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DUALITY FOR INFINITE HERMITE SPLINE INTERPOLATION

T.N.T. GOODMAN

1. Introduction. Let $x = (x_i)_{-\infty}^{\infty}$, $\xi = (\xi_i)_{-\infty}^{\infty}$ be non-decreasing sequences in $\mathbf{R}$ satisfying

$$(1) \qquad |\{i \mid x_i = t\}| + |\{i \mid \xi_i = t\}| \leqq n + 1,$$

where $|S|$ denotes the number of elements in a set S.

For a positive integer n, we denote by (n, x, ξ) the problem of interpolatting data at x by spline functions of degree n with knots at ξ. To make this precise we define for each integer i,

$$(2) \qquad \mu_i = |\{k < i \mid x_k = x_i\}|, \nu_i = |\{k < i \mid \xi_k = \xi_i\}|.$$

Then the space of spline functions of degree n with knots at ξ is defined to be

$$\begin{aligned}\zeta_n(\xi) := \{f\colon (\xi_{-\infty}, \xi_\infty) \to \mathbf{R} \mid & \text{ for any integer } i \text{ with} \\ & \xi_i < \xi_{i+1}, f \text{ coincides on } (\xi_i, \xi_{i+1}) \text{ with a} \\ & \text{polynomial of degree} \leqq n \text{ and } f^{(j)} \text{ is continuous} \\ & \text{at } \xi_i, 0 \leqq j \leqq n - \nu_i - 1\},\end{aligned}$$

where $\xi_{\pm\infty} = \lim_{i \to \pm\infty} \xi_i$.

We shall say (n, x, ξ) is *solvable* if for any bounded sequence $(y_i)_{-\infty}^{\infty}$ in $\mathbf{R}$ there is a unique bounded spline f in $\zeta_n(\xi)$ satisfying

$$(3) \qquad f^{(\mu_i)}(x_i) = y_i \ (i \in \mathbf{Z}).$$

For this to make sense we must have $x_i \in (\xi_{-\infty}, \xi_\infty)$ $(i \in \mathbf{Z})$.

We note that condition (1) ensures that we do not interpolate at a discontinuity. Defining

$$(4) \qquad \Delta x_i = \min\{x_j - x_i \mid x_j > x_i\},$$

we define the *global mesh ratio* of x as

$$(5) \qquad \sup \{\Delta x_i / \Delta x_j \mid i, j \in \mathbf{Z}\}.$$

A similar definition holds for ξ. We shall prove the following.

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THEOREM. *If the global mesh ratios of x and ξ are finite and if (n, x, ξ) is solvable, then (n, ξ, x) is solvable.*

We remark that this result is known if x and ξ are periodic **[4]**, and also for the corresponding problem when x and ξ are finite **[5]**. Indeed in both these cases the duality extends to more general Birkhoff spline interpolation. Finally we note that if n is odd and x is strictly increasing with finite global mesh ratio, then (n, x, x) is solvable **[2]**.

2. Proof of the theorem.

LEMMA 1. *For any interval I, let $x(I) = |\{i|x_i \in I\}|$ and $\xi(I) = |\{i|\xi_i \in I\}|$. Then if (n, x, ξ) is solvable, $x(I)$ is finite if and only if $\xi(I)$ is finite, and if they are finite then $|x(I) - \xi(I)| \leqq n + 1$.*

PROOF. Suppose (n, x, ξ) is solvable. Take I with $\xi(I)$ finite. Then for any bounded vector $\{y_i|x_i \in I\}$ there is a spline f in $\zeta_n(\xi)|I$ with $f^{(\mu_i)}(x_i) = y_i$ whenever $x_i \in I$. But $\dim \zeta_n(\xi)|I \leqq \xi(I) + n + 1$ and so $x(I)$ is finite with $x(I) \leqq \xi(I) + n + 1$.

Next take I with $x(I)$ finite. Let ζ denote the space of splines in $\zeta_n(\xi)$ which vanish outside I. If $\dim \zeta > x(I)$, there would be a non-trivial element f of ζ with $f^{(\mu_i)}(x_i) = 0$ for all x_i in I, and hence for all integers i. Since f is bounded this would contradict (n, x, ξ) being solvable. Thus $\dim \zeta \leqq x(I)$. But $\dim \zeta \geqq \xi(I) - n - 1$ and so $\xi(I)$ is finite with $\xi(I) \leqq x(I) + n + 1$.

We now introduce the 'normalised B-splines' defined by

$$(6) \qquad N(t|\xi_i, \ldots, \xi_{i+n+1}) := (\xi_{i+n+1} - \xi_i)[\xi_i, \ldots, \xi_{i+n+1}](\cdot - t)_+^n,$$

where as usual $[\xi_i, \ldots, \xi_{i+n+1}]f$ denotes the divided difference of f at these points. We shall denote $N(.|\xi_i, \ldots, \xi_{i+n+1})$ by N_i. It is well known that N_i is in $\zeta_n(\xi)$ and $N_i(t) \geqq 0$ for all t, with $N_i(t) > 0$ if and only if $\xi_i < t < \xi_{i+n+1}$. Moreover any spline f in $\zeta_n(\xi)$ can be expressed uniquely in the form

$$(7) \qquad f(t) = \sum_{-\infty}^{\infty} \beta_j N_j(t)$$

where the sum converges locally uniformly since locally it has only a finite number of non-zero terms, see **[3]**. Thus for any integer i,

$$(8) \qquad f^{(\mu_i)}(x_i) = y_i \Leftrightarrow \sum_{j=-\infty}^{\infty} N_j^{(\mu_i)}(x_i)\beta_j = y_i.$$

It is shown in **[1]** that there is a positive constant C_n, independent of ξ, such that for any $\beta = (\beta_i)_{-\infty}^{\infty} \in \ell_\infty$,

$$(9) \qquad C_n\|\beta\|_\infty \leqq \|\sum_{-\infty}^{\infty} \beta_i N_i\|_\infty \leqq \|\beta\|_\infty.$$

Thus f in $\zeta_n(\xi)$ is bounded if and only if the sequence β of its B-spline coefficients is bounded and so by (8), (n, x, ξ) is solvable if and only if the matrix

$$N := (N_{ij})_{i,j=-\infty}^{\infty}, \quad N_{ij} := N_j^{(\mu_i)}(x_i), \tag{10}$$

represents a bijective map on ℓ_∞.

LEMMA 2. *If (n, x, ξ) is solvable, then there is an integer m such that for any i, j, $N_{ij} \neq 0$ only when $m - n \leqq i - j \leqq m + n$, i.e., all the non-zero elements of N are contained within $2n + 1$ consecutive diagonals.*

PROOF. Take any i, j, k, ℓ with $i - j \leqq k - \ell$ and $N_{ij} \neq 0 \neq N_{k\ell}$. Then $\xi_j < x_i < \xi_{j+n+1}$, $\xi_\ell < x_k < \xi_{\ell+n+1}$. First suppose $\xi_j < \xi_{\ell+n+1}$. Then applying Lemma 1 with $I = (\xi_j, \xi_{\ell+n+1})$ gives $k - i + 1 \leqq \ell + n - j + n + 1$ and so $k - \ell \leqq i - j + 2n$. Next suppose $\xi_j \geqq \xi_{\ell+n+1}$. Then Lemma 1 with $I = [\xi_{\ell+n+1}, \xi_j]$ gives $i - k - 1 \geqq j - \ell - n - (n + 1)$ and so again $k - \ell \leqq i - j + 2n$. Thus in all cases $0 \leqq (k - \ell) - (i - j) \leqq 2n$ and the result follows.

LEMMA 3. *For any f in $\zeta_n(x)$, let*

$$\gamma_j = ((-1)^{\mu_j}/n!)\{f^{(n-\mu_j)}(x_j^+) - f^{(n-\mu_j)}(x_j^-)\} \quad (j \in \mathbf{Z}). \tag{11}$$

Then for any integer i,

$$\sum_{j=-\infty}^{\infty} N^{(\mu_j)}(x_j|\xi_i, \ldots, \xi_{i+n+1})\,\gamma_j = (\xi_{i+n+1} - \xi_i)[\xi_i, \ldots, \xi_{i+n+1}]f. \tag{12}$$

PROOF. Take f in $\zeta_n(x)$, i in $\mathbf{Z}$, and choose any k, ℓ with $x_k \leqq \xi_i$, $\xi_{i+n+1} \leqq x_\ell$. Then for some polynomial p of degree $\leqq n$,

$$f(t) = p(t) + \sum_{j=k+1}^{\ell-1} \frac{n!}{(n-\mu_j)!}(-1)^{\mu_j}\gamma_j(t - x_j)_+^{n-\mu_j}, \quad x_k \leqq t < x_\ell.$$

Thus, recalling (6),

$$\begin{aligned}
&(\xi_{i+n+1} - \xi_i)[\xi_i, \ldots, \xi_{i+n+1}]f \\
&\quad = \sum_{j=k+1}^{\ell-1} \frac{n!}{(n-\mu_j)!}(-1)^{\mu_j}\gamma_j(\xi_{i+n+1} - \xi_i) \times [\xi_i, \ldots, \xi_{i+n+1}](\cdot - x_j)_+^{n-\mu_j} \\
&\quad = \sum_{j=-\infty}^{\infty} \gamma_j N^{(\mu_j)}(x_j \,|\, \xi_i, \ldots, \xi_{i+n+1}).
\end{aligned}$$

LEMMA 4. *Take points $t_0 \leqq t_1 \leqq \cdots \leqq t_{n+1}$ with $t_0 < t_{n+1}$. Suppose the distinct elements of $\{t_0, \ldots, t_{n+1}\}$ are $z_1, \ldots, z_m$ with multiplicities $\alpha_1, \ldots, \alpha_m$ respectively, and write*

$$[t_0, \ldots, t_{n+1}]f = \sum_{i=1}^{m}\sum_{j=0}^{\alpha_i - 1} \lambda_{ij} f^{(j)}(z_i). \tag{13}$$

Then

(14) $$|\lambda_{ij}| \leqq \binom{n-j}{n-\alpha_i+1} \Big/ j!M_i^{n+1-j},$$

where $M_i = \min \{|z_i - z_k| \, | k = 1, \ldots, m, k \neq i\}$.

PROOF. We first show that

(15) $$\lambda_{ij} = \phi_i^{(\alpha_i-1-j)}(z_i)/j!(\alpha_i - 1 - j)!,$$

where $\phi_i(t) := \prod_{k\neq i}(t - z_k)^{-\alpha_k}$.

It is easily verified that for any sufficiently smooth function f, the poly nomial

(16) $$p(t) = \sum_{i=1}^{m} \frac{1}{\phi_i(t)} \sum_{j=0}^{\alpha_i-1} \frac{(t-z_i)^j}{j!} \left[\frac{d^j}{dt^j}(f(t)\phi_i(t))\right]_{t=z_i}$$

satisfies

(17) $$p^{(j)}(z_i) = f^{(j)}(z_i), \; j = 0, \ldots, \alpha_i - 1, i = 1, \ldots, m.$$

For $\nu = 0, \ldots, n + 1$, we put $f(t) = p_\nu(t) := t^\nu$ in (16). Then (17) tells us $p(t) \equiv p_\nu(t)$ and equating powers of t^{n+1} gives:

$$\delta_{\nu,n+1} = \sum_{i=1}^{m} \frac{1}{(\alpha_i-1)!}\left[\frac{d^{\alpha_i-1}}{dt^{\alpha_i-1}}(p_\nu(t)\phi_i(t))\right]_{t=z_i}$$

$$= \sum_{i=1}^{m}\sum_{j=0}^{\alpha_i-1} \frac{\phi_i^{(\alpha_i-1-j)}(z_i)}{j!(\alpha_i-1-j)!} p_\nu^{(j)}(z_i).$$

Comparing with (13) then gives (15).

Now $\phi_i'(t) = - \phi_i(t) \sum_{j\neq i} \alpha_j(t - z_j)^{-1}$, and so

$$\phi_i''(t) = \phi_i(t) \sum_{j\neq i} \alpha_j(t - z_j)^{-1} \sum_{k\neq i} (\alpha_k + \delta_{kj})(t - z_k)^{-1}.$$

Repeating this procedure we see that for $\nu = 0, 1, 2, \ldots,$

$$|\phi_i^{(\nu)}(t)| \leqq |\phi_i(t)| \frac{(n + \nu + 1 - \alpha_i)!}{(n + 1 - \alpha_i)!}\{\min_{k\neq i}|t - z_k|\}^{-\nu}.$$

Substituting into (15) gives (14).

PROOF OF THE THEOREM. We assume the global mesh ratios of x and ξ are finite and (n, x, ξ) is solvable. Without loss of generality we can number the indices of x and ξ so that, from Lemma 2, $N_{ij} \neq 0$ only when $|i - j| \leqq n$. We have seen that the matrix N represents a bijection on ℓ_∞, which we denote by A. Since the global mesh ratio of ξ is finite, we see from (6) and Lemma 4 that N_{ij} is uniformly bounded, and hence A is a bounded map. The Open Mapping Theorem then tells us that A^{-1} is also bounded. Now it is easily seen that N^T, the transpose of N, represents a bounded map B on ℓ_1 whose adjoint is A. But it can be shown that if a bounded, linear map on a Banach space has a boundedly invertible adjoint, then it must also be boundedly invertible. Hence B is boundedly invertible.

For any f in $\zeta_n(x)$ we define $\gamma(f) = (\gamma_j)_{-\infty}^{\infty}$ by (11), and $\eta(f) = (\eta_j)_{-\infty}^{\infty}$ by

$$\eta_j = (\xi_{j+n+1} - \xi_j)[\xi_j, \ldots, \xi_{j+n+1}]f. \tag{18}$$

Then Lemma 3 tells us

$$N^T\gamma(f) = \eta(f) \tag{19}$$

We shall first prove uniqueness for the problem (n, ξ, x); that is we take any bounded element f of $\zeta_n(x)$ satisfying $f^{(\nu_i)}(\xi_i) = 0$, $i \in \mathbf{Z}$, and we shall show $f \equiv 0$. Now $N^T\gamma(f) = \eta(f) = 0$. Since N^T represents a bounded and boundedly invertible map on ℓ_1, we can apply Theorem 3 of [2] to show that $\gamma(f)$ is either zero or increases exponentially in at least one direction. More precisely, if $\gamma(f) \neq 0$, then for some index μ and positive constants K, Λ, with $\Lambda > 1$, we have either for all $i > \mu$ or else for all $i < \mu$:

$$\sum_{2ni < j \leqq 2n(i+1)} |\gamma_j| \geqq K\,\Lambda^{|i-\mu|}.$$

For any integer i we write $\tilde{N}_i(t) = N(t\,|\,x_i, \ldots, x_{i+n+1})$. Now for integers, i, j with $x_i \leqq x_j \leqq x_{i+n+1}$, we see from Lemma 4 that, since the global mesh ratio of x is finite, there is a constant K_1, independent of i and j, such that

$$|\tilde{N}_i^{(n-\mu_j)}(x_j^+) - \tilde{N}_i^{(n-\mu_j)}(x_j^-)| \leqq K_1. \tag{20}$$

Letting $f = \sum_{-\infty}^{\infty} \beta_i \tilde{N}_i$, we then have for any integer j,

$$\begin{aligned} |\gamma_j| &= \left| \frac{1}{n!} \sum_{i=j-2n-1}^{j+n} \beta_i \{\tilde{N}_i^{(n-\mu_j)}(x_j^+) - \tilde{N}_i^{(n-\mu_j)}(x_j^-)\} \right| \\ &\leqq \frac{K_1}{n!} \sum_{i=j-2n-1}^{j+n} |\beta_i|\,. \end{aligned}$$

Since f is bounded, β_i is uniformly bounded and so $\gamma(f)$ cannot increase exponentially in either direction. Hence $\gamma(f) = 0$. So f is a polynomial which vanishes infinitely often and so $f \equiv 0$.

We shall next construct the fundamental functions for the problem (n, ξ, x). Take any integer k and let $\eta = \eta(g_k)$, where g_k denotes any function satisfying $g_k^{(\nu_i)}(\xi_i) = \delta_{ik}$, $i \in \mathbf{Z}$. Choose L_k in $\zeta_n(x)$ with $\gamma(L_k) = B^{-1}\eta$. By altering L_k by a polynomial of degree $\leqq n$ we may assume $L_k^{(\nu_i)}(\xi_i) = g_k^{(\nu_i)}(\xi_i)$, $i = \ell, \ldots, \ell + n$, where ℓ is any integer with $\nu_\ell = 0$. But by (19), $\eta(L_k) = N^T\gamma(L_k) = B\gamma(L_k) = \eta = \eta(g_k)$ and so $[\xi_i, \ldots, \xi_{i+n+1}](L_k - g_k) = 0$, $i \in \mathbf{Z}$. Thus for any integer i, $L_k^{(\nu_i)}(\xi_i) = g_k^{(\nu_i)}(\xi_i) = \delta_{ik}$.

Next we make estimates on $L_k(t)$. Suppose t is in $(\xi_{\ell-1}, \xi_\ell)$ for $\ell \geqq k$. Then by (12) with $\gamma(L_k) = (\gamma_i)_{-\infty}^{\infty}$,

$$\sum_{j=-\infty}^{\infty} N^{(\mu_j)}(x_j | t, \xi_{\ell+n+1}, \ldots, \xi_{\ell+2n+1})\gamma_j$$
$$= (\xi_{\ell+2n+1} - t)[t, \xi_{\ell+n+1}, \ldots, \xi_{\ell+2n+1}]L_k$$
$$= -L_k(t)(t - \xi_{\ell+n+1})^{-1} \cdots (t - \xi_{n+2n+1})^{-1}.$$

Recalling (6), Lemma 4 and that the global mesh ratio of ξ is bounded, we see there is a constant K_2, independent of k and ℓ, such that

$$(21) \qquad |L_k(t)| \leqq K_3 \sum_{j=\ell-n-1}^{\ell+2n} |\gamma_j|.$$

By applying a similar argument for t in $(\xi_{\ell-1}, \xi_\ell)$, $\ell \leqq k$, we see there is a constant K_3 such that for any integers k and ℓ, and any t in $(\xi_{\ell-1}, \xi_\ell)$,

$$(22) \qquad |L_k(t)| \leqq K_3 \sum_{j=t-3n-2}^{\ell+2n} |\gamma_j|.$$

Now Theorem 2 of **[2]** tells us that if the matrix which represents B^{-1} is denoted by (b_{ij}), then there are positive constant K_4, λ, with $\lambda < 1$, such that for all i, j,

$$(23) \qquad |b_{ij}| \leqq K_4 \lambda^{|i-j|}.$$

Since $\gamma(L_k) = B^{-1}\eta$ and $\eta = \eta(L_k)$, on recalling (18) we see for any integer i,

$$(24) \quad \gamma_i = \sum_{j=-\infty}^{\infty} b_{ij}\eta_j = \sum_{j=k-2n-1}^{k+n} b_{ij}(\xi_{j+n+1} - \xi_j)[\xi_j, \ldots, \xi_{j+n+1}]L_k.$$

From (23), (24) and Lemma 4, noting that the global mesh ratio of ξ is bounded, there is a constant K_5 such that for any integers i and k.

$$(25) \qquad |\gamma_i| \leqq K_5 \lambda^{|i-k|}.$$

Combining (22) and (25) gives a constant K_6 such that

$$(26) \qquad |L_k(t)| \leqq K_6 \lambda^{|\ell-k|} (t \in [\xi_{\ell-1}, \xi_\ell), k, \ell \in \mathbf{Z}).$$

Finally we take any bounded sequence $(y_i)_{-\infty}^{\infty}$. By (26) the series $\sum_{-\infty}^{\infty} y_i L_i(t)$ converges uniformly on bounded sets to a bounded function f. Clearly f lies in $\zeta_n(x)$ and satisfies $f^{(\nu_j)}(\xi_j) = y_j$, $j \in \mathbf{Z}$. Thus (n, ξ, x) is solvable.

Remark. If x and ξ are strictly increasing, then the above proof can be easily modified to cover the possibility of x and ξ having infinite global mesh ratios, provided there are positive constants A, α such that

$$\Delta x_i/\Delta x_j \leqq A|i-j|^\alpha,\ \Delta\xi_i/\Delta\xi_j \leqq A|i-j|^\alpha\ (i,j\in\mathbb{Z},\ i\neq j).$$

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DEPARTMENT OF MATHEMATICAL SCIENCES, THE UNIVERSITY, DUNDEE DD1 4HN, SCOTLAND