ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
VOLUME 15, Number 2, Spring 1985

SOME COMBINATORIAL IDENTITIES AND ARITHMETICAL APPLICATIONS

JOHN A. EWELL

ABSTRACT. On the strength of the Gauss-Jacobi triple-product identity the author presents a method for successively expanding infinite products

$$\prod_1^\infty (1 - x^{2n})^e(1 + ax^{2n-1})^e(1 + a^{-1}x^{2n-1})^e$$

e = 2, 4 where $a, x \in \mathbf{C}$, $a \neq 0$ and $|x| < 1$. Some arithmetical applications are noted.

1. Introduction. The mainspring of our discussion is the Gauss-Jacobi triple-product identity

$$\prod_1^\infty (1 - x^{2n})(1 + ax^{2n-1})(1 + a^{-1}x^{2n-1}) = \sum_{-\infty}^{\infty} x^{n^2}a^n \tag{1}$$

valid for each pair of complex number a, x such that $a \neq 0$ and $|x| < 1$. For a proof see [**3**, p. 282]. Our objective is the presentation of an elementary method for successively raising this identity to the second and fourth powers. To this end, we recall that, for an arbitrary pair of integers k, n, with $k \geqq 2$ and $n \geqq 0$, $r_k(n)$ denotes the cardinality of the set

$$\{(x_1, \ldots, x_k) \in \mathbf{Z}^k \mid n = x_1^2 + \cdots + x_k^2\}.$$

Our results can now be stated as

THEOREM 1. *For each pair of complex numbers a, x such that $a \neq 0$ and $|x| < 1$,*

$$\begin{aligned}\prod_1^\infty (1 - x^{2n})^2(1 + ax^{2n-1})^2(1 + a^{-1}x^{2n-1})^2 \\ = \sum_{-\infty}^{\infty} x^{2n^2} \sum_{-\infty}^{\infty} x^{2n^2}a^{2n} + x\sum_{-\infty}^{\infty} x^{2n(n+1)} \sum_{-\infty}^{\infty} x^{2n(n+1)}a^{2n+1}\end{aligned} \tag{2}$$

and

Received by the editors on May 24, 1983.

$$\prod_1^\infty (1 - x^{8n})^4(1 + ax^{8n-4})^4(1+a^{-1}x^{8n-4})^4$$

$$(3) \qquad = \sum_0^\infty r_3(n)\, x^{8n} \sum_{-\infty}^\infty x^{(4m)^2} a^{4m} + \sum_0^\infty r_3(2n + 1)x^{8n+4} \sum_{-\infty}^\infty x^{(4m+2)^2} a^{4m+2}$$
$$+ \left\{x \prod_1^\infty (1-x^{16n})(1+x^{8n}) \cdot \sum_1^\infty r_2(2n+1)x^{4n+2}\right\} \sum_{-\infty}^\infty x^{(2m+1)^2} a^{2m+1}.$$

In §2 we prove this theorem, state a corollary containing two additional identities, and illustratively prove one of them. Our concluding remarks are concerned with arithmetical applications of these identities.

2. Proof of Theorem 1. On the strength of (1) and the elementary identity

$$u^2 + v^2 = \frac{1}{2}\{(u + v)^2 + (u - v)^2\}$$

we write

$$\prod_1^\infty (1 - x^{2n})^2(1 + ax^{2n-1})^2(1 + a^{-1}x^{2n-1})^2$$
$$= \sum x^{i^2+j^2} a^{i+j}$$
$$= \sum x^{[(i+j)^2+(i-j)^2]/2} a^{i+j},$$

the summation extending over all $(i, j) \in \mathbf{Z}^2$ for each of the two sums.

At this point the following completely transparent lemma naturally facilitates our discussion.

LEMMA. *The function F: $\mathbf{Z}^2 \to \mathbf{Z}^2$, defined by $F(i, j) = (i + j, i - j)$, is one-to-one from $\mathbf{Z}^2$ onto the set*

$$E := \{(r, s) \in \mathbf{Z}^2 \mid r \ \&\ s \text{ have the same parity}\}.$$

Accordingly, we continue our proof, realizing that the last-mentioned sum equals

$$\sum_{(r,s)\in E} x^{[r^2+s^2]/2} a^r$$
$$= \sum_{m=-\infty}^\infty \sum_{n=-\infty}^\infty x^{[4m^2+4n^2]/2} a^{2m} + \sum_{m=-\infty}^\infty \sum_{n=-\infty}^\infty x^{[4m^2+4m+1+4n^2+4n+1]/2} a^{2m+1}$$
$$= \sum_{-\infty}^\infty x^{2n^2} \sum_{-\infty}^\infty x^{2m^2} a^{2m} + x \sum_{-\infty}^\infty x^{2n(n+1)} \sum_{-\infty}^\infty x^{2m(m+1)} a^{2m+1}$$

This proves identity (2).

To establish identity (3) we, first of all, let $x \to x^2$ in (2) to get

$$\prod_1^\infty (1 - x^{4n})^2(1 + ax^{4n-2})^2(1 + a^{-1}x^{4n-2})^2$$
$$= \alpha \sum_{-\infty}^{\infty} x^{(2n)^2}a^{2n} + \beta \sum_{-\infty}^{\infty} x^{(2n+1)^2}a^{2n+1},$$

where $\alpha = \alpha(x) := \sum x^{(2n^2)}$ and $\beta = \beta(x) := \sum x^{(2n+1)^2}$, both sums extending from $-\infty$ to ∞. We then square this identity to get

$$\prod_1^\infty (1 - x^{4n})^4(1 + ax^{4n-2})^4(1 + a^{-1}x^{4n-2})^4$$
$$= \alpha^2 \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} x^{(2i)^2+(2j)^2}a^{2(i+j)} + \beta^2 \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} x^{(2i+1)^2+(2j+1)^2}a^{2(i+j+1)}$$
$$+ 2\alpha\beta \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} x^{(2i)^2+(2j+1)^2}a^{2(i+j)+1}$$
$$= S_0 + S_1 + S_2, \text{ say.}$$

Now, by the method used to prove (2) we develop each of the series S_0, S_1, S_2 to get

$$S_0 = [\alpha(x)]^2\alpha(x^2) \sum_{-\infty}^{\infty} x^{2(2m)^2}a^{4m} + [\alpha(x)]^2\beta(x^2) \sum_{-\infty}^{\infty} x^{2(2m+1)^2}a^{4m+2}$$
$$S_1 = [\beta(x)]^2\alpha(x^2) \sum_{-\infty}^{\infty} x^{2(2m+1)^2}a^{4m+2} + [\beta(x)]^2\beta(x^2) \sum_{-\infty}^{\infty} x^{2(2m)^2}a^{4m}$$
$$S_2 = \alpha(x)\beta(x)\beta(x^{1/2}) \sum_{-\infty}^{\infty} x^{(2n+1)^2/2}a^{2n+1}.$$

Letting $x \to x^2$ and rearranging, we have

$$\prod_1^\infty (1 - x^{8n})^4(1 + a^{8n-4})^4(1 + a^{-1}x^{8n-4})^4$$
$$= \{[\alpha(x^2)]^2\alpha(x^4) + [\beta(x^2)]^2\beta(x^4)\} \sum_{-\infty}^{\infty} x^{(4m)^2}a^{4m}$$
$$+ \{[\alpha(x^2)]^2\beta(x^4) + [\beta(x^2)]^2\alpha(x^4)\} \sum_{-\infty}^{\infty} x^{(4m+2)^2}a^{4m+2}$$
$$+ \alpha(x^2)\beta(x^2)\beta(x) \sum_{-\infty}^{\infty} x^{(2n+1)^2}a^{2n+1}$$
$$= A(x) \sum_{-\infty}^{\infty} x^{(4m)^2}a^{4m} + B(x) \sum_{-\infty}^{\infty} x^{(4m+2)^2}a^{4m+2}$$
$$+ C(x) \sum_{-\infty}^{\infty} x^{(2n+1)^2}a^{2n+1},$$

where $A(x)$, $B(x)$ and $C(x)$ represent obvious definitions to abbreviate.

We now simplify $A(x)$, $B(x)$ and $C(x)$ by generous appeal to (1), (2) and the definitions of $\alpha(x)$ and $\beta(x)$. Under the substitutions $a \to 1$, $x \to x^4$, (1) becomes

$$(4)\qquad \prod_1^\infty (1 - x^{8n})(1 + x^{8n-4})^2 = \alpha(x^4) + \beta(x^4),$$

while (2) becomes

$$(5)\qquad \prod_1^\infty (1 - x^{8n})^2(1 + x^{8n-4})^4 = [\alpha(x^2)]^2 + [\beta(x^2)]^2.$$

Multiplying (4) and (5), we have

$$(6)\qquad \prod_1^\infty (1 - x^{8n})^3(1 + x^{8n-4})^6 = A(x) + B(x).$$

Similarly, we let $a \to -1$, $x \to x^4$ to obtain two identities which we then multiply to get

$$(7)\qquad \prod_1^\infty (1 - x^{8n})^3(1 - x^{8n-4})^6 = A(x) - B(x).$$

Adding (6) and (7), we get

$$A(x) = \frac{1}{2}\left\{\prod_1^\infty (1 - x^{8n})^3(1 + x^{8n-4})^6 + \prod_1^\infty (1 - x^{8n})^3(1 - x^{8n-4})^6\right\}.$$

Subtracting (7) from (6), we get

$$B(x) = \frac{1}{2}\left\{\prod_1^\infty (1 - x^{8n})^3(1 + x^{8n-4})^6 - \prod_1^\infty (1 - x^{8n})^3(1 - x^{8n-4})^6\right\}.$$

However, if in (1) we let $a \to 1$ and raise the resulting identity to the kth power, $k \geqq 2$, we get

$$\prod_1^\infty (1 - x^{2n})^k(1 + x^{2n-1})^{2k} = \left\{\sum_{-\infty}^\infty x^{n^2}\right\}^k = \sum_0^\infty r_k(n)x^n.$$

Hence, with $k = 3$, we have

$$A(x) = \sum_0^\infty r_3(2n)x^{8n}, \qquad B(x) = \sum_0^\infty r_3(2n + 1)x^{8n+4}.$$

To get a simplified expression for $C(x)$, we successively let $a \to 1$ and $a \to -1$ in (1); and, then square the resulting identities to get

$$\prod_1^\infty (1 - x^{2n})^2(1 + x^{2n-1})^4 = [\alpha(x)]^2 + 2\alpha(x)\beta(x) + [\beta(x)]^2,$$

$$\prod_1^\infty (1 - x^{2n})^2(1 - x^{2n-1})^4 = [\alpha(x)]^2 - 2\alpha(x)\beta(x) + [\beta(x)]^2,$$

whence

$$\alpha(x^2)\beta(x^2) = \frac{1}{4}\left\{\prod_1^{\beta}(1 - x^{4n})^2(1 + x^{4n-2})^4 - \prod_1^{\infty}(1 - x^{4n})^2(1 - x^{4n-2})^4\right\}.$$

Also,

$$\beta(x) = 2x\prod_1^{\infty}(1 - x^{16n})(1 + x^{8n}).$$

(This is a form of Gauss's identity [**3**, p. 284].) Multiplying the last two identities and simplifying, we have

$$C(x) = x\prod_1^{\infty}(1 - x^{16n})(1 + x^{8n})\sum_0^{\infty} r_2(2n + 1)x^{4n+2}.$$

Substituting these values for $A(x)$, $B(x)$, $C(x)$, we have thus proved identity (3).

COROLLARY. *For each complex number x such that $|x| < 1$,*

$$\prod_1^{\infty}(1 - x^n)^6 = \sum_{-\infty}^{\infty} x^{n^2}\sum_0^{\infty}(2n+1)^2x^{n(n+1)} - \sum_{-\infty}^{\infty} x^{n(n+1)}\sum_1^{\infty}(2n)^2x^{n^2}, \tag{8}$$

$$\begin{aligned}192\prod_1^{\infty}(1 - x^{8n})^{12} &= A(x)\sum_0^{\infty}(8n + 4)^4x^{16n(n+1)} \\ &+ B(x)\sum_1^{\infty}(8n)^4x^{4(2n-1)(2n+1)} - C(x)\sum_0^{\infty}(4n + 2)^4x^{(2n-1)(2n+3)}\cdot\end{aligned}\tag{9}$$

PROOF: We establish (9) by appeal to the method of proving (8) in [**1**]. Accordingly, we check directly that the identity is valid for $x = 0$. Hence, let $a = -x^4e^{2it}$, so that the left side of (3) becomes $(1 - e^{-2it})^4f(t)$, where $f(t) = f(x, t) := \prod_1^{\infty}(1 - x^{8n})^4\{1 - 2x^{8n}\cos 2t + x^{16n}\}^4$. We then multiply both sides of the resulting identity by $(e^{it}/2i)^4$, so that the left side becomes $(\sin^4 t)f(t)$. At this juncture we differentiate both sides of the resulting identity four times with respect to t (after transforming the right side), and follow this by putting $t = 0$, to get

$$\begin{aligned}24\prod_1^{\infty}(1 - x^{8n})^{12} &= \frac{1}{8}A(x)\sum_0^{\infty}(8m + 4)^4x^{16m(m+1)} \\ &+ \frac{1}{8}B(x)\sum_1^{\infty}(8m)^4x^{4(2m-1)(2m+1)} - \frac{1}{8}C(x)\sum_0^{\infty}(4m + 2)^4x^{(2m-1)(2m+3)}.\end{aligned}$$

This proves identity (9).

CONCLUDING REMARKS. Arithmetical consequences of the triple-product identity are well known. For some of these see [**3**, pp. 282–296]. Arithmetical applications of identity (2), especially its corollary identity (8), are given by the author in the papers [**1**], [**2**]. For example, identity

(8) and two well-known special cases of (1) combine to give a simple derivation of Jacobi's formula for the number $r_4(n)$ of representations of a natural number n by sums of four squares. Identity (8) is also used in [1] to give an alternate proof of Ramanujan's theorem concerning divisibility of certain values of the partition function by the modulus 7.

Although the arithmetical functions r_2 and r_3 enter naturally into the expression of each of the identities (3) and (9), direct applications of these identities to number theory have not yet been found. However, the methods of this discussion can be applied to express the infinite product $\prod(1-x^n)^{24}$ as a series (or sum of series), whose exact form must be investigated. In any event, such investigation could lead to enlightenment concerning Ramanujan's tau function.

References

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Department of Mathematical Sciences Northern Illinois University DeKalb, IL 60115-2854