{G}$, for all $\mathscr{G} \in X^\pi$.

Definition 4.14. An extension $(\Psi, (Y, k))$ is a compactification if (Y, k) is a compact closure space. Čech [**1**; 41 A3, p. 785] defines a closure space to be compact if and only if every filter has a cluster point. Using his definition of a cluster point we are led to the following statement.

A closure space (X, c) is compact if and only if, for every filter $\mathscr{F}$ on X,

$$\bigcap\{c(F)\colon F \in \mathscr{F}\} \neq \phi.$$

Let (X, c) be a closure space and $\mathscr{G}$ be a grill on X. Then $x \in X$ is said to be a cluster point of $\mathscr{G}$ if $\mathscr{G}^+ \cap \mathscr{G}_c^+(x) \neq \phi$. It has been shown in **[2]** that (X, c) is compact if and only if $\{\mathscr{G}_c^+(x)\colon x \in X\}$ covers $\Omega(X)$. Using this result one can easily deduce that a closure space (X, c) is compact if and only if each grill on X has a cluster point. It can also be verified that a continuous image of a compact closure space is compact.

DEFINITION 4.15. A family $\mathfrak{S}$ of grills on X is said to be a **-family* if the following condition holds.

Given $\mathscr{A} \subset \mathscr{P}(X)$, if each finite subfamily of $\mathscr{A}$ is contained in an element of $\mathfrak{S}$ then $\mathscr{A}$ is contained in an element of $\mathfrak{S}$.

LEMMA 4.16. *Let X^π be the set of all maximal π-families and all $\pi^{-1}(x)$, $x \in X$. Then X^π is a *-family.*

PROOF. The result follows immediately from Lemma 4.9 and Theorem 4.11.

THEOREM 4.17. *Let (X, π) be a quasi-proximity space and let X^π be the set of all maximal π-families and $\pi^{-1}(x)$, for all $x \in X$. Then (X^π, s) is a compact topological space.*

PROOF. Let $\mathscr{A}^\pi$ be a subfamily of $\{A^\pi\colon A \subset X\}$ such that, for any finitely many $A_1^\pi, A_2^\pi, \ldots\ldots\ldots\ldots, A_n^\pi$ of $\mathscr{A}^\pi$,

$$A_1^\pi \cap A_2^\pi \cap \ldots\ldots\ldots\ldots \cap A_n^\pi \neq \varnothing.$$

Set $\mathscr{A} = \{A \subset X\colon A^\pi \in \mathscr{A}^\pi\}$. Since $\mathscr{A}^\pi$ has the finite intersection property, it follows that each finite subfamily of $\mathscr{A}$ is contained in a π-family. Hence, by Lemma 4.16, $\mathscr{A}$ is contained in a maximal π-family, i.e., $\mathscr{A} \subset \mathscr{G}$ and $\mathscr{G} \in X^\pi$. Hence $\mathscr{G} \in \bigcap\{A^\pi\colon A^\pi \in \mathscr{A}^\pi\}$. Since $\{A^\pi\colon \mathrm{A} \subset X\}$ is a base for the closed sets of the topological space (X^π, s), it follows that (X^π, s) is compact.

THEOREM 4.18. *Let (X, π) be a quasi-proximity space. (X^π, h_r) is the closure space defined as in Theorem* 4.11 *such that $s(\alpha) \subset h_r(\alpha)$ for all $\alpha \subset X^\pi$. Then $(\varphi, (X^\pi, h_r))$ is a compact extension of (X, c_π); in particular, $(\varphi, (X^\pi, h_s))$ is a D_0-compactification of (X, c_π).*

PROOF. From Theorem 4.11, we have $(\varphi, (X^\pi, h_r))$ is an extension of (X, c_π). From above, we know that (X^π, s) is a compact topological space.

Since $s(\alpha) \subset h_r(\alpha)$, for all $\alpha \subset X^\pi$, it follows that the identity mapping from (X^π, s) onto (X^π, h_r) is continuous and, hence, $(\varphi, (X^\pi, h_r))$ is a compact extension of (X, c_π).

It can be easily verified that $hs(\alpha) = (\varphi^{-1}(\alpha))^\pi \cup s(\alpha)$. Consequently $h_d(\alpha) \supset s(\alpha)$ for all $\alpha \subset X^\pi$ and, hence, as above, $(\varphi, (X^\pi, h_s))$ is a compactification of (X, c_π).

To show that (X^π, h_s) is D_0, let us consider that $\mathscr{G}_1 \in h_s(\mathscr{G}_2)$ and $\mathscr{G}_2 \in h_s(\mathscr{G}_1)$ for two elements $\mathscr{G}_1$, $\mathscr{G}_2$ of X^π.

(i). Suppose both of $\mathscr{G}_1$, $\mathscr{G}_2$ belong to $\varphi(X)$. Then there exist x, y in X such that $\mathscr{G}_1 = \pi^{-1}(x)$, $\mathscr{G}_2 = \pi^{-1}(y)$. Hence $\pi^{-1}(x) \in h_s(\pi^{-1}(y)) = \{y\}^\pi$. Similarly $\pi^{-1}(y) \in \{x\}^\pi$. These together imply that $(x, y) \in \pi \cap \pi^{-1}$. Since π is separated, $x = y$ and hence $\mathscr{G}_1 = \mathscr{G}_2$.

(ii) Suppose that at least one of $\mathscr{G}_1$, $\mathscr{G}_2$ does not belong to $\varphi(X)$. Let $\mathscr{G}_2 \notin \varphi(X)$. Since $\mathscr{G}_1 \in h_s(\mathscr{G}_2) = \bigcap \{A^\pi\colon A \in \mathscr{G}_2\}$ it follows that $\mathscr{G}_2 \subset \mathscr{G}_1$. Since $\mathscr{G}_1$ is a π-family and $\mathscr{G}_2$ is a maximal π-family, it follows that $\mathscr{G}_1 = \mathscr{G}_2$. This completes the proof.

REMARK 4.19. In general, there is no proximkty compatible with a given D_0-closure space. In order to have a proximity compatible with a given D_0-space, one must assume that the underlying space is symmetric. But compatible quasi-proximities are always available for a given D_0-closure space (See Example 2.20). Thus, from the result discussed above, it is clear that we have introduced a method of compactification of all D_0-closure spaces. But, at this moment, it is not clear to us what compactifications could be achieved by our method. So it remains an open problem to find the class of compactifications that can be realized by our method.

REMARK 4.20. F. Riesz in 1908 asked to determine the class of proximity spaces (X, π) for which the following condition holds.

There exists an extension $(\Psi, (Y, k))$ of the closure space (X, c_π) such that $(A, B) \in \pi$ if and only if $k(\Psi(A)) \cap k(\Psi(B)) \neq \phi$.

While proving the remarkable result that there is a one-to-one correspondence between the class of all T_2-compactifications of a Tychonoff space and the class of EF-proximities compatible with the Tychonoff space, Smirnov **[18]** has pointed out that separated EF-proximities satisfy the Riesz property. Thron **[21]** has proved that proximities belonging to the class of separated LO-proximities, have the property of Riesz. The class of LO-proximities is larger than the class of EF-proximities. Chattopadhyay and Thron **[3]** have pointed out that proximities belonging to the class of separated RI-proximities satisfy the Riesz property. The latter class is even larger than the class of LO-proximities.

A meaningful formulation of Riesz' problem in the context of quasi-proximities should have been the following.

What quasi proximities (X, π) have the property that there exists an extension $(\Psi, (Y, k))$ of the closure space (X, c_π) such that

$$(A, B) \in \pi \text{ if and only if } k(\Psi(A)) \cap \Psi(B) \neq \emptyset?$$

Now it is easy to verify that $c_\pi(A) \cap B \neq \emptyset$ if and only if $k(\Psi(A)) \cap \Psi(B) \neq \phi$. Thus, given a D_0-closure space, the smallest quasi-proximity compatible with it satisfies Riesz' property (in the context of quasi-proximities). This shows that the investigation of Riesz' problem in the context of quasi-proximities does not seem to be that interesting, as its interest lies in the theory of proximities.

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