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SOME RESULTS ON REAL ALGEBRAIC CYCLES

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Dedicated to the memory of Gus Efraymson

Introduction. Let φ: $V \to W$ be a differentiable map between two compact differentiable manifolds. We shall say that φ has a weak algebraic approximation if it is possible to find two structures of real algebraic varieties V_a, W_a on V_a, W such that φ: $V_a \to W_a$ can be approximated by algebraic maps. It was conjectured by Akbulut and King [**1**] that any differentiable map should have weak algebraic approximation. Bendetti and Dedo [**4**] have proved that for any $n \geqq 11$ there exists a differentiable map φ: $V_{n-2} \to W_n$, dim $V = n - 2$, dim $W = n$, that has no weak algebraic approximation. It seems that one of the main problems in this area of real algebraic geometry is to characterise differentiable maps that have weak algebraic approximation. These notes are devoted to giving some results in this direction.

In §1 we fix the notations and we recall some Known results that show that the above problem is equivalent to characetrizing the algebraic homology classes. The fact that a differentiable map φ: $V \to W$, $W \subset \mathbf{R}^n$, between algebraic compact varieties has, in general, no algebraic approximation depends on the non existence of an algebraic tubular neighbourhood of W in $\mathbf{R}^n$.

In §2 we shall prove that, if on W there exists a topological line bundle $F \to W$ that is not trivial, then W never has a neighbourhood $W \subset U \subset \mathbf{R}^n$ and an algebraic retraction π: $U \to W$.

In §3 we disprove the "strong Nash conjecture". In fact we prove (theorem 4) that for any $n \geqq 11$ there exists a compact, connected, differentiable manifold W_n that has no algebraic structure W_n^a birationally equivalent to $P_n(\mathbf{R})$. On the other side we prove (Theorem 5) that the strong Nash conjecture is true for $n = 2$.

In §4 we recall some results and definitions about the strongly algebraic vector bundles. Using the notions of §4 and the theory of weak complete intersections, in §5, we give some positive answers to our main problem. The results of §2 are not yet published, §3 is contained in a paper (to appear) by Benedetti and Tognoli. The results of §5 well appear in a paper of the author.

1. Definitions and problems. By algebraic variety (or manifold if it is regular) we shall mean affine, reduced real algebraic variety (i.e., the locus of zeroes of polynomials functions in $\mathbf{R}^n$). Algebraic map means rational regular map.

Let $\varphi: V \to W$ be a continuous map between algebraic varieties, let V be compact, $V \subset \mathbf{R}^n$, $W \subset \mathbf{R}^m$. We shall say that the map φ has algebraic approximation if for any $\varepsilon > 0$, $p \in \mathbf{N}$, there exists an algebraic map $\varphi_\varepsilon: V \to W$ such that $\|\varphi - \varphi_\varepsilon\|^p < \varepsilon$, where $\| \quad \|^p$ is the norm of C^p functions.

If W is an algebraic variety we shall say that $\alpha \in H_p(W, \mathbf{Z}_2)$ is algebraic if there exists an algebraic map $\varphi: V \to W$, V a regular, compact algebraic variety, such that: $\alpha = \varphi_*([V])$, where $[V]$ is the fundamental class of V.

We shall denote by $H_1^a(W, \mathbf{Z}_2)$ the subgroup of $H_p(W, \mathbf{Z}_2)$ generated by algebraic classes.

Let W be an algebraic variety and $\varphi: V \to W$ a differentiable map, where V is a compact differentiable manifold. We shall say that φ is bordant to an algebraic map if there exists a map $\phi: B \to W$ such that

(i) B is a differentiable manifold with boundary $\partial B = V \cup V_0$, $\phi|_V = \varphi$, and

(ii) V_0 has an algebraic structure and $\phi|_{V_0}$ is algebraic.

Let $\varphi: V \to W$ be a differentiable map between compact differentiable manifolds. We shall say that φ has a weak algebraic approximation if there exist two diffeomorphisms $g: V \to V_a$, $h: W \to W_a$, where V_a, W_a are regular algebraic varieties, such that $h \circ \varphi \circ g^{-1}: V_a \to W_a$ has algebraic approximation.

One of the most interesting problems in real algebraic geometry seems to be the following.

PROBLEM. Characterize the differentiable maps $\varphi: V \to W$ that have weak algebraic approximation.

The following theorem contains some information about the above problem.

THEOREM 1. [11] *Let $\varphi: V \to W$ be a differentiable map between algebraic regular varieties and let us suppose that V is compact.*

Then for any $\varepsilon > 0$, $p \in \mathbf{N}$ there exists an algebraic variety $\hat{V}$ and an analytic isomorphism $J: V' \to V$, where V' is open and closed in $\hat{V}$ and an algebraic map $\varphi: \hat{V} \to W$ such that $\|\varphi|_{V'} - \varphi \circ J\|^p < \varepsilon$, If one of the following conditions is satisfied we may suppose $\hat{V} = V'$ (and hence y has a weak algebraic approximation).

α) $\varphi_*([V]) \in H_p^a(W, \mathbf{Z}_2)$, $[V]$ = fundamental class of V;

(β) φ is bordant to an algebraic map.

An algebraic variety W such that $H_*(W, \mathbf{Z}_a) = H_*^a(W, \mathbf{Z}_2)$ shall be called totally algebraic.

COROLLARY 1. *Let W be a totally algebraic variety. Then any differentiable map φ: $V \to W$, where V is a compact differentiable manifold, has weak algebraic approximation.*

Let W be a differentiable manifold, we shall denote by $H_*^{sw}(W, \mathbf{Z}_2)$ the subring of $H_*(W, \mathbf{Z}_2)$ generated by the elements α having one of the following properties:

1) α is represented by the fundamental class of some submanifold of W;
2) α is dual of the Stiefel-Whitney class of some vector bundle $F \to W$.

We have (see [3]).

PROPOSITION 1. *Let V be a compact differentiable manifold. Then V has an algebraic structure V_a, such that*

$$H_*^a(V_a, \mathbf{Z}_2) \supset H_*^{sw}(V_a, \mathbf{Z}_2).$$

COROLLARY 2. *Let W be a compact differentiable manifold. If* dim $W \leqq 6$, *then W has an algebraic structure that is totally algebraic.*

PROOF. By a result of Thom ([10]), if dim $W > 6$, we have $H_*(W, \mathbf{Z}_2) = H_*^{sw}(W, \mathbf{Z}_2)$. Hence the corollary follows from proposition 1.

2. The tubular neighbourhood. Let W be an algebraic subvariety of $\mathbf{R}^n$, we shall say that the embedding i: $W \hookrightarrow \mathbf{R}^n$ is perfect if there exists a Zariski open set $U \supset W$ and an algebraic map π: $U \to W$, such that $\pi|_W = \text{id}$.

Let W be an algebraic (differentiable) manifold. We shall say that W is perfect (has a model that is perfect) if any differentiable map φ: $V \to W$, where V is a compact algebraic variety, has algebraic approximation (if W has an algebraic structure W_a such that W_a is perfect).

LEMMA 1. *Let i: $W \hookrightarrow \mathbf{R}^n$ be a perfect embedding. Then W is perfect.*

PROOF. Let π: $U \to W$ be the algebraic retraction and φ: $V \to W$ a differentiable map. From the Weierstrass approximation theorem we know that we can approximate φ by polynomial maps ψ_n: $V \to U$, hence $\pi \circ \psi_n$: $V \to W$ are algebraic approximations of φ.

LEMMA 2. *Let W be a compact algebraic variety. If W (considered as differentiable manifold) has a perfect model, then any topological line bundle $F \to W$ has a strongly algebraic structure* (For the definition of strongly algebraic vector bundle see §4.)

PROOF. Let W_a be a perfect model of W. Any differentiable divisor $D \to W_a$ can be approximated by an algebraic divisor $D_a \to W_a$; hence any topological line bundle $F' \to W_a$ has a strongly algebraic structure.

Let now $F \to W$ be a topological line bundle and h: $W \to W_a$ a diffeomorphism.

If $(h^{-1})^*(F)_a$ is an algebraic structure on $(h^{-1})^*(F)$ and $\hat{h}$: $W \to W_a$ is an algebraic diffeomorphism near to h we have that $\hat{h}^*((h^{-1})^*(F)_a)$ is a strong algebraic structure on F.

THEOREM 2. *Let W be a compact algebraic variety such that one of the* following conditions is staisfied:

i) *W is not connected*;

ii) *There exists a topological nontrivial line bundle $F \to W$. Then W has no perfect embedding.*

PROOF. Let us suppose W not connected and that $W = \bigcup_{i=1}^q W_i$, $W_i \neq \phi$, dim $W_1 \leqq \dim \bigcup_{i \neq 1} W_i$, is the decomposition of W into connected components. Any Zarisky open set U of $\mathbf{R}^n$ is irreducible hence the theorem is proved if W is reducible.

Now suppose W is irreducible and let φ: $W \to W$ be a differentiable map such that $\varphi|_{W_1} = \mathrm{id}$, $\varphi(W) \subset W_1$. From the remark d) of **[12]** we know that φ has no algebraic approximation and hence W is not perfect. By lemma 1 the theorem is proved in this case.

Let $F \to W$ be a non trivial topological line bundle. From lemma 2 we know that if W has a perfect embedding $W \hookrightarrow \mathbf{R}^n$, π: $U \to W$, then F has a strongly algebraic structure $F_a \to W$. Let $\hat{F}_a = \pi^*(F_a) \to U$ be the strongly algebraic line bundle pullback of F_a. If D' is the (algebraic) divisor associated to $\hat{F}_a$ then D' can be extended to $S^n \supset \mathbf{R}^n \supset U$ hence $\hat{F}_a$ is the restriction of a line bundle $\tilde{F} \to S^n$. Clearly dim $W > 0$ and hence $n \geqq 2$. So the line bundle $\tilde{F}$ should be trivial but this cannot happen because $\tilde{F}|_W$ is not.

COROLLARY 1. *If W is a compact algebraic variety, then $W \times S^1$ has no perfect embedding. If* dim $W \geqq 2$ *nad $\tilde{W}$ is obtained from W blowing up a point, then $\tilde{W}$ has no perfect embedding.*

3. The strong Nash conjecture. The following result proves that, in general, a differentiable map φ: $V \to W$ has no weak algebraic approximation.

THEOREM 3. *For any $n \geqq 11$ there exists a compact connected differentiable manifold W_n and an element $\alpha_n \in H_{n-2}(W_n, \mathbf{Z}_2)$ such that for any algebraic structure W_n^a on W_n, α is not algebraic.*

See **[4]** for the proof.

COROLLARY 1. *For any $n \geqq 11$ there exists a differentiable map φ: $V \to W_n$ that has no weak algebraic approximation.*

PROOF. Let $\varphi: V \to W_n$ be a differentiable map such that $\alpha_n = \varphi_*([V])$ (see [10] for the existence of φ). By theorems 2 and 3 it follows that φ has no weak algebraic aproximation.

Let V, W be two real (or complex) algebraic varieties. We shall say that V is birationally equivalent to W if there exist dense open sets $U_v \subset V$, $U_w \subset W$ and an algebraic isomorphism $\varphi: U_v \to U_w$; V shall be called birational if it is birationally equivalent to $P_n(\mathbf{R})$. It is well known (see [12] p. 178) that the number of connected components (of maximal dimention) is a birational invariant. Nash [9] has stated the following problem.

PROBLEM. Let V be a compact connected differentiable manifold. Does there exist a birational structure V_a on V?

Now we shall prove

THEOREM 4. *For any $n \geqq 11$ there exists a compact connected n-dimensional differentiable manifold W_n that has no birational structure.*

The above theorem is a consequence of theorem 3 and of the following proposition.

PROPOSITION 2. *Let W be a compact algebraic variety birationally equivalent to $P_n(\mathbf{R})$. Then W is totally algebraic.*

PROOF. Let $U \subset W$ be an open set isomorphic to an open set $U' \subset P_n(\mathbf{R})$ and $\varphi: U \to U'$ an isomorphism. We shall denote by $\Gamma \subset W \times P_n(\mathbf{R})$ the closure of the graph of φ and by $\Gamma \to^{\pi'} P_n(\mathbf{R})$, $\Gamma \to^{\pi} W$ the natural projections. Using the "main lemma" of Hironaka [8] we deduce that there exists a sequence of blow ups (with smooth centers) $\vartheta: X \to P_n(\mathbf{R})$ and a birational algebraic map $\psi: X \to \Gamma$ such that the following diagram is commutative.

$$\begin{array}{ccc} X & \xrightarrow{\psi} & \Gamma \\ & \vartheta \quad \pi' & \\ & P_n(\mathbf{R}) & \end{array} \qquad (1)$$

It is well-Known that $P_n(\mathbf{R})$ is totally algebraic, hence by the proposition 6.1 of [9], X is also totally algebraic. Let us now consider the surjective algebraic map $\pi \circ \psi: X \to W$. $\pi \circ \psi$ is surjective of degree 1, hence by Poincare duality we deduce that $(\pi \circ \psi)_*: H_*(X, \mathbf{Z}_2) \to H_*(W, \mathbf{Z}_2)$ is surjective. This proves that X totally algebraic implies that W has the same property.

In dimension 2 we have

THEOREM 5. *Let W be a compact, connected, two dimensional regular variety. The following conditions are equivalent*:

1) *Any topological, compact, connected, two dimensional manifold T has an algebraic structure T_a birationally equivalent to W.*

2) *W is birationally equivalent to an algebraic surface T homeomorphic to the sphere S^2.*

3) *$H_1(W, \mathbf{Z}_2)$ is algebraic.*

4) *Any topological line bundle $F \to W$ has a strongly algebraic structure. (For the definition of strongly algebraic vector bundle see* §4.)

PROOF. We recall some well known facts about topological classification of compact surfaces.

By $P_2(\mathbf{R})$, S^2, T_1 we shall denote the projective plane, the two sphere and the two dimensional torus. If X, Y are manifolds $Z \# Y$ is the connected sum.

Let

$$T_g = \frac{T_1 \# \cdots \# T_1}{g \text{ times}} \text{ and } U_g = \frac{P_2(\mathbf{R}) \# \cdots \# P_2(\mathbf{R})}{g \text{ times}}$$

be the orientable and the non orientable surfaces obtained by iterated connected sums of T_1 and $P_2(\mathbf{R})$.

If V is any connected compact two dimensional manifold we have (see [5]):

i) V is homeomorphic to some of the following models S^2, T_g, U_g,

ii) $T_g \# P_2(\mathbf{R}) \cong U_{2g+1}$ where $\cong$ means homeomorphic, and

iii) $V \# P_2(\mathbf{R})$ is homeomorphic to the surface obtained from V by blowing up a point.

Now we prove the equivalence of the various conditions.

3) $\Leftrightarrow$ 4). This is well-known see [13] of [3].

2) $\Rightarrow$ 3). The fact that T is homeomorphic to S^2 implies that T is totally algebraic. By the proposition 6.1 of [2] we deduce, using the same arguments of proposition 2, that W is totally algebraic.

3) $\Rightarrow$ 2). By the topological classification we know that one can find a finite number of differentiable curves, $S_1, \ldots, S_q$ in general position in W such that W/R is homeomorphic to S^2, where R is the equivalence relation $x \sim^R y \Leftrightarrow x = y$ or $x \cup y$ $C(\bigcup_{i=1}^q S_i)$. The condition 3) implies (see Theorem 1) that we can approximate S_i by regular algebraic curves S_i'. From Proposition 3.4 of [5] we deduce that there exists an algebraic variety W' and an algebraic surjective map $\pi\colon W \to W'$ such that: $\pi | W - \bigcup_{i=1}^q S_i'$ is an isomorphism, $\pi(\bigcup_{i=1}^q S_i')$ is a point. Clearly W' is homeomorphic to a sphere and birationally equivalent to W.

3) $\Rightarrow$ 1). We have proved that if 3) holds, then W is birationally equivalent to W' homeomorphic to a sphere. Blowing up W' we prove that

W is birationally equivalent to surfaces homeomorphic to any U_g, $g \in \mathbf{N}$. Let now T_g be an orientable surface of genus g. By the previous remark we know that $T_g \# P_2(\mathbf{R}) \simeq U_{2g+1}$ has an algebraic structure $\hat{U}_{2g+1}$ birationally equivalent to W'. Hence $\hat{U}_{2g+1}$ is totally algebraic. Now we can find a differentable curve $S \subset \hat{U}_{2g+1}$ such that if we contract this curve we find a topological madel of T_g. By the usual argument we deduce that S can be chosen to be algebraic and the quotient ($\simeq T_g$) birationally equivalent to W.

REMARK 1. In the proof of the previous theorem, to find the algebraic approximations of the curves S_i we use the following well known argument. The divisor S_i is algebraic the line bundle F_i associated to S_i is strongly algebraic $\Rightarrow$ the differentiable sections of F_i can be approximated by algebraic sections $\Rightarrow$ S_i has algebraic approximation. We remark explicitely that this argument works also in case W is not regular.

REMARK 2. Let W be an algebraic regular variety homeomorphic to $P_2(\mathbf{R})$. Then W is totally algebraic. In fact the generator of $H_1(W, \mathbf{Z}_2)$ is the dual of the Stiefel Whitney class of the tangent bundle; hence (see [3]) is algebraic.

4. Strongly algebraic vector bundles and complete intersections. Let V be an algebraic variety and $F \to^{\pi} V$ be an algebraic vector bundle. Hence F is an abstract algebraic variety. We shall say that F is strongly algebraic if F is an affine variety. The following result is proved in [3].

PROPOSITION 3. *Let V be a compact real algebraic variety and $F \to^{\pi} V$ be an algebraic vector bundle. The following conditions are equivalent:*

1) *F is strongly algebraic.*

2) *There exists an algebraic vector bundle $F' \to V$ such that $F \oplus F'$ is algebraically isomorphic to the trivial bundle.*

3) *There exists an algebraic map φ: $V \to G_{n,q}$ such that F is algebraically isomorphic to φ^* (tautological bundle).*

REMARK 1. An algebraic vector bundle in general is not strongly algebraic (see [13]).

In the following we shall use

LEMMA 3. *Let V be a compact algebraic variety and $F \to V$ be a strongly algebraic vector bundle. Any differentiable section γ: $V \to F$ can be approximated by algebraic sections.*

See [13] for the proof.

Let now V be a differentiable manifold, $F \to^{\pi} V$ be a differentiable vector bundle and S a subvariety of V. We shall say that S is a weak complete intersection in V, with respect to F, if there exists a differentiable section $\gamma\colon V \to F$ such that

i) $S = \{x \in V | \gamma(x) = 0\}$, and

ii) γ is transverse to the zero section of F. γ shall be called an equation of S. If V, F, γ are algebraic, S is called an algebraic weak complete intersection.

REMARK 2. Let S be a weak complete intersection. Then S is a smooth submanifold and dim V-dim S = dim$(F_x = \pi^{-1}(x))$. If F is trivial we have the usual notion of complete intersection.

Let S be a weak complete intersection in V with respect to the vector bundle F and the equation $\gamma\colon V \to F$. Let us suppose that a scalar product is defined on F and denote:

1) $F_\gamma \to V - S$ as the subbundle of $F|_{V-S}$ generated by $\gamma(x)$, $x \in V - S$, and

2) $F_\gamma^\perp$ as the orthogonal comple ment of F_γ in $F|_{V-S}$.

The following results are proved in **[6]**.

PROPOSITION 4. *Let S be a submanifold of the differentiable manifold V. Then*

a) *If S is a weak complete intersection in V with respect to the vector bundle $F \to V$, then $F|_S$ is isomorphic to the normal bundle of S in V.*

b) *S is weak complete intersection in the tubular neighbourhood of S in V.*

c) *Let $N_S \to U_S$ be the normal bundle of S extended to a tubular neighbourhood U_S and $\gamma\colon U_S \to N_S$ an equation of S in U_S. S is a weak complete intersection in V is and only if $(N_S)_\gamma^\perp \to U_S - S$ can be extended to a vector bundle $F \to V - S$.*

d) *Let S, V, $N_S \to U_{S'}$, γ be as defined in* c) *and ∂U_S the boundary of U_S in V. Then $\partial U_S \to^p S$ is a fiber bundle and the fibers are spheres. The fiber bundle $(N_S)_\gamma^\perp|_{\partial U_S}$ is isomorphic to the subbundle of the tangent bundle of V given by vectors that are tangent to the fibers of $p\colon \partial U_S \to S$.*

REMARK 3. The following facts are easy consequences of proposition 4 (see **[14]**).

α) One point x_0 is a weak complete intersection in the sphere S^n if and only if $n = 1, 2, 4, 8$.

β) Let $S \subset \mathbf{R}^3$ a compact curve, then S is a complete intersection.

γ) One point x_0 is a weak complete intersection in the projective space $P_n(\mathbf{R})$, but it is not a complete intersection. The point x_0 is a weak complete intersection with respect to the vector bundle $F \simeq \bigoplus_{i=1}^n F_i$ where the F_i are the line bundles associated to the hyperplane sections.

Finally we have

PROPOSITION 5. *Let V be an algebraic variety and $F \to V$ a strongly* *algebraic vector bundle. Let S be a compact, differentiable submanifold of V that is a weak complete intersection with respect to F. Then S can be approximated by regular algebraic subvarieties of V that are weak complete intersection with respect to F.*

PROOF. It follows from Lemma 3.

5. Lie groups and Nash homology classes. If V is a differentiable manifold, we shall say that V has q umbilics, and write $\mathcal{U}(V) = q$ if there exists a subset $S = \bigcup_{i=1}^{q} x_i$, $q > 0$, of q points that is a weak complete intersection but any nonempty set of less then q points is not a weak complete intersection.

We need

LEMMA 4. *Let V be a differentiable manifold and $S \subset V$ a weak complete intersection with respect to the vector bundle $F \to V$. Let φ: $W \to V$ be a differentiable map, U'_S a neighbourhood of $\varphi^{-1}(S)$ such that φ: $U'_S \to \varphi(U'_S)$ is a diffeomorphism. Under these hypotheses $\varphi^{-1}(S)$ is a weak complete intersection in W respect to the vector bundle $\varphi^*(F)$.*

The proof is easy, see [**14**].

Now we have

THEOREM 6. *Let V be a differentiable manifold. Then we have $\mathcal{U}(V) \leqq 2$ and, if V is not compact, $\mathcal{U}(V) = 1$, and the set of one point is a complete intersection in V.*

SKETCH OF THE PROOF. I) Let $S^n = \{(x_1, \ldots x_{n+1}) \in \mathbf{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\}$ be the n-sphere. The set $S = (0, \ldots 0, 1) \cup (0, \ldots 0, -1)$ is a complete intersection and $x_1 = x_2 = \cdots = x_n = 0$ are equations of S. So we have (see also remark 3 of §4) $\mathcal{U}(S^n) = 1$ if $n = 1, 2, 4, 8$ and $\mathcal{U}(S^n) = 2$ if $n \neq 1, 2, 4, 8$.

Now let V be an n-dimensional manifold and $D \subset V$ an open set such that D is diffeomorphic to the open n-disk and $\bar{D}$ to the closed n-disk. We can construct a differentiable surjective map ϕ: $V \to S^n$ such that $\phi|_D$ is a diffeomorphism. From the results of lemma 4 the map ϕ proves $\mathcal{U}(V) \leqq \mathcal{U}(S^n) \leqq 2$.

II) Now suppose that V is connected and noncompact. It is possible to define a differentiable map J: $\mathbf{R} \to V$ such that

i) J: $\mathbf{R} \to J(\mathbf{R})$ is an homeomorphism, and

ii) $J(\mathbf{R})$ is a closed differentiable submanifold of V and J is a proper map. See [**14**]. Theorem 2, for the details of this construction.

Let U be a tubular neighbourhood of $J(\mathbf{R})$. $J(\mathbf{R})$ is contractible; hence U is diffeomorphic to $\mathbf{R} \times \mathbf{R}^{n-1}$. It is now possible to project V onto $\mathbf{R} \times S^{n-1}$ (we parametrize by $\mathbf{R}$ the construction used in the first part of the theorem). Using Lemma 4 we are reduced to proving that $\mathscr{U}(\mathbf{R} \times S^{n-1}) = 1$ and this is easy. See **[14]** for the details.

Finally we can use the above results to prove the

THEOREM 7. *Let G be a real algebraic, non compact, group and $\alpha \in H_p(G, Z_2)$. There exists an analytic component V' of an algebraic subvariety V of G such that α = fundamental class of V'.*

PROOF. Clearly we can suppose G is connected. By a well known result of Thom **[10]** there exists a differentiable map φ: $V \to G$, where V is a compact differentiable manifold, such that $\alpha = \varphi_*$ (fundamental class of V).

Let $\Gamma_\varphi = \{(x, y) \in V \times G | y = \varphi(x)\}$, $y_0 \in G$ and $\Gamma_0 = \{(x, y) \in V \times G | y = y_0\}$. The couple $(V \times G, \Gamma_\varphi)$ is diffeomorphic to $(G \times V, \Gamma_0)$, the map $(x, y) \to (x, y_0 \cdot \varphi(x)^{-1}y)$ is a diffeomoprhism. By Theorem 6, Γ_0 is a complete intersection in $V \times G$; hence Γ_y has the same property. Let $U \supset \varphi(V)$ be a compact neighbourhood of $\varphi(V)$ in G. Using proposition 5 of §4 we can approximate Γ_φ in $V \times U$ by an analytic component $\hat{\Gamma}$ of an algebraic variety $W \in V \times G$. We may suppose $W \cap (V \times U) = \hat{\Gamma}$ and hence the image of the foundamental class of $\hat{\Gamma}$ under π: $V \times G \to G$ is α. The theorem is proved.

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