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A METHOD FOR CONSTRUCTION OF SURFACES UNDER TENSION

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ABSTRACT. We describe and develop the properties of a method for interpolation of scattered data based on a generalization of the univariate spline under tension defined on a triangulation of the domain. Examples are given showing that the surface responds in a predictable way to the application of tension. As in the univariate case, tension parameters offer the promise of a way for the user to control the behavior of surfaces which have steep gradients implied by the data.

1. Introduction. We consider the problem of interpolation of scattered data, (x_i, y_i, z_i), $i = 1, 2, \ldots, N$. This requires the definition of a bivariate function F such that $F(x_i, y_i) = z_i$, $i = 1, 2, \ldots, N$. This problem has been addressed by several authors (see [5] and the references therein). To date, there are a number of methods available which work quite well on a wide variety of data sets, but these general purpose methods fail to yield acceptable results when steep gradients are implied by the data. A typical example is the data given in Table 1. The modified quadratic Shepard's method [6] is a good general purpose method. The result of applying it to the data of Table 1 is shown in Figure 1 and is similar to that obtained from other general purpose methods.

Our purpose here is to devise a scheme which allows some control over the behavior of the interpolant. In particular we wish to allow the capability to suppress, or at least to dampen the overshoot and undershoot of the surface near steep gradients. In the univariate case this has been achieved by various means, one being the spline under tension [20]. One type of bivariate analog of the univariate cubic spline is the minimum norm network of Nielson [15]. It is our goal to combine the two ideas in an attempt to obtain a certain type of bivariate analog of the spline under tension.

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X	Y	Z	X	Y	Z
0.040	0.040	0.500	0.400	0.040	0.500
0.600	0.040	0.500	0.800	0.000	0.500
0.960	0.080	0.500	0.000	0.280	0.500
0.200	0.270	0.500	0.400	0.200	0.500
0.880	0.200	0.500	0.160	0.400	0.500
0.280	0.380	0.500	0.480	0.360	0.500
0.680	0.320	0.500	0.800	0.370	0.500
1.000	0.360	0.500	0.400	0.440	0.327
0.640	0.440	0.196	0.840	0.420	0.093
1.000	0.440	0.000	0.720	0.560	0.094
0.920	0.600	0.022	0.480	0.640	0.117
0.960	0.760	0.004	0.640	0.760	0.036
0.400	0.880	0.015	0.800	0.880	0.005
0.600	1.000	0.000	1.000	1.000	0.000
0.080	0.480	0.491	0.000	0.600	0.444
0.040	0.800	0.278	0.000	1.000	0.000
0.280	1.000	0.000			

TABLE 1.

Our approach then, is as follows: (1) Triangulate the region of interest, using the points $(x_i, y_i) = V_i$ (and only those points) as the vertices of the triangulation. (2) Determine a curve network defined over the edges of the triangulation which minimizes an analog of the functional minimized by the spline under tension. (3) Extend the curve network to a function defined on the interior of the triangles by use of a C^1 triangular interpolant. The basic idea of using C^1 elements over a triangulated grid is not new. Barnhill **[2]**, Klucewicz **[8]**, Little **[2]**, **[10]**, Lawson **[9]** and Akima **[1]** are some of the first to study and popularize this type of method. In **[16]**, we dealt in general with this type of method and made some comparisons.

In §2 we give a brief resume of notation and properties of the univariate spline under tension. We then define a functional for the curve network, obtain its minimizer, and give some results concerning properties of the curve network. In §3 we discuss ways to extend the curve network to a C^1 function defined over the entire domain. Examples are given which show that standard triangular interpolants do not yield desirable results. A new element with tension is described and the results of applying it are shown for a number of cases. In the final section we discuss some practicalities and some problems which remain unsolved.

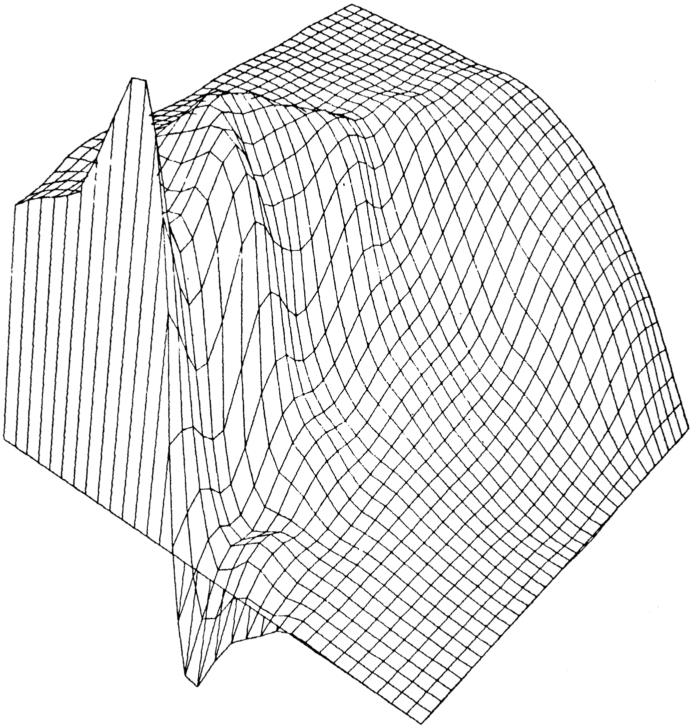


FIGURE 1.

2. The tension network. We first review some material concerning univariate splines under tension. Let $a = t_1 < t_2 < \cdots < t_n = b$ be a knot sequence and f_i, $i = 1, 2, \ldots n$ the ordinates for interpolation. Let $\mathscr{H}[a, b] = \{f: f'$ is absolutely continuous on $[a, b]$ and $f'' \in L^2[a, b]\}$. It can be shown (see **[12]**) that the unique minimizer of

$$\int_a^b [f''(t)]^2 dt + \alpha^2 \int_a^b [f'(t)]^2 dt \tag{2.1}$$

in $\mathscr{H}[a, b]$, subject to the interpolation requirements $f(t_i) = f_i$, $i = 1, 2, \ldots, n$ is a function T_α, characterized as follows:

(i) $T_\alpha(t) = a_i + b_i t + c_i e^{\alpha t} + d_i e^{-\alpha t}$, $t \in [t_i, t_{i+1}]$, $i = 1, \ldots, n - 1$,

(ii) $T_\alpha \in C^2[a, b]$,

(iii) $T''_\alpha(a) = T''_\alpha(b) = 0$,

(iv) $T_\alpha(t_i) = f_i$, $i = 1, 2, \ldots, n$.

T_α is called the natural spline under tension and it is well known that $\mathrm{Lim}_{\alpha \to 0} T_\alpha = S$, $\mathrm{Lim}_{\alpha \to \infty} T_\alpha = P$ where S is the natural cubic spline of interpolation and P is the piecewise linear interpolant.

An efficient method for computing a spline under tension is based upon the Hermite form

$$\begin{aligned} H_\alpha(s) &= H_\alpha[y_1, y_0, y_1', y_0'](s) \\ &= g_\alpha(1-s)[y_1 - y_0 - y_0'] - g_\alpha(s)[y_1 - y_0 - y_1'] \\ &\qquad + (1-s)y_0 + sy_1 \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} g_\alpha(s) &= \beta(e^{-\alpha s} + \alpha s - 1) + \gamma(e^{\alpha s} - \alpha s - 1), \\ \beta &= (e^\alpha - \alpha - 1)/\delta, \gamma = (1 - \alpha - e^{-\alpha})/\delta, \\ \delta &= \alpha^2(e^{-\alpha} - e^\alpha) + 2\alpha(e^{-\alpha} + e^\alpha - 2). \end{aligned} \tag{2.3}$$

We refer to H_α as an exponential Hermite interpolant and note that $g_\alpha(0) = g_\alpha(1) = g_\alpha'(0) = 0$, $g_\alpha'(1) = 1$, so that $H_\alpha(0) = y_0$, $H_\alpha(1) = y_1$, $H_\alpha'(0) = y_0'$, $H_\alpha'(1) = y_1'$. We also point out that

$$H_\alpha'''' - \alpha^2 H_\alpha'' = 0. \tag{2.4}$$

It is this property that has served as the characterization of splines under tension for several authors (see **[20]**, **[4]**, **[17]** and **[21]**). With

$$\begin{aligned} T_\alpha(t) &= H_{\alpha h_i}[f_{i+1}, f_i, T_\alpha'(t_{i+1}), T_\alpha'(t_i)]\left(\frac{t - t_i}{t_{i+1} - t_i}\right), \\ &\quad t \in [t_i, t_{i+1}], h_i = t_{i+1} - t_i, \\ &\quad i = 1, 2, \ldots, n-1 \end{aligned}$$

the condition that $T_\alpha \in C^2[a, b]$ leads to a tridiagonal system of equations in the unknowns $T_\alpha'(t_i)$, $i = 1, 2, \ldots, n$.

We also alert the reader to the taut splines of de Boor [3], where the effect of tension is accomplished by the use of additional knots.

We now proceed with the development of our bivariate analog. We describe our notation and give some definitions before we set out to characterize a network of curves which is analogous to a univariate spline under tension. Consider a set of nondegenerate triangles in the $x - y$ plane whose interiors do not intersect, with vertices V_i, $i = 1, \ldots, N$. Denote the triangle with vertices V_i, V_j, V_k, for $i \neq j \neq k \neq i$, by T_{ijk}, and the edge connecting V_i and V_j by e_{ij}. Let N_e be a set of double indices corresponding to edges in the triangulation. For our discussion, it makes no difference whether N_e contains ij or ji, so long as only one is included for each edge. The set of indices j such that V_i is joined to V_j by an edge in the triangulation will be denoted by N_i. The union of all triangles T_{ijk} in the triangulation is the region D over which the final interpolant will be defined. The domain of the curve network will be union of all edges,

$$E = \bigcup_{ij \in N_e} e_{ij}.$$

Since we eventually desire the final interpolant to be contained in $C^1[D]$, we define

$$C[E] = \{F\colon F = G|_E \text{ for some } G \in C^1[D],\ F|_{e_{ij}} \in \mathscr{H}[e_{ij}]\}.$$

Analogous to (2.1), we define for $F \in C[E]$

$$\sigma_\alpha(F) = \sum_{ij \in N_e} \int_{e_{ij}} \left[\frac{\partial^2 F}{\partial e_{ij}^2}\right]^2 de_{ij} + (\alpha_{ij})^2 \int_{e_{ij}} \left[\frac{\partial F}{\partial e_{ij}}\right]^2 de_{ij}$$

where de_{ij} is the element of arc length along e_{ij}, α is an $|N_e|$-tuple with components α_{ij}, $ij \in N_e$ and

$$\frac{\partial F}{\partial e_{ij}} = \frac{(x_j - x_i)F_x + (y_j - y_i)F_y}{|e_{ij}|}.$$

THEOREM 2.1. *The unique minimizer of $\sigma_\alpha(F)$ in $C[E]$ subject to $F(V_i) = z_i$, $i = 1, 2, \ldots, N$ is the piecewise exponential Hermite interpolant of the form*

$$T^\alpha(x, y) = a_{ij} + b_{ij}u + c_{ij}e^{\alpha_{ij}u} + d_{ij}e^{-\alpha_{ij}u},$$

$$(x, y) \in e_{ij},\ u = \sqrt{(x - x_i)^2 + (y - y_i)^2},$$

which has first order partial derivatives satisfying the $2N \times 2N$ linear system:

$$\sum_{j \in N_i} \frac{(x_j - x_i)}{|e_{ij}|^2}\left[g_{ij}''(1)\left[\frac{\partial T^\alpha}{\partial e_{ij}}(V_i) - \frac{z_j - z_i}{|e_{ij}|}\right] - g_{ij}''(0)\left[\frac{\partial T^\alpha}{\partial e_{ij}}(V_j) - \frac{z_j - z_i}{|e_{ij}|}\right]\right] = 0$$

(2.5)

$$\sum_{j \in N_i} \frac{(y_j - y_i)}{|e_{ij}|^2}\left[g_{ij}''(1)\left[\frac{\partial T^\alpha}{\partial e_{ij}}(V_i) - \frac{z_j - z_i}{|e_{ij}|}\right] - g_{ij}''(0)\left[\frac{\partial T^\alpha}{\partial e_{ij}}(V_j) - \frac{z_j - z_i}{|e_{ij}|}\right]\right] = 0,$$

$$i = 1, 2, \ldots, N.$$

Here $g_{ij}(s) = g_{\alpha_{ij}|e_{ij}|}(s)$, $0 \leqq s \leqq 1$, where g_α is given in (2.3).

PROOF. We find it convenient to represent each $F \in C[E]$ as a collection of univariate functions

$$f_{ij}(s) = F((1 - s)V_i + sV_j),\ ij \in N_e,\ 0 \leqq s \leqq 1.$$

With this, we have that

$$\sigma_\alpha(F) = \sum_{ij \in N_e} \frac{1}{|e_{ij}|^3}\left[\int_0^1 [f_{ij}''(s)]^2\, ds + (\alpha_{ij}|e_{ij}|)^2 \int_0^1 [f_{ij}'(s)]^2\, ds\right].$$

We now assume that F is any element of $C[F]$ such that $F(V_i) = z_i$,

$i = 1, 2, \ldots, N$, and that (2.5) has a solution $T_x^\alpha(V_i)$, $T_y^\alpha(V_i)$, $i = 1, 2, \ldots, N$ leading to the network T^α with

$$t_{ij}^\alpha(s) = H_{\alpha_{ij}|e_{ij}|}\left[z_j, z_i, |e_{ij}| \frac{\partial T^\alpha}{\partial e_{ij}}(V_j), |e_{ij}| \frac{\partial T^\alpha}{\partial e_{ij}}(V_i)\right](s), \quad 0 \leqq s \leqq 1. \tag{2.6}$$

It is easy to verify that

$$\begin{aligned}\sigma_\alpha(F) - \sigma_\alpha(T^\alpha) &= \sigma_\alpha(F - T^\alpha) \\ &\quad -2 \sum_{ij \in N_e} \frac{1}{|e_{ij}|^3}\left[\int_0^1 (t_{ij}^\alpha)''(s)[(t_{ij}^\alpha)''(s) - f_{ij}''(s)]\, ds\right. \\ &\quad \left. +(\alpha_{ij}|e_{ij}|)^2 \int_0^1 (t_{ij}^\alpha)'(s)[(t_{ij}^\alpha)'(s) - f_{ij}'(s)]\, ds\right].\end{aligned}$$

Using integration by parts and the fact that both F and T^α interpolate we have that

$$\begin{aligned}\sigma_\alpha(F) - \sigma_\alpha(T^\alpha) &= \sigma_\alpha(F - T^\alpha) \\ &\quad - 2 \sum_{ij \in N_e} \frac{1}{|e_{ij}|^3}\left\{(t_{ij}^\alpha)''\left[(t_{ij}^\alpha)' - f_{ij}'\right]_0^1\right. \\ &\quad + \int_0^1 [(t_{ij}^\alpha)''''(s) - (\alpha_{ij}|e_{ij}|)^2(t_{ij}^\alpha)''(s)] \\ &\quad \left.\times [t_{ij}^\alpha(s) - f_{ij}(s)]\, ds\right\}.\end{aligned}$$

Because t_{ij}^α satisfies (2.4) on e_{ij}, the last term in the above equation vanishes and

$$\sigma_\alpha(F) - \sigma_\alpha(T^\alpha) = \sigma_\alpha(F - T^\alpha) - 2 \sum_{ij \in N_e} \frac{1}{|e_{ij}|^3}\left[(t_{ij}^\alpha)''[(t_{ij}^\alpha)' - f_{ij}']\right]_0^1.$$

Note that $(t_{ij}^\alpha)''(1) = (t_{ji}^\alpha)''(0)$ and $(t_{ij}^\alpha)'(1) - f_{ij}'(1) = -(t_{ji}^\alpha)'(0) + f_{ji}'(0)$. Use of this fact, along with a rearrangement of the order of the summation yields

$$\sigma_\alpha(F) - \sigma_\alpha(T^\alpha) = \sigma_\alpha(F - T^\alpha) - 2 \sum_{i=1}^N \sum_{j \in N_i} \frac{1}{|e_{ij}|^3}\{(t_{ij}^\alpha)''(0)[(t_{ij}^\alpha)'(0) - f_{ij}'(0)]\}.$$

Now, replacing the directional derivatives with partial derivatives in the x and y directions, and rearranging slightly we find

$$\begin{aligned}\sigma_\alpha(F) - \sigma_\alpha(T^\alpha) &= \sigma_\alpha(F - T^\alpha) \\ &\quad - 2 \sum_{i=1}^N \left\{\left[\sum_{j \in N_i} \frac{(x_j - x_i)}{|e_{ij}|^3}(t_{ij}^\alpha)''(0)\right](T_x^\alpha(V_i) - F_x(V_i))\right. \\ &\quad \left. + \left[\sum_{j \in N_i} \frac{y_j - y_i}{|e_{ij}|^3}(t_{ij}^\alpha)''(0)\right](T_y^\alpha(V_i) - F_y(V_i))\right\}.\end{aligned}$$

Recognizing that the terms in brackets above are just the left sides of equations (2.5) we obtain $\sigma_\alpha(F) - \sigma_\alpha(T^\alpha) = \sigma_\alpha(F - T^\alpha) \geqq 0$. Thus if T^α exists, it minimizes σ_α. To establish the existence and uniqueness of T^α we consider a solution of the homogeneous version of (2.5) (i.e., $z_i = 0$, $i = 1, \ldots, N$). Let H be a piecewise exponential Hermite network associated with this data. Replacing both T^α and $F - T^\alpha$ with H we obtain $\sigma_\alpha(H) = 0$. This implies that $h_{ij}(s)$ is linear (constant if $\alpha_{ij} \neq 0$). Since $h_{ij}(0) = h_{ij}(1) = 0$ we may conclude that $h'_{ij}(0) = h'_{ij}(0) = 0$, $\forall ij \in N_e$ and so

$$(x_j - x_i)H_x(V_i) + (y_j - y_i)H_y(V_i) = 0, \quad j \in N_i, \quad i = 1, 2, \ldots, N.$$

Because each V_i is the vertex of some nondegenerate triangle, we have that $H_x(V_i) = H_y(V_i) = 0$, $i = 1, 2, \ldots, N$ and so the homogeneous system has only the trivial solution. This establishes the existence and uniqueness of T^α.

COROLLARY 2.2. *The tension network T^α reproduces linear functions.*

PROOF. We assume (x_i, y_i, z_i) lie on a plane $P(x, y) = a + bx + cy$ so that $z_i = a + bx_i + cy_i$, $i = 1, 2, \ldots, N$. This implies that

$$z_j - z_i = b(x_j - x_i) + c(y_j - y_i), j \in N_i, i = 1, 2, \ldots N.$$

If we make this substitution into (2.5) we can see that $T^\alpha_x(V_i) = b$, $T^\alpha_y(V_i) = c$, $i = 1, 2, \ldots N$, is a solution and due to the uniqueness we have $T^\alpha = P|_E$.

For the following asymptotic property, it is convenient to use a normalized, uniform set of tension parameters α with $\alpha_{ij} = \bar{\alpha}/|e_{ij}|$, $ij \in N_e$.

THEOREM 2.3. $\text{Lim}_{\bar{\alpha}\to\infty} T^\alpha = L$ *where L is the piecewise linear network.*

PROOF. We show that on each edge T^α converges to a linear function. On the edge e_{ij}

$$\begin{aligned} &T^\alpha(x, y) \\ &= g_{\bar{\alpha}}(1-s)\Big[z_j - z_i - \frac{\partial T^\alpha}{\partial e_{ij}}(V_i)\Big] - g_{\bar{\alpha}}(s)\Big[z_j - z_i - \frac{\partial T^\alpha}{\partial e_{ij}}(V_j)\Big] + (1-s)z_i + sz_j, \\ &\quad (x, y) \in e_{ij},\ s = \sqrt{(x - x_i)^2 + (y - y_i)^2}/|e_{ij}|. \end{aligned}$$

Since $\text{Lim}_{\bar{\alpha}\to\infty} \text{Max}_{0\leq s\leq 1} |g_{\bar{\alpha}}(s)| = 0$, we need only show that $T^\alpha_x(V_i)$, $T^\alpha_y(V_i)$, $i = 1, 2, \ldots, N$ remain bounded as $\bar{\alpha} \to \infty$. Actually these values have limits which are interesting in their own right. In the case of uniform, normalized tension values, (2.5) can be rewritten as

$$(2+\nu)\begin{pmatrix} a_i & c_i \\ c_i & b_i \end{pmatrix}\begin{pmatrix} T_x^\alpha(V_i) \\ T_y^\alpha(V_i) \end{pmatrix} + \sum_{j\in N_i}\begin{pmatrix} a_{ij} & c_{ij} \\ c_{ij} & b_{ij} \end{pmatrix}\begin{pmatrix} T_x^\alpha(V_j) \\ T_y^\alpha(V_j) \end{pmatrix}$$
$$= (3+\nu)\sum_{j\in N_i}\frac{1}{|e_{ij}|^3}\begin{pmatrix} (x_j-x_i)(z_j-z_i) \\ (y_j-y_i)(z_j-z_i) \end{pmatrix}, \quad i = 1, 2, \ldots, N$$

where

$$\nu = \left[\bar{\alpha}\frac{e^{\bar{\alpha}} + e^{-\bar{\alpha}} - 2}{e^{\bar{\alpha}} - e^{\bar{\alpha}} - 2\bar{\alpha}}\right] - 3,$$

$$a_{ij} = \frac{(x_j-x_i)^2}{|e_{ij}|^3}, \quad b_{ij} = \frac{(y_j-y_i)^2}{|e_{ij}|^3}, \quad c_{ij} = \frac{(x_j-x_i)(y_j-y_i)}{|e_{ij}|^3}$$

$$a_i = \sum_{j\in N_i} a_{ij}, \quad b_i = \sum_{j\in N_i} b_{ij}, \quad c_i = \sum_{j\in N_i} c_{ij}.$$

Since $\nu \to \infty$ as $\alpha \to \infty$ we have that $T_x^\alpha(V_i) \to T_x^\infty(V_i)$, $T_y^\alpha(V_i) \to T_x^\infty(V_i)$, $i = 1, 2, \ldots, N$ where

$$\begin{pmatrix} a_i & c_i \\ c_i & b_i \end{pmatrix}\begin{pmatrix} T_x^\infty(V_i) \\ T_y^\infty(V_i) \end{pmatrix} = \sum_{j\in N_i}\frac{1}{|e_{ij}|^3}\begin{pmatrix} (x_j-x_i)(z_j-z_i) \\ (y_j-y_i)(z_j-z_i) \end{pmatrix},$$

$i = 1, 2, \ldots, N$. In other words, the limits $T_x^\infty(V_i)$, $T_y^\infty(V_i)$, are the slopes of a local (triangular neighbors), inverse distance weighted, least squares plane passing through (x_i, y_i, z_i), for each $i = 1, 2, \ldots, N$.

We conclude this section with an example illustrating the effect of tension on the curve network. The data is given in Table 2. D is the convex hull of V_i, $i = 1, 2, \ldots, N$ and the triangulation is the optimal max-min triangulation associated with the Dirichlet tessellation. A uni-

X	Y	Z	X	Y	Z
0.000	0.000	0.300	0.000	0.600	0.360
0.000	1.000	0.300	0.240	0.040	0.291
0.160	0.320	0.300	0.200	0.600	0.330
0.200	0.800	0.300	0.200	1.000	0.240
0.400	0.400	0.300	0.360	0.560	0.300
0.320	0.760	0.075	0.400	1.000	0.150
0.480	0.480	0.300	0.520	0.520	0.075
0.520	0.640	0.075	0.600	0.080	0.210
0.600	0.400	0.225	0.600	0.800	0.075
0.680	0.560	0.075	0.800	0.200	0.150
1.000	0.000	0.120	0.960	0.600	0.060
1.000	1.000	0.000			

TABLE 2.

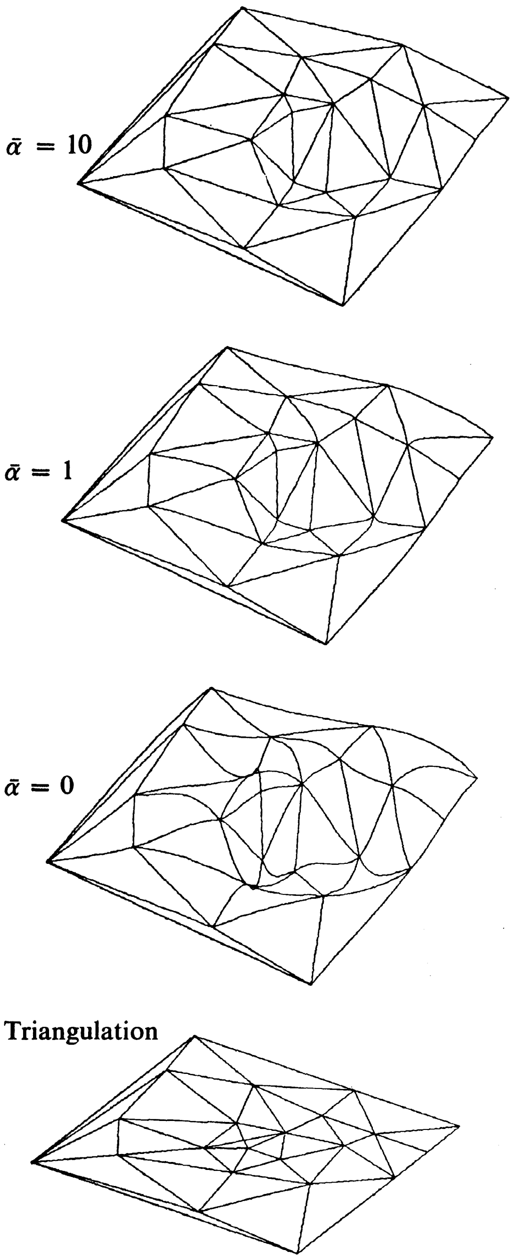


FIGURE 2.

form, normalized tension value is used with the results for $\bar{\alpha} = 0, \bar{\alpha} = 1$ and $\bar{\alpha} = 10$ shown in Figure 2.

3. Extension of the Tension network. In order to extend the tension network T^α defined on E to a C^1 function defined over D, we first must provide values of the normal derivative on E. We let

$$\frac{\partial T^\alpha}{\partial n_{ij}} = \frac{(y_j - y_i)\, T_x^\alpha - (x_j - x_i)\, T_y^\alpha}{|e_{ij}|}$$

denote the normal derivative on e_{ij}, $ij \in N_e$. Our approach here is to assume that the normal derivative has the same form as the tangential derivative $\partial T^\alpha/\partial e_{ij}$, provided by the tension network, with the additional constraint that

$$\frac{\partial T^\alpha}{\partial n_{ij}}(M_{ij}) = \frac{1}{2}\left[\frac{\partial T^\alpha}{\partial n_{ij}}(V_i) + \frac{\partial T^\alpha}{\partial n_{ij}}(V_j)\right],$$

$$M_{ij} = \frac{1}{2}(V_i + V_j).$$

This is analogous to quadratic derivatives in the cubic case, where the constraint of average value at the midpoint leads to a linear normal derivative. Here, it leads to

$$\frac{\partial T^\alpha}{\partial n_{ij}}(s) = \frac{\partial T^\alpha}{\partial n_{ij}}(V_i)\left[\frac{1 - g_{ij}'(s) + g_{ij}'(1 - s)}{2}\right]$$
$$+ \frac{\partial T^\alpha}{\partial n_{ij}}(V_j)\left[\frac{1 - g_{ij}'(s) + g_{ij}'(1 - s)}{2}\right].$$

With this selection of the normal derivative and the information provided by the tension network, we have completely defined T^α, T_x^α and T_y^α on E. All that is needed to complete the definition of the final interpolant is a description of how to interpolate in a single triangle, say T_{ijk}. This requires a nine-parameter, C^1 element (interpolant) which (i) assumes the values $T^\alpha(V_\ell)$, $T_x^\alpha(V_\ell)$, $T_y^\alpha(v_\ell)$, $\ell = i, j, k$; (ii) has position values on the edge e_{ij} of the form

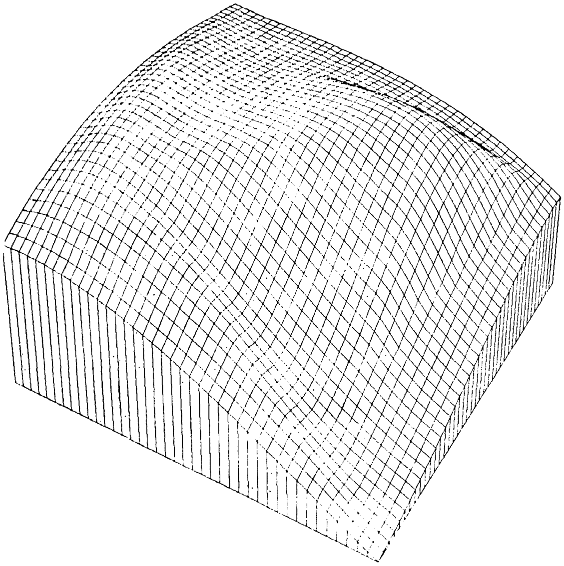
$$a + bu + ce^{\alpha_{ij}u} + de^{-\alpha_{ij}u},$$

$$u = \sqrt{(x - x_i)^2 + (y - y_i)^2}$$

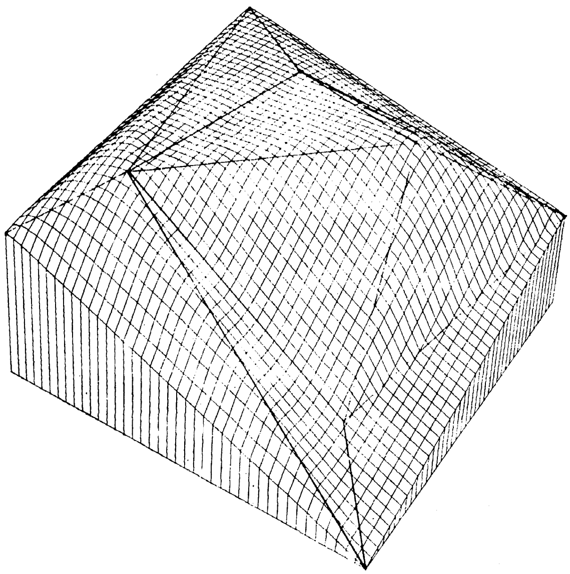
and (iii) normal derivatives of the form

$$a + b[(e^{-\alpha_{ij}u} - e^{\alpha_{ij}u})$$
$$+ 2(e^{\alpha_{ij}(u - .5|e_{ij}|)} - e^{-\alpha_{ij}(u - .5|e_{ij}|)})$$
$$+ (e^{\alpha_{ij}(|e_{ij}| - u)} - e^{-\alpha_{ij}(|e_{ij}| - u)}).$$

One way to obtain such an element is to discretize a transfinite C^1 tri-



(3*a*) $\alpha_{ij} = 0,\ ij \in N_e$



(3*b*) $\alpha_{ij} = 50/|e_{ij}|,\ ij \in N_e$

FIGURE 3.

angular interpolant which assumes arbitrary position and slope on the boundary of a triangle. Because the curve network will generally not have compatibility of cross partials, we must use a transfinite element which does not require it. We also want the element to have linear precision so that the overall interpolation scheme will preserve the linear precision of the curve network. A transfinite interpolant with these properties is the minimum norm triangular interpolant described in **[14]** and used in a similar context in the minimum norm network method for interpolating scattered data **[15]**. While this method of interpolating in a triangle works well in conjunction with a minimum norm network, the results here are disappointing. The application of this technique to a variety of examples quickly points out some undesirable behavior as the tension values are increased. Examples of this are shown in Figure 3 and Figure 4. The data for Figure 3 is given in Table 3, while that for Figure 4 is given by Table 2. The curve network has been sketched on the surface in Figures 3*b* and 4 to emphasize that the interpolant on the interior of the triangle exhibits none of the effects of tension and in many cases this results in a puffy appearing surface. The failure of the effects of tension to be propagated into the interior of the triangle is probably not unique to this scheme. We suspect this would be the case for any triangular interpolant that does not explicitly involve the tension parameters.

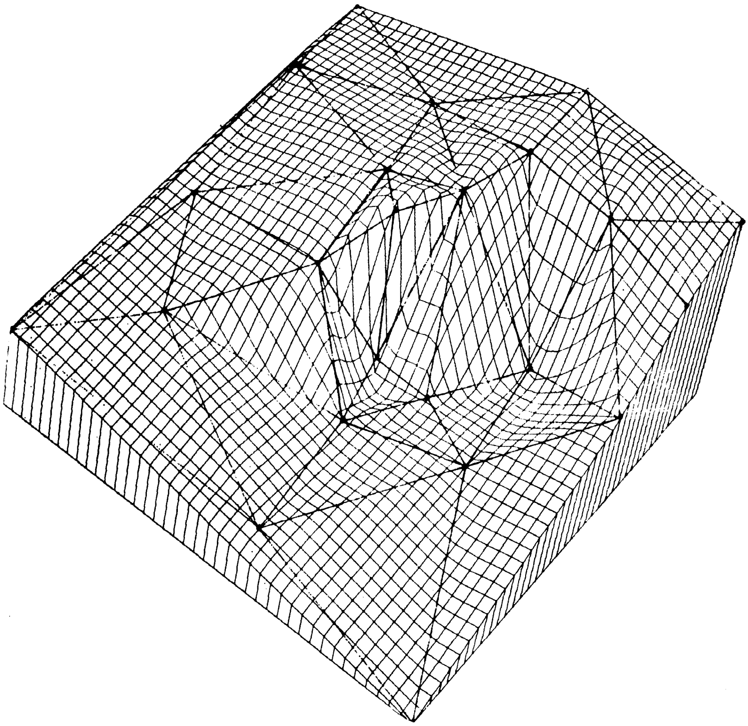
X	Y	Z	X	Y	Z
0.00	0.00	0.40	1.00	0.00	0.30
0.00	1.00	0.20	0.20	0.30	0.40
0.30	0.70	0.40	0.80	0.20	0.40
0.80	0.82	0.10	1.00	1.00	0.00

TABLE 3.

In order to correct this situation, we have used a triangular interpolant that was specifically developed for this application. It is based upon the ideas of the side-vertex method originally presented in **[13]**. For $V = (x, y) \in T_{ijk}$, we let

$$S_i = S_i(x, y) = \left(\frac{x - x_i b_i}{1 - b_i},\ \frac{y - y_i b_i}{1 - b_i}\right)$$

denote the point of intersection of the edge e_{jk} and a line emanating from V_i passing through V. See Figure 5. The values b_i, b_j and b_k represent the barycentric coordinates of V given by $x = b_i x_i + b_j x_j + b_k x_k$, $y = b_i x_i + b_j y_j + b_k y_k$, $1 = b_i + b_j + b_k$.



$\alpha_{ij} = 10/|e_{ij}|,\ ij \in N_e$

FIGURE 4.

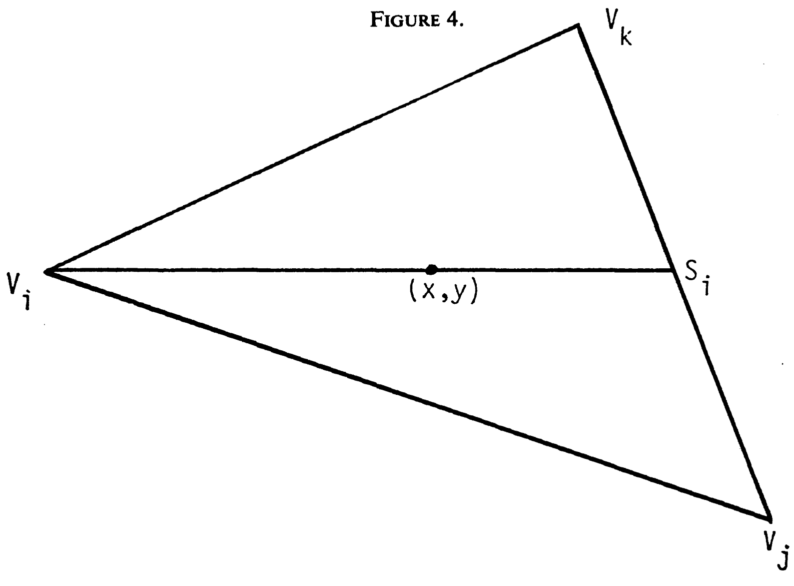
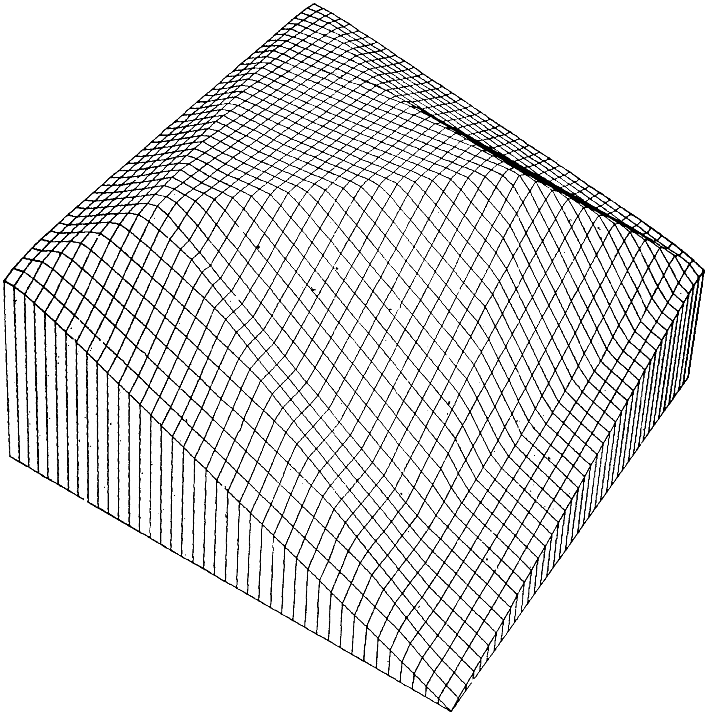
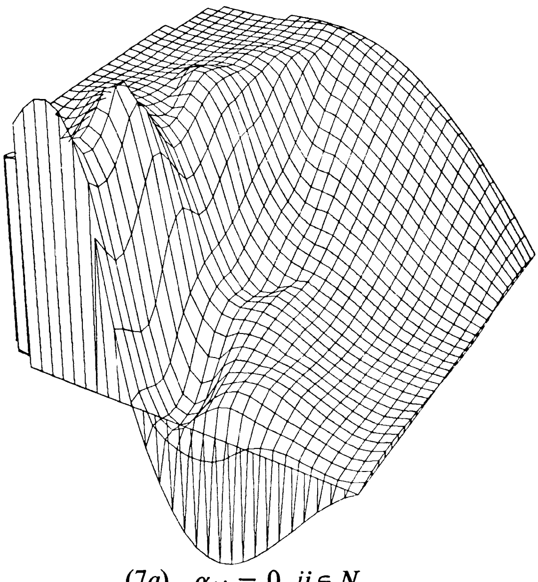


FIGURE 5.

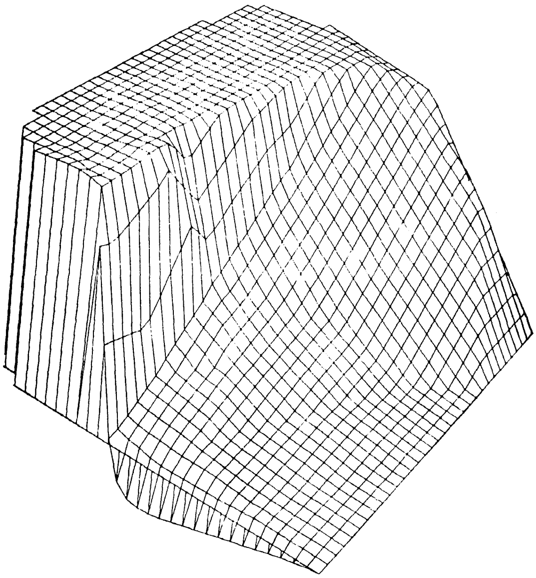


$\alpha_{ij} = 20/|e_{ij}|,\ ij \in N_e$

FIGURE 6.

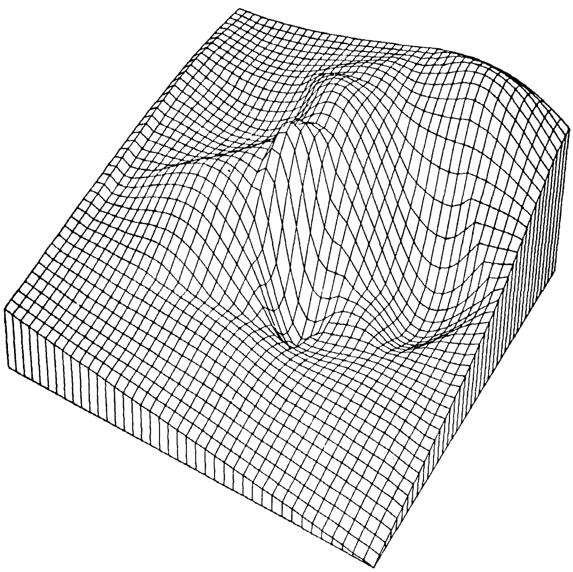


(7a) $\alpha_{ij} = 0,\ ij \in N_e$

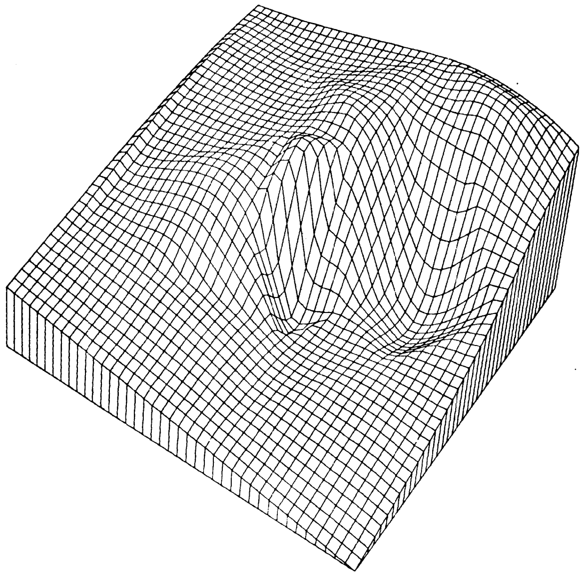


(7*b*) $\alpha_{ij} = 10/|e_{ij}|$, $ij \in N_e$

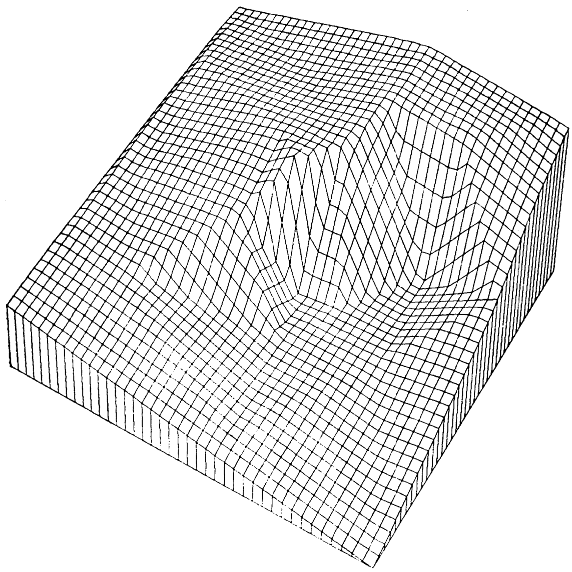
FIGURE 7.



(8*a*) $\alpha_{ij} = 0$, $ij \in N_e$



(8*b*) $\alpha_{ij} = 1/|e_{ij}|,\ ij \in N_e$



(8*c*) $\alpha_{ij} = 10/|e_{ij}|,\ ij \in N_e$

FIGURE 8.

On the line segment $\overline{V_iS_i}$, we define an interpolant based upon the exponential Hermite interpolant of (2.2). Since $|\overline{VV_i}|/|\overline{V_iS_i}| = 1 - b_i$, we define

$$D_i(x, y) = H_{\beta_i}[T^\alpha(S_i), T^\alpha(V_i), \delta T^\alpha(S_i), \delta T^\alpha(V_i)](1 - b_i)$$

where

$$\delta T^\alpha(S_i) = \frac{(x - x_i)T^\alpha_x(S_i) + (y - y_i)T^\alpha_y(S_i)}{1 - b_i},$$

$$\delta T^\alpha(V_i) = \frac{(x - x_i)T^\alpha_x(V_i) + (y - y_i)T^\alpha_y(V_i)}{1 - b_i},$$

and

$$\beta_i = \frac{b_k\alpha_{ik} + b_j\alpha_{ij}}{1 - b_i}.$$

This interpolant has the properties that

$$D_i\Big|_{\partial T_{ijk}} = T^\alpha\Big|_{\partial T_{ijk}}, \quad \frac{\partial D_i}{\partial x}\Big|_{e_{jk}} = T^\alpha_x\Big|_{e_{jk}}$$
$$\text{and } \frac{\partial D_i}{\partial y}\Big|_{e_{jk}} = T^\alpha_y\Big|_{e_{jk}}.$$

Utilizing this interpolant and the corresponding ones for the other two vertices, we form the convex combination $D = (b_jb_kD_i + b_ib_kD_j + b_ib_jD_k)/(b_jb_k + b_ib_k + b_ib_j)$. It can be verified that this operator has the requisite interpolatory properties on all three edges. Unlike the minimum norm element, this triangular interpolant has the ability to propagate the effects of tension to the interior of a triangle in a desirable manner. This behavior is shown in Figure 6.

We have applied this method to a variety of examples where tension might be beneficial and have found the effects of tension parameters to be predictable and desirable. An example is shown in Figure 7 where we have used the same data as in Figure 1. Our final example in Figure 8 illustrates the effects of increasing the tension parameters. The data is the same as that of Figure 2.

4. Remarks and conclusions. We have developed a scheme for the construction of a surface under tension. The effects of increasing tension values are as one would expect, and are subject to the underlying triangulation.

Much still remains to be done. While the tension parameters appear to be very useful in controlling the behavior of the surface in the presence of steep gradients, little is known about how tension parameters should be chosen. For the examples included here, this selection was made on an interactive, trial and error basis. As a practical matter we have specified tension values α_i, at the vertices V_i, since the edges are unknown, a priori. We then set $\alpha_{ij} = (\alpha_i + \alpha_j)/2|e_{ij}|$ for normalized tension. It is probably desirable to set tension values in the vicinity of steep gradients to different values than in relatively flat regions. In some situations, it would be desirable to have tension values set automatically so as to achieve certain effects. The ideas and techniques put forward by Lynch **[11]** for achieving monotonicity and convexity for univariate splines under tension are potentially useful in this context. These matters will be dealt with elsewhere.

In the case of data lying on rectangular grid lines (not necessarily at grid points), Salkauskas **[18]** has been concerned with eliminating the ringing phenomenon usually present in bicubic splines applied to rapidly varying data. He utilizes the tensor product of a C^1 cubic spline which minimizes $\int w(x)[f''(x)]^2dx$. The weight, w is chosen so as to dampen the effects of ringing.

We also wish to mention some related work on the incorporation of fault lines and creases into scattered data interpolation. In [7], we describe techniques for imposing C^0 and C^1 discontinuities on the surface of interpolation. These methods require specific information concerning the fault line (or crease) and its location and in this context work well. While creases are certainly achievable by the present method, they would have to be confined to locations where data is provided and some intervention into the triangulation step must take place in order to ensure that the proper edges are included in the triangulation. Achieving fault lines with the present method requires more substantial modifications.

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