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ON THE CHARACTER THEORY OF FULLY RAMIFIED SECTIONS

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1. Introduction. As corollaries of some lengthy and relatively difficult arguments one can derive several character theoretic facts which are quite often useful in the study of solvable groups. The purpose of this paper is to present a method which yields simple and direct proofs for some of these. As might be expected, we do not obtain the strongest possible results this way, but what we shall prove here is sufficient for many applications, some of which are discussed in the final section of the paper.

A configuration which often arises, and which will be our primary object of attention, is the following.

BASIC CONFIGURATION 1.1. *Let* $L \subseteqq K \lhd G$ *with* $L \lhd G$ *and* K/L *abelian. Let* $\varphi \in \mathrm{Irr}(L)$ *be invariant in* G *and assume* $\varphi^K = e\theta$ *for some* $\theta \in \mathrm{Irr}(K)$ *and integer* e.

Note that in this situation, $\theta_L = e\varphi$ and a computation of $\varphi^K(1)$ yields that $e^2 = |K: L|$. Also, θ is necessarily G-invariant. We occasionally discribe this situation by saying that K/L is a *fully ramified section in* G.

Frequently in these circumstances, we have in mind a particular subgroup H, such that $HK = G$ and $H \cap K = L$. (In other words, H is a complement for K in G relative to L.) We would like to obtain information which relates the irreducible characters of G which lie over φ with those of H which lie over φ. Generally, one must assume some additional hypotheses before being able to draw conclusions of the desired type.

The following is an example of a result of this kind. (We use the notation $\mathrm{Irr}(X|\varphi)$ to denote the set of irreducible constituents of φ^X where φ is an irreducible character of some subgroup of X.)

THEOREM 1.2. *Assume the Basic Configuration* (1.1) *and in addition suppose that at least one of* $|K: L|$ *or* $|G: K|$ *is odd. Then there exists* $U \subseteqq G$ *and a character* ψ *of* G *with* $K \subseteqq \ker \psi$ *such that*

a) $UK = G$ *and* $U \cap K = L$,

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b) *if* $\chi \in \operatorname{Irr}(G \mid \varphi)$, *then there exists a unique* $\xi \in \operatorname{Irr}(U \mid \varphi)$ *such that* $\chi_U = \psi_U \xi$,

c) *if* $\xi \in \operatorname{Irr}(U \mid \varphi)$, *then there exists a unique* $\chi \in \operatorname{Irr}(G \mid \varphi)$ *such that* $\xi^G = \bar{\psi}\chi$,

d) *the maps* $\chi \mapsto \xi$ *and* $\xi \mapsto \chi$ *defined by the previous two parts are inverse bijections between* $\operatorname{Irr}(G \mid \varphi)$ *and* $\operatorname{Irr}(U \mid \varphi)$, *and*

e) $|\psi(g)|^2 = |\mathbf{C}_{K/L}(g)|$ *for all* $g \in G$. *In particular,* $\psi(1) = e$.

This result, which includes parts of Theorem 9.1 and Corollary 9.2 of **[3]** does not say anything about the characters of an arbitrary complement H for K relative to L in G; it only says that for some complement, good conclusions hold. However, when Theorem 1.2 is applied, we often have available additional information, sufficient to guarantee that there is a unique conjugacy class of relative complements, and in this situation, it is no loss to assume that $H = U$.

A frequently occurring example of such additional information is given by the following lemma.

LEMMA 1.3. *Let* $L \subseteq K \lhd G$ *with* $L \lhd G$ *and* K/L *abelian, as in the Basic Configuration. Assume in addition*

(∗) *there exists* $M \lhd G$ *with* $K \subseteq M$ *and* $(|K/L|, |M/K|) = 1$ *such that* $\mathbf{C}_{K/L}(M) = 1$.

Then there exists a unique conjugacy class of complements for K *in* G *relative to* L.

The proof of this result, using the Schur-Zassenhaus Theorem in M/L and the Frattini argument, is quite routine and will not be given here.

A result parallel to Theorem 1.2 which avoids the oddness hypothesis of that theorem is this result of Dade.

THEOREM 1.4. *Assume the Basic Configuration and condition* (∗) *of Lemma* 1.3. *Then there exists* $U \subseteq G$ *such that*

a) $UK = G$, $U \cap K = L$, *and*

b) *there is a bijection* π: $\operatorname{Irr}(U|\varphi) \to \operatorname{Irr}(G|\varphi)$ *with* $\xi^\pi(1) = e\xi(1)$ *for* $\xi \in \operatorname{Irr}(U|\varphi)$.

This is essentially Theorem 5.10 of **[1]**. A simplified proof in the case where M/K is solvable can be found in **[5]**. Note that condition (*) is not being used in Theorem 1.4 merely to guarantee uniqueness (up to conjugacy) of complement; it is an essential hypothesis.

Observe that in situations where both Theorems 1.2 and 1.4 apply, the conclusions of Theorem 1.4 form a proper subset of those of Theorem 1.2. What is more, the character ψ of Theorem 1.2 can be constructed in an invariant way, so that the character bijections it determines are "canonical". In the situation of Theorem 1.4, there is no uniquely defined bijection given, and in general, none exists.

Since Theorem 1.4 is much easier to prove than Theorem 1.2, especially if M/K is solvable, and since its conclusions are strong enough for many purposes, it probably provides the quicker route to most applications. Nevertheless, there is a very useful corollary of Theorem 1.2 which does not even mention the character bijections and which is not available from the point of view of Theorem 1.4.

COROLLARY 1.5. *Assume the hypotheses of Theorem* 1.2 *and let U be the subgroup given by that theorem. Suppose* $|K/L| > 1$.

a) *If* $\xi \in \mathrm{Irr}(U|\varphi)$, *then* ξ^G *is reducible.*

b) *If* $\chi \in \mathrm{Irr}(G|\varphi)$, *then* χ_U *is reducible.*

Proof. If either (a) or (b) fails, then the character ψ given by Theorem 1.2 must be irreducible by (c) or (b) of that theorem, respectively. Thus $1 = [\psi, \psi] = [\psi\bar{\psi}, 1_G]$. However, by (1.2) (e), $\psi\bar{\psi}$ is the permutation character of G on K/L, acting by conjugation. Since $|K/L| > 1$, this action cannot be transitive (since the identity is fixed) and so $[\psi\bar{\psi}, 1_G] > 1$. This contradiction completes the proof.

It is the principal goal of this paper to give a self-contained and elementary proof of Corollary 1.5 (at least in the case where $|K\colon L|$ is odd) while avoiding the complexities which appear to be necessary to get the full strength of Theorem 1.2 (as in [3]). In addition, the present method of proof (which incorporates a few of the ideas in [3]) is sufficient to obtain the equality of the cardinalities of $\mathrm{Irr}(G|\varphi)$ and $\mathrm{Irr}(U|\varphi)$, though it does not construct an explicit correspondence and the author has been unable to obtain from this argument the fact that the degrees of the characters in the two sets are proportional. Our main result is the following theorem.

THEOREM 1.6. *Assume the Basic Configuration with* $2 \nmid |K\colon L|$. *Then there exists* $U \subseteq G$ *such that*

a) $UK = G$ *and* $U \cap K = L$,

b) $|\mathrm{Irr}(G|\varphi)| = |\mathrm{Irr}(U|\varphi)|$,

c) *if* $|K/L| > 1$ *and* $\xi \in \mathrm{Irr}(U|\varphi)$, *then* ξ^G *is reducible, and*

d) *if* $|K/L| > 1$ *and* $\chi \in \mathrm{Irr}(G|\varphi)$, *then* χ_U *is reducible.*

Before proceeding with the proof of Theorem 1.6, we mention that it definitely can fail if $|K\colon L|$ is even. An example is provided by $G = \mathrm{GL}(2, 3)$ with $K = Q_8$ and $L = \mathbf{Z}(K)$. In this case, condition (*) of (1.3) holds, and so there is a unique conjugacy class of relative complements and (b) holds by Theorem 1.4. Both (c) and (d) fail.

2. Character triple isomorphism. The first step in the proof of Theorem

1.6 is to use the theory of projective representations in order to reduce to the situation where $L \subseteq \mathbf{Z}(G)$. This can be done without explicit reference to projective representations by using the facts about character triple isomorphism which are presented in Chapter 11 of [**4**]. We summarize this information briefly here.

We say that (G, L, φ) is a *character triple* if $L \lhd G$ and $\varphi \in \mathrm{Irr}(L)$ is invariant in G. Two triples, (G, L, φ) and (Γ, N, θ) are *isomorphic* provided that $G/L \cong \Gamma/N$ and that the character theory of G over φ corresponds in a certain precise way to the character theory of Γ over θ. In particular, if H/L and K/N are corresponding subgroups under the given isomorphism between G/L and Γ/N, then there exists a fixed bijection $\sigma_H\colon \mathrm{Irr}(H|\varphi) \to \mathrm{Irr}(K|\theta)$, and these bijections respect restriction in that if $L \subseteq V \subseteq U \subseteq G$, $\xi \in \mathrm{Irr}(U|\varphi)$ and $\eta \in \mathrm{Irr}(V|\varphi)$, then

$$[\xi_V, \eta] = [\sigma_U(\xi)_W, \sigma_V(\eta)]$$

where W/N corresponds to V/L. (See Definition 11.23 of [**4**] for more details.)

It follows in the above situation that if $\xi_V = \eta$, then $\sigma_U(\xi)_W = \sigma_V(\eta)$ and if, on the other hand, $\eta^U = \xi$, then $\sigma_V(\eta)^X = \sigma^U(\xi)$, where X/N corresponds to U/L.

The key result here is that every character triple (G, L, φ) is isomorphic to one, (Γ, Z, λ) where $Z \subseteq \mathbf{Z}(\Gamma)$ and λ is faithful. (See Theorem 11.28 and Lemma 11.26 of [**4**].) It follows that in proving Theorem 1.6, it is no loss to assume that $L \subseteq \mathbf{Z}(G)$ and L is cyclic.

Observe that in the Basic Configuration, we automatically have $\mathbf{Z}(K) \subseteq L$ since $\theta = (1/e)\varphi^K$ vanishes on $K - L$. We may thus assume in the Basic Configuration that $L = \mathbf{Z}(K) \subseteq \mathbf{Z}(G)$.

3. A group theoretic result. Most of the work in the proof of Theorem 1.6 is contained in the following non-character theoretic result.

THEOREM 3.1. *Let $K \lhd G$ and assume that $L = \mathbf{Z}(K)$ is cyclic and that $L \subseteq \mathbf{Z}(G)$. Suppose K/L is Abelian of odd order. Then there exists $U \subseteq G$ such that*

a) *$UK = G$ and $U \cap K = L$,*
b) *$U \supseteq \mathbf{C}_G(K)$,*
c) *if $u \in U$ and $u^k \in U$ for $k \in K$, then $u^k = u$, and*
d) *if $u, v \in V \subseteq U$ are conjugate in VK, then they are conjugate in V.*

We begin with some lemmas.

LEMMA 3.2. *Let $C = \mathbf{Z}(K)$ where C is cyclic and K/C is abelian. Then every automorphism of K which is trivial on C and on K/C is inner.*

PROOF. Let $\mathscr{A} = \{\sigma \in \mathrm{Aut}(K) \mid [K, \sigma] \subseteq C$ and $[C, \sigma] = 1\}$ and note that $\mathrm{Inn}(K) \subseteqq \mathscr{A}$. Since $|\mathrm{Inn}(K)| = |K/C|$, it suffices to show that $|\mathscr{A}| \leqq |K/C|$.

For each $\sigma \in \mathscr{A}$, there is a well defined map θ_σ: $K/C \to C$ defined by $(Ck)\theta_\sigma = [k, \sigma]$, and we have $\theta_\sigma \in \mathrm{Hom}(K/C, C)$. The map $\mathscr{A} \to \mathrm{Hom}(K/C, C)$ defined by $\sigma \mapsto \theta_\sigma$ is injective since if $\theta_\sigma = \theta_\tau$, then $[k, \sigma] = [k, \tau]$ for all $k \in K$. It follows that $k^\sigma = k^\tau$ for all k and so $\sigma = \tau$. Thus $|\mathscr{A}| \leqq |\mathrm{Hom}(K/C, C)|$.

Since C is cyclic, it follows easily from the fundamental theorem of abelian groups applied to K/C, that $|\mathrm{Hom}(K/C, C)| \leqq |K/C|$. The result follows.

LEMMA 3.3. *Let $C \subseteqq \mathbf{Z}(K)$ with K/C abelian. Assume that every coset of C in K contains an element of order equal to the order of the coset, viewed as an element of K/C. Then there exists $\sigma \in \mathrm{Aut}(K)$, trivial on C and inverting all elements of K/C.*

PROOF. Write K/C as a direct product $\langle Cu_1\rangle \times \cdots \times \langle Cu_r\rangle$ of cyclic subgroups and choose the representatives u_i of the generating cosets so that $o(u_i) = o(Cu_i)$. Then every $k \in K$ is uniquely of the form $k = x_1x_2 \cdots x_rz$ with $x_i \in \langle u_i\rangle$ and $z \in C$. Define σ: $K \to K$ by

$$k^\sigma = x_1^{-1}x_2^{-1} \cdots x_r^{-1}z.$$

It is clear that σ is a permutation of the elements of K and the proof will be complete when we show that σ is a homomorphism. To this end, let $l \in K$ and write $l = y_1y_2 \cdots y_rt$ with $y_i \in \langle u_i\rangle$ and $t \in C$. Then

$$kl = (x_1y_1)(x_2y_2) \cdots (x_ry_r)ztc$$

where

$$c \prod_{i>j} [x_i, y_j] \in K' \subseteqq C.$$

We have

$$k^\sigma l^\sigma = (x_1^{-1}y_1^{-1}) \cdots (x_r^{-1}y_r^{-1})ztc'$$

where

$$c' = \prod_{i>j} [x_i^{-1}, y_j^{-1}].$$

Since $[x^{-1}, y^{-1}] = [x, y]$ for $x, y \in K$, we have $c' = c$ and so $k^\sigma l^\sigma = (kl)^\sigma$ as desired.

PROOF OF THEOREM 3.1. Let $m = \exp(K)$ and let C be a cyclic group of order m. Let $C_0 \subseteqq C$ be the subgroup of order equal to $|L|$ and let K^* be the central product of K and C with L and C_0 identified. Note that $C = \mathbf{Z}(K^*)$.

If $k \in K^*$, let r be the order of the coset Ck in K^*/C. Then $k^r \in C$ and $o(k) = ro(k^r)$. Since $o(k)$ divides $m = |C|$, we see that $o(k^r)$ divides m/r and so there exists $c \in C$ with $k^r = c^r$. Thus $(kc^{-1})^r = 1$ and the coset Ck contains an element of order r. By Lemma 3.3, there exists $\sigma \in \mathrm{Aut}(K^*)$ such that σ inverts K^*/C and is trivial on C.

Every automorphism of K which is trivial on L induces a uniquely defined automorphism of $K^* = KC$ which is trivial on C. In particular, since $L \subseteq \mathbf{Z}(G)$, the conjugation action of G on K induces an action of G on K^*. We therefore have a homomorphism $\theta\colon G \to A$ where $A = \{\alpha \in \mathrm{Aut}(K^*) \mid [C, \alpha] = 1\}$. Note that $\ker\theta = \mathbf{C}_G(K)$.

Now $\sigma \in A$ and we let U be the complete inverse image in G of $\mathbf{C}_A(\sigma)$. Thus $\mathbf{C}_G(K) = \ker\theta \subseteq U$ and (b) is immediate.

Let $I = \mathrm{Inn}(K^*) \lhd A$ and observe that $I = \theta(K)$. Since σ inverts K^*/C, it follows that σ inverts I by conjugation. However, $|I| = |K^*/C| = |K/L|$ is odd, and thus $I \cap \mathbf{C}_A(\sigma) = 1$. It follows that $K \cap U \subseteq \ker\theta \cap K = \mathbf{Z}(K) = L$, as desired.

If $\sigma' \in A$ is conjugate to σ, it is easy to check that σ' also inverts L^*/C and thus $\sigma\sigma'$ acts trivially on K/C. By Lemma 3.2, we see that $\sigma\sigma' \in I$ and we conclude that $I\langle\sigma\rangle \lhd A$. Since $|I|$ is odd, the Frattini argument yields that $I\mathbf{C}_A(\sigma) = A$ and since $I = \theta(K) \subseteq \theta(G)$, we have $\theta(G) = I(\theta(G) \cap \mathbf{C}_A(\sigma)) = \theta(KU)$. Since $\ker\theta \subseteq U$, we have $G = KU$ and (a) is proved.

Now suppose $u \in U$ and $k \in K$ with $u^k \in U$. Then $[u, k] \in U \cap K = L$ and so, working in the holomorph of K^*, we have $[k, \alpha] \in C$, where $\alpha = \theta(u) \in \mathrm{Aut}(K^*)$. Thus $[\langle k\rangle, \langle\alpha\rangle, \langle\sigma\rangle] = 1$. Also, since $u \in U$, we have $\alpha \in \mathbf{C}_A(\sigma)$ and so $[\langle\alpha\rangle, \langle\sigma\rangle, \langle k\rangle] = 1$. By the three subgroups lemma, we conclude that $[\langle\sigma\rangle, \langle k\rangle, \langle\alpha\rangle] = 1$ and so α acts trivially on $[\langle\sigma\rangle, \langle k\rangle]C$. This group, however, contains k^2 since $k^\sigma \in k^{-1}C$, and thus it contains k, because $|K^*/C|$ is odd. Thus $k^\alpha = k$ and so u centralizes k. Part (c) now follows.

Finally, suppose $u^g = v$ for some $u, v \in V \subseteq U$ and $g \in VK$. Write $g = wk$ with $w \in V$ and $k \in K$, and observe that $u^w \in U$ and $(u^w)^k = v \in U$. By (c), $v = (u^w)^k = u^w$ and (d) is proved.

4. A little character theory. In order to prove Theorem 1.6, we need some sufficient conditions for induced and restricted characters to be reducible.

LEMMA 4.1. *Let $H \subseteq G$ and let χ be any class function of G. Then $(\chi_H)^G = (1_H)^G\chi$.*

PROOF. Compute, using the definition of induced class functions.

LEMMA 4.2. *Let $H < G$ and suppose $\chi \in \mathrm{Irr}(G)$ vanishes on all $g \in G$ not conjugate to elements of H. Then χ_H is reducible.*

PROOF. (D. Gluck). We have $[\chi_H, \chi_H] = [\chi, (\chi_H)^G] = [\chi, (1_H)^G\chi]$. Since $(1_H)^G(g) \geqq 1$ whenever $\chi(g) \neq 0$ (by hypothesis) and $(1_H)^G(1) > 1$, we have $[\chi_H, \chi_H] > [\chi, \chi] = 1$.

LEMMA 4.3. *Let $H < G$ and suppose $\xi \in \operatorname{Irr}(H)$ has the property that $\xi(h) = \xi(k)$ whenever $h, k \in H$ are conjugate in G. Then ξ^G is reducible.*

PROOF. Define the function α on G by setting $\alpha(g) = \xi(h)$ if g is conjugate to $h \in H$ and $\alpha(g) = 0$ if g is not conjugate to any element of H. Then α is a well defined class function on G and $\alpha_H = \xi$. We have

$$[\xi^G, \xi^G] = [\xi^G, (\alpha_H)^G] = [\xi^G, (1_H)^G\alpha] = [\xi, ((1_H)^G)_H\xi].$$

However, $((1_H)^G)_H(h) \geqq 1$ for all $h \in H$ and $(1_H)^G)_H(1) > 1$. Thus $[\xi^G, \xi^G] > [\xi, \xi] = 1$.

Note that a consequence of Lemma 4.2 is that if $\chi \in \operatorname{Irr}(G)$ is induced from any class function of $H < G$, then χ_H is reducible. Similarly, by (4.3) if $\xi \in \operatorname{Irr}(H)$ is the restriction of any class function of G, then ξ^G is reducible.

5. The proof.

PROOF OF THEOREM 1.6. By the remarks of §2, we may assume that $L = \mathbf{Z}(K) \subseteq \mathbf{Z}(G)$ and L is cyclic. Since we are assuming that $|K/L|$ is odd, Theorem 3.1 applies and provides us with a certain subgroup $U \subseteq G$. Part (a) of Theorem 1.6 is now immediate. Part (c) follows from Lemma 4.3 since if $u, v \in U$ are conjugate in G, then they are conjugate in U by (3.1) (d), and thus $\xi(u) = \xi(v)$ for all $\xi \in \operatorname{Irr}(U|\varphi)$.

Now, for each subgroup X, with $K \subseteq X \subseteq G$, let $V(X)$ be the complex vector space of class functions of X spanned by $\operatorname{Irr}(X|\varphi)$, and similarly, if $L \subseteq Y \subseteq U$, let $W(Y)$ be the span of $\operatorname{Irr}(Y|\varphi)$. Suppose $Y = X \cap U$. Since $\chi_Y \in W(Y)$ for all $\chi \in \operatorname{Irr}(X|\varphi)$, restriction clearly maps $V(X)$ into $W(Y)$. Similarly, if $\xi \in \operatorname{Irr}(Y|\varphi)$, then $(\xi^X)_K = (\xi_L)^K = a\varphi^K$ for some integer a. Thus $(\xi^X)_L = a(\varphi^K)_L = a|K: L|\varphi$. Thus $\xi^X \in V(X)$ and so induction maps $W(Y)$ into $V(X)$. Obviously, the restriction and induction maps between $V(X)$ and $W(Y)$ are linear transformations.

Suppose we knew that induction defined a vector space isomorphism from $W(U)$ onto $V(G)$. Then $|\operatorname{Irr}(G|\varphi)| = \dim V(G) = \dim W(U) = |\operatorname{Irr}(U|\varphi)|$ and (b) would follow. Also, if $\chi \in \operatorname{Irr}(G|\varphi)$, then $\chi \in V(G)$, and so χ is induced from some class function of U. Thus χ vanishes on elements of G which are not conjugate to elements of U, and (d) follows by Lemma 4.2.

We work now toward proving that induction does define an isomorphism of $W(U)$ onto $V(G)$. First we argue that for every X with $K \subseteq X \subseteq G$, restriction maps $V(X)$ onto $W(Y)$ where $Y = X \cap U$. To see this, let

$\beta \in W(Y)$ and define α on X by $\alpha(x) = 0$, if x is not conjugate in X to an element of Y, and $\alpha(x) = \beta(y)$, if x is conjugate in X to $y \in Y$. Note that α is a well defined class function of X by (3.1) (d). Write $\alpha = \alpha_1 + \alpha_2$ where $\alpha_1 \in V(X)$ and α_2 is a linear combination of $\psi \in \mathrm{Irr}(X)$ with $\psi \notin \mathrm{Irr}(X|\varphi)$. Then $\beta = \alpha_Y = (\alpha_1)_Y + (\alpha_2)_Y$ and so

$$(\alpha_2)_Y = \beta - (\alpha_1)_Y \in W(Y).$$

However, by the definition of α_2 we see that $(\alpha_2)_Y$ is a linear combination of $\eta \notin \mathrm{Irr}(Y|\varphi)$. We conclude that $(\alpha_2)_Y = 0$ and thus $\beta = (\alpha_1)_Y$. It follows that restriction maps $V(X)$ onto $W(Y)$, as claimed.

With X and Y as above, we show next that the induction map from $W(Y)$ to $V(X)$ has trivial kernel. Let $\beta \in W(Y)$ and suppose $\beta^X = 0$. Then $0 = [\beta^X, \psi] = [\beta, \psi_Y]$ for all $\psi \in \mathrm{Irr}(X|\varphi)$. Since these characters ψ span $V(X)$ which maps by restriction onto $W(Y)$, it follows that $[\beta, \gamma] = 0$ for all $\gamma \in W(Y)$. In particular, $[\beta, \beta] = 0$ and so $\beta = 0$.

To complete the proof of the theorem, it now suffices to show that $\dim V(G) \leqq \dim W(U)$, and to this end, we show that the restriction map $V(G) \to W(U)$ has trivial kernel. Suppose then $\alpha \in V(G)$ and $\alpha_U = 0$. Let $g \in G$. We must show that $\alpha(g) = 0$.

Let $X = \langle K, g \rangle$ so that X/K is cyclic. Then θ extends to X (see, for instance, Corollary 11.22 of **[4]**) and there is a bijective correspondence between $\mathrm{Irr}(X|\theta)$ and $\mathrm{Irr}(X/K)$. (This result of Gallagher is Corollary 6.17 or **[4]**.) Since $\mathrm{Irr}(X|\theta) = \mathrm{Irr}(X|\varphi)$ and $|\mathrm{Irr}(X/K)| = |X/K|$, we conclude that $\dim V(X) = |X/K|$. Similarly, if $Y = X \cap U$, then Y/L is cyclic, φ extends to L and $\dim W(Y) = |Y/L| = |X/K| = \dim V(X)$.

Since we already know that the restriction map $V(X) \to W(Y)$ is surjective, it follows that it has trivial kernel. However, $(\alpha_X)_Y = (\alpha_U)_Y = 0$ and thus $\alpha_X = 0$ since $\alpha_X \in V(X)$. In particular, $\alpha(g) = 0$, and the proof is complete.

6. Applications. A number of results in the literature have been proved by appeals to **[1]**, **[3]** or **[5]**. For many of these, Theorem 1.6 is sufficient, and for some others, Theorem 1.6 provides a quick route to a similar, but somewhat less precise result. Consider, for example, the following theorem.

THEOREM 6.1. *Let G be solvable of odd order and let A act on G with $(|A|, |G|) = 1$. Let $C = \mathbf{C}_G(A)$. Then the number of A-fixed irreducible characters of G is equal to the total number of irreducible characters of C.*

In **[3]**, this result was proved as an example of an application of the techniques of that paper. In fact a specific and uniquely defined bijection was constructed between the two sets of characters. In **[5]**, this result was reproved, without constructing a specific bijection, but without the necessity of assuming that $|G|$ is odd. This latter proof works just as well to derive Theorem 6.1 from part (b) of Theorem 1.6.

Another application occurs as Theorem 3.1 of [**6**].

THEOREM 6.2. *Let G be p-solvable with $p \neq 2$ and let H be a maximal subgroup of index divisible by p. Suppose $\chi \in \mathrm{Irr}(G)$ is induced from a character of H. Let $N \lhd G$ and write $L = \mathrm{core}_G(H \cap N)$. If χ_L is homogeneous, then $H \cap N \lhd G$.*

In [**6**] this result is proved by appeal to [**3**]. All that is actually used, however is Corollary (1.5) (a) and so Theorem 1.6 part (c) provides a far more elementary proof.

The two remaining applications which we shall discuss both concern the following situation.

HYPOTHESIS 6.3. Assume the Basic Configuration with $|K: L|$ odd. Let $R \subseteq G$ have the following properties:

(i) $(|R|, |K: L|) = 1$,

(ii) $\mathbf{C}_{K/L}(R) = 1$, and

(iii) $RK \lhd G$.

Assume $H \subseteq G$ with $H \supseteq RL$.

The following is essentially Lemma 4.4 of Dade's paper [**2**]. A version of it was also found by T. R. Berger.

THEOREM 6.4. *Assume Hypothesis 6.3. Let $\xi \in \mathrm{Irr}(H|\varphi)$ and suppose $\xi^G \in \mathrm{Irr}(G)$. Let $X = K \cap H$ and assume ξ_X is a multiple of $\alpha \in \mathrm{Irr}(X)$. Then $\alpha^K = \theta$ and $|K: X| \leq |X: L|$.*

PROOF. We may assume that $KH = G$ and thus $X \lhd G$. Let $T = I_G(\alpha)$, the inertia group, so that $T \supseteq H$. Write $Y = T \cap K = I_K(\alpha)$ and note that $Y \lhd G$, $Y \cap H = X$ and $YH = T$.

Since $Y = I_K(\alpha)$, induction defines a bijection (for instance, by Theorem 6.11 of [**4**]) from $\mathrm{Irr}(Y|\alpha)$ onto $\mathrm{Irr}(K|\alpha) \subseteq \mathrm{Irr}(K|\varphi) = \{\theta\}$. It follows that $\mathrm{Irr}(Y|\alpha)$ is also a singleton set, say $\mathrm{Irr}(Y|\alpha) = \{\beta\}$. Then $\beta^K = \theta$ and $\alpha^Y = a\beta$ for some integer a. We therefore have the Basic Configuration reproduced, with $(T, Y, X, \beta, \alpha, a)$ in place of $(G, K, L, \theta, \varphi, e)$.

Let $M = RY$. Then $M = T \cap RK \lhd T$ since $RK \lhd G$ and the hypotheses of Lemma 1.3 are satisfied, and H is a representative of the unique conjugacy class of complements for Y in T relative to X.

Since $\xi \in \mathrm{Irr}(H|\alpha)$ and ξ^T is irreducible, we conclude by Theorem 1.6, part (c), that $Y = X$. Thus $\beta = \alpha$ and $\alpha^K = \theta$ as claimed. Also,

$$e\varphi(1) = \theta(1) = |K: X|\,\alpha(1) \geq |K: X|\varphi(1)$$

so that $|K: X| \leq e$. Since $|K: L| = e^2$, it follows that $|K: X| \leq e \leq |X: L|$.

Next, we present our final application of Theorem 1.6.

THEOREM 6.5. *Assume Hypothesis* 6.3. *Let* $\chi \in \mathrm{Irr}(G|\varphi)$ *and suppose* $\chi_H \in \mathrm{Irr}(H)$. *Then* $H \supseteqq K$.

PROOF. Again we may assume that $HK = G$ and we write $X = H \cap K \lhd G$. Since $\mathbf{C}_{K/X}(R) = 1$ and $(|R|, |K/X|) = 1$, it follows that θ_X has a unique R-invariant irreducible constituent α. (See Corollary 2.4 of **[5]** or Theorem 13.27 and Corollary 13.9 of **[4]**.) Since $RX \lhd H$, the uniqueness of α guarantees that α is invariant in H.

Now $\chi \in \mathrm{Irr}(G|\theta)$ and so α is a constituent of χ_X. Because χ_H is irreducible, we see that χ_X is a multiple of α and so α is invariant in G. Reasoning as in the proof of Theorem 6.4 (except that here we know that $Y = K$ and $T = G$), we see that the Basic Configuration is reproduced with X replacing L and α replacing φ. Also, H represents the unique conjugacy class of complements for K in G relative to X.

Since χ_H is irreducible, we have $|K/X| = 1$ by part (d) of Theorem 1.6 and the result follows.

REFERENCES

1. E.C. Dade, *Characters of groups with nomral extra-special subgroups*, Math. Zeit. **152** (1976), 1–31.

2. ———, *Monomial characters and normal subgroups*, Math. Zeit. **178** (1981), 401–420.

3. I.M. Isaacs, *Characters of solvable and symplectic groups*, Amer. J. Math. **95** (1973), 594–635.

4. ———, *Character theory of finite groups*, Academic Press, New York, 1976.

5. ———, *Character correspondences in solvable groups*, Advances in Math. **43** (1982), 284–306.

6. ———, *Primitive characters, normal subgroups and M-groups*, Math. Zeit. **177** (1981), 267–284.

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