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SOME REMARKS ON NASH RINGS

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Dedicated to the memory of Gus Efroymson

Let k be any field. We try to extend the notion of Nash functions in the following way (cf. [3], [5]). Let X be an irreducible affine algebraic variety defined over k, U be a subset of X and B be an integral domain such that:

i) $\Gamma_X \hookrightarrow B$ is an injective homomorphism of k-algebras,

ii) there is an injection $U \hookrightarrow \Omega_k(B)$ such that if $x \mapsto M_x \subset B$ then $M_x \cap \Gamma_X = m_x = \{f \in \Gamma_X \mid f(x) = 0\}$.

iii) for each $x \in U$ we have $(\Gamma_{X,x})\hat{} \simeq (B_{M_x})\hat{}$ ($\hat{}$ denotes the completion),

iv) let $f \in B$; if $f \notin M_x$ for all $x \in U$ then $1/f \in B$.

In this setting we say that (U, B) is a Nash pair and that the algebraic closure D of Γ_X in B is the Nash ring associated with (U, B). This notion applies notably, when k is a complete valued field, to the case where U is a connected open domain and B is the ring of global analytic k-valued functions (provided that B has no zero-divisors). We will denote $B = \mathscr{I}(U)$ and $D = \mathscr{N}(U)$. The above definition also applies when U is a compact subset and B is the ring of germs of analytic functions defined on (some neighborhood of) U.

From this definition it is possible to obtain, in a purely algebraic way, some information about D and about the homomorphism $D \hookrightarrow B$.

We will assume that X has a unique branch at every point $x \in U$ (e.g., X normal), we will also assume $k \neq \bar{k}$.

1. In [3] the following results for D are given.

THEOREM 1. $U \sim \Omega(D) \sim \Omega_k(B)$.

THEOREM 2. *Suppose X normal. If* (U, B) *verifies the condition*:

$$(C) \textit{ for every } \underline{p} \in \operatorname{Spec}\Gamma_X,\ h^\circ(B/\underline{p}B) < +\infty,$$

then D is noetherian.
Moreover: *D is noetherian if and only if for every* $\underline{q} \in \operatorname{Spec}\Gamma_X$, $h^\circ(D/\underline{q}D) < +\infty$.

We remark that if $k = \mathbf{C}$ then the first statement is false, while the second is unknown.

2. We are interested in the following conjectures (cf. **[2]**): (a) $D \hookrightarrow B$ is flat; (b) for every $\underline{p} \in \operatorname{Spec} D$, $\underline{p}B \in \operatorname{Spec} B$. We will deal with the case $k = \mathbf{R}$ (in **[5]** some partial result is obtained also in the general case). We suppose that U is coherent and that $\mathcal{O}(U)$ verifies the condition (C) of Th. 2. (e.g., U is semi-algebraic).

THEOREM 3. *$\mathcal{N}(U) \hookrightarrow \mathcal{O}(U)$ is faithfully flat.*

Suppose now X normal, U semi-algebraic, and let $\underline{p} \subset \mathcal{N}(U)$ be a defining prime ideal.

THEOREM 4. *There exist a smialgebraic set $U' \subset U x \mathbf{R}^m$ and a prime ideal $\underline{q} \subset \mathcal{N}(U')$ such that $\underline{p} = \underline{q} \cap \mathcal{N}(U)$ and $Z(\underline{q}) \to Z(\underline{p})$ is a normalization map.*

CONJECTURE. $\underline{p}\mathcal{O}(U) = q\mathcal{O}(U') \cap \mathcal{O}(U)$.

THEOREM 5. *The extended ideal $\underline{p}\mathcal{O}(U)$ is prime if $\mathcal{N}(U)/\underline{P}$ is integrally closed or if the conjecture holds.*

We refer to **[5]** for Th. 3. and to **[4]** for the first assertion of Th. 5.

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