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RESEARCH PROBLEMS

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1. Communicated by M.V. Subbarao, University of Alberta, from original manuscripts by the late Leo Moser.

"The squares of sides 1/2, 1/3, 1/4, . . . can be accommodated in a square of side 5/6, and this is 'best possible'. Can they be accommodated in some rectangle of area $(\pi^2/6) - 1$?"

2. Proposed by E.G. Straus, University of California—Los Angeles.

For the integer n, define $A = \operatorname{lcm}(n + 1, \ldots, n + k)$ and $B = \operatorname{lcm}(n - k, \ldots, n - 1)$. How likely is it that $A < B$?
It is known that the set $S = \{n: A < B \text{ for all } k\}$ has density 0. Are there infinitely many $n \in S$?

3. Proposed by A. Schinzel, Polish Academy of Science.

Find a sequence $\{a_i\}$ of integers such that no 3 of the a_i are in arithmetic progression and $\sum 1/a_i$ converges to a sum $\geqq 3.0085$. This will improve a result obtained by the greedy algorithm which yields a series converging to a sum > 3.0078. [See J. Gerver and L. T. Ramsey, Math. Comput. 33, 1353–1359 (1970); also, a construction of J. Wroblewski, Math. Comput. (to appear 1984), which gives a sum > 3.0084.]

4. Proposed by Paul Erdös, Hungarian Adademy of Sciences.

The following three problems are concerned with the divisors $0 < d_1 < \cdots < d_{\tau(n)}$ of the integer n.

A. There exists a constant C such that there are infinitely many n for which

$$\sum_{i=1}^{\tau(n)} \left(\frac{d_{i+1}}{d_i} - 1\right) < C.$$

What is the best possible C for which the inequality holds for infinitely many n?

B. There exists a constant C such that

$$\sum_{d_i < \sqrt{n}} (d_{i+1} - d_i)^2 < \frac{n}{(\log n)^c}$$

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for infinitely many n. What is the best possible C?

C. Is it true that $d_{i+1} < 2d_i$ for almost all d_i?

5. Proposed by John Brillhart, University of Arizona.

Let $a_1 < a_2 < \cdots$ be positive integers. Iseki [Math. Sem. Notes Kobe Univ. 7 no. 1, 183–184 (1979)] easily proved that the number

$$\alpha = \frac{1}{a_1} + \frac{1}{a_1 a_2} + \cdots + \frac{1}{a_1 a_2 \cdots a_n} + \cdots$$

is irrational. It is easy to see that α may be transcendental—for example, if $a_n = n$, then $\alpha = e - 1$. On the other hand, α may be algebraic, if $a_n = F_{2^n}/F_{2^{n-1}}$, where $\{F_n\}$ is the Fibonacci sequence ($F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$), then $\alpha = (5 - \sqrt{5})/2$. [See Good, Fibonacci Quart. **12** no. 4, 346 (1974); or Hoggatt and Bicknell, Fibonacci Quart. **14** no. 3, 272–276 (1976).]

How likely is it that α is transcendatal?

ED. NOTE. The proposer has communicated another example: if $a_1 = 3$ and $a_{n+1} = a_n^2 - 2$, $a_1 = 3$, then $\alpha = (3 - \sqrt{5})/2$.

6. Proposed by Hugh Edgar, San Jose State University.

For the diophantine equation

$$1 + a + a^2 + \cdots + a^{x-1} = p^y,$$

$a > 1$, p an odd prime, $x \geqq 3$, $y \geqq 1$:

A. is it true that $y = 1$ or y is prime?

B. if a is a prime power, can it happen that $a > p$?

7. Communicated by Carl Pomerance, University of Georgia, and credited to the late Leo Moser.

Can the plane be tiled with all the integer squares? It is known that a tilling of the plane is possible using only integer squares; is it possible to use almost all the integers?

8. Proposed by M.V. Subbarao, University of Alberta.

Define $\psi(n) = n - \varphi(n)$, where $\varphi(n)$ is the Euler totient. It is easy to see that $\psi(n)$ is prime if

(1) $n = p^2$, prime p, or

(2) $n = pq$, prime p, q such that $p + q - 1$ is prime (which occurs infinitely often).

Moreover, if $\psi(n)$ is prime and n is not the square of a prime, then n is squarefree.

A. Find necessary and/or sufficient conditions (other than those stated above) for the primality of $\psi(n)$.

B. Let n be the product of k distinct primes, $k \geqq 3$. For each such k, is $\psi(n)$ prime at least once?

ED. NOTE The proposer has shown that if $p < q < r$ are primes and $\psi(pqr)$ is prime, then $q \not\equiv 1 \pmod p$, $r \not\equiv 1 \pmod p$, and $r \not\equiv 1 \pmod q$.

The question posed in B has been answered in the affirmative for $k = 3$ and $k = 4$ by Hardy with the examples $\psi(3 \cdot 5 \cdot 17) = 127$ and $\psi(3 \cdot 17 \cdot 29 \cdot 41) = 24{,}799$.

9. Proposed by V.C. Harris, San Diego State University.
Let

$$S = \{a_1, a_2, a_3, \ldots, a_n, \ldots\}$$

be an infinite sequence with $a_n \neq 0$, $n = 1, 2, 3, \ldots$, and set

$$r(a_n) = r^1(a_n) = a_{n+1}/a_n.$$

Define

$$r^{k+1}(a_n) = r(r^k(a_n)), \; k = 1, 2, 3, \ldots$$

If S is such that

$$r^k(a_n) = c, \text{ a constant, } n \geqq 1, \; a_1, a_2, \ldots, a_k \text{ given}$$

then $S = S(c, k; a_1, a_2, \ldots, a_k)$ is by definition a geometric series of order k. We let

$$S_m(k) = S_m(c, k; a_1, a_2, \ldots, a_k)$$

represent the sum of the first m terms of S.

Assuming a geometric series of order k and $a_1, a_2, \ldots, a_k$ are integers or powers of one variable:

A. determine what integers are representable by partial sums of a given S.

B. determine whether a given S contains an infinite number of terms which are primes.

C. determine which S represent an infinite number of r-th powers.

D. discuss congruence properties of the partial sums of a given S.

Examples supplied by the proposer:

$S_7(1,3; a, a^2, a^4) = a + a^2 + a^4 + a^{11} + a^{16} + a^{22}$,
and $S_7(1, 3; 2, 3, 9) = 2 + 3 + 3^2 + 3^2 \cdot 2 + 3^4 \cdot 2^3 + 3^6 \cdot 2^6 + 3^6 \cdot 2^{10}$.

10. Proposed by V.C. Harris, San Diego State University.
[See L.E. Bush, Amer. Math. Monthly **37**, 353–357 (1930).]

Let a, b, k, m, n be positive integers with $(a, b) = 1$, $m \leqq n$, and let

$$S_{m,n}(a, b, k) = \sum_{i=m}^{n} (ai + b)^k.$$

Determine

A. For fixed k, the number of ways a positive integer N can be so represented (including trivial [one-term] representations).

B. The requirements on a, b, k so that

$$S = \{S_{m,n}(a, b, k)\}_{n=1}^{\infty}$$

contains infinitely many terms which are primes.

C. The requirements on a, b, k that S contains infinitely many r-th powers for fixed r.

Proposer's Note: The case $S_{m,n}(a, b, l)$ is well known.