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EXTENSIONS OF SEVERAL SUMMATION FORMULAE OF RAMANUJAN USING THE CALCULUS OF RESIDUES

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1. Introduction. Using the theory of modular transformations, Berndt [1, 2] has recently generalized many of Ramanujan's summation formulae, showing them to be particular examples within a large class of similar results. Berndt's approach is of further interest for the fact that most of the large number of summation theorems contained in [1] and [2] are consequences of a few main theorems which thus provides a unification of many summation theorems that had in the past been established using a variety of unrelated methods.

Our aim in this paper is similar to that of Berndt in that Ramanujan's summation formulae will be rederived and generalized using a few main theorems. However, as our chief tool will be Cauchy's theorem, our extensions will mostly be in a different direction to that of Berndt. To illustrate the extensions obtained here, consider Ramanujan's formula

$$\sum_{n=1}^{\infty} \frac{n^{4M+1}}{e^{2\pi n}-1} = \frac{B_{4M+2}}{2(4M+2)}, \tag{1.1}$$

where B_j denotes the j^{th} Bernoulli number, and M is used here and throughout to denote any positive integer. We will show (1.1) results from the same summation formula as do previously unknown sums such as

$$\sum_{n=1}^{\infty} \frac{n^{4(2M+1)-1}(e^{-\pi n}\cosh\sqrt{2}\,\pi n + e^{\pi n}\cos\sqrt{2}\,\pi n)}{\sinh \pi n(\cosh\sqrt{2}\,\pi n - \cos\sqrt{2}\,\pi n)} = \frac{B_{4(2M+1)}}{4(2M+1)} \tag{1.2}$$

(take $k = 2, 4$ in (2.15) to obtain (1.1), (1.2) respectively).

A curious result deducible immediately from (1.1) is

$$\sum_{n=1}^{\infty} \frac{n^{4M+1}}{e^{2\pi n}-1} = \int_0^{\infty} \frac{x^{4M+1}}{e^{2\pi x}-1}\,dx. \tag{1.3}$$

We shall deduce (1.3) without using (1.1). From this derivation we discover other equalities between series and integrals, of which

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$$(1.4)\qquad \sum_{n=1}^{\infty} n^{4M+1} \cos \pi n \frac{e^{-2\pi n}}{\sinh \pi n} = 2 \int_0^{\infty} x^{4M+1} \cos \pi x \frac{e^{-2\pi x}}{\sinh \pi x} dx$$

is typical (take $\ell = 1, 2$ in (3.8) to obtain (1.3), (1.4) respeciively).

2. One-dimensional summation formulae. We will formulate our main summation theorem immediately.

THEOREM 2.1. *Suppose the following hypotheses are satisfied*:

(i) $f(z)$ *is an even function with a countably infinite number of zeros, all simple, except maybe at the origin where higher order zeros are allowed, and* $1/f(z)$ *is analytic except for poles occurring at the zeros of* f. *We label the zeros* z_0 $(= 0$, *if appropriate*$)$, $\pm z_1, \pm z_2, \ldots$.

(ii) *If* $w = e^{\pi i/k}$ $(k$ *an integer* $> 1)$, *then the zeros of* $f(w^\ell z)$ $(\ell = 0, 1, \ldots, k-1)$ *do not intersect, except maybe at the origin.*

(iii) *There exists a sequence of contours* Γ_K *such that as* $K \to \infty$, Γ_K *is unbounded in all directions, and* $1/\prod_{\ell=0}^{k-1} f(w^\ell z) = O(e^{-c|z|})$ $(c > 0)$ *on* Γ_K *for all* K *large enough.*

Let N_0 *be the smallest integer such that*

$$\lim_{z\to 0} z^{2kN_0-1} \Big/ \prod_{\ell=0}^{k-1} f(w^\ell z) = 0;$$

the existence of N_0 *following from hypothesis* (i). *Then*

$$(2.1)\qquad \sum_{n=1}^{\infty} z_n^{2kN-1}/f'(z_n)\Big(\prod_{\ell=1}^{k-1} f(w^\ell z_n)\Big) = 0,$$

for all integers $N \geqq N_0$.

PROOF. Consider the integral

$$I_K = \frac{1}{2\pi i} \int_{\Gamma_K} z^{2kN-1} \Big/ \prod_{\ell=0}^{k-1} f(w^\ell z)\, dz.$$

By hypothesis (iii)

$$(2.2)\qquad \lim_{K\to\infty} I_K = 0.$$

By hypotheses (i) and (ii) the poles of the integrand are simple for $N \geqq N_0$ and occur at $\pm w^{2k-\ell} z_n$, $n = 1, 2, \ldots$, with corresponding residue

$$z_n^{2kN-1}/f'(z_n)\Big(\prod_{\ell=1}^{k-1} f(w^\ell z_n)\Big).$$

Hence by Cauchy's residue theorem,

$$(2.3)\qquad \lim_{K\to\infty} I_K = 2k \sum_{n=1}^{\infty} z_n^{2kN-1}/f'(z_n)\Big(\prod_{\ell=1}^{k-1} f(w^\ell z_n)\Big),\ N \geqq N_0.$$

Equating (2.3) and (2.2), (2.1) is immediate.

A choice of function particularly well suited to application in theorem 2.1 is $f(z) = z^{-\nu}J_\nu(z)$, $\nu > -1$, where J_ν denotes the Bessel function of order ν. For then it is known [3, Ch. 17] that all the zeros are real and unequal and $f(z)$ is an even analytic function so hypotheses (i) and (ii) are satisfied. Furthermore, from the large z asymptotic expansion of $J_\nu(z)$, hypothesis (iii) is satisfied by selecting a circle with circumference bisecting the K^{th} and $(K+1)^{\text{th}}$ zero, and since at $z = 0$ $z^{-\nu}J_\nu(z) = 1/2^\nu\Gamma(\nu+1)$, $N_0 = 1$. Recalling $(d/dz)\{z^{-\nu}J_\nu(z)\} = -z^{-\nu}J_{\nu+1}(z)$, we have from theorem 2.1

$$\sum_{n=1}^{\infty} z_n^{2kM+k\nu-1}/J_{\nu+1}(z_n)\Big(\prod_{\ell=1}^{k-1} J_\nu(z_n w^\ell)\Big) = 0, \tag{2.4}$$

where z_n denotes that n^{th} positive zero of $J_\nu(z)$. For example, when $\nu = -1/2$ so that $J_\nu = (2/\pi z)^{1/2}\cos z$, and k is odd, (2.4) reduces to

$$\sum_{n=1}^{\infty} \frac{(-1)^n(n-1/2)^{2kM-1}}{\prod_{\ell=1}^{(k-1)/2}\left(\cosh\left\{2\pi\left(n-\frac{1}{2}\right)\sin\frac{\pi\ell}{k}\right\}+\cos\left\{2\pi\left(n-\frac{1}{2}\right)\cos\frac{\pi\ell}{k}\right\}\right)} = 0 \tag{2.5}$$

while if k is even

$$\sum_{n=1}^{\infty} \frac{(-1)^n(n-1/2)^{2kM-1}}{\cosh\pi\left(n-\frac{1}{2}\right)\prod_{\ell=1}^{(k/2)-1}\left(\cosh\left\{2\pi\left(n-\frac{1}{2}\right)\sin\frac{\pi\ell}{k}\right\}+\cos\left\{2\pi\left(n-\frac{1}{2}\right)\cos\frac{\pi\ell}{k}\right\}\right)} = 0 \tag{2.6}$$

When $\nu = 1/2$ so that $J_\nu(z) = (2/\pi z)^{1/2}\sin z$, (2.4) shows for k odd

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^{k(2M+1)-1}}{\prod_{\ell=1}^{(k-1)/2}(\cosh\{2\pi n\sin\pi\ell/k\} - \cos\{2\pi n\cos\pi\ell/k\})} = 0 \tag{2.7}$$

and for k even

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^{k(2M+1)-1}}{\sinh\pi n\prod_{\ell=1}^{(k/2)-1}(\cosh\{2\pi n\sin\pi\ell/k\}-\cos\{2\pi n\cos\pi\ell/k\})} = 0. \tag{2.8}$$

The case $k = 2$ in (2.6) was stated as a problem by Ramanujan [7] (see also [8, p. 326]), and the cases $k = 2$ of (2.8) and $k = 3$ of (2.5) are due to Cauchy [3, p. 362 and p. 317 resp.] who also used the calculus of residues to obtain the results.

To obtain a generalization of (1.1), we choose $f(z) = \sin\pi z/z\cos\pi\mu z$, $0 < \mu < 1$ in theorem 2.1. By subtracting then adding the term $(-1)^n n^{k(2M+1)-1}\cos\pi\mu n(\prod_{\ell=1}^{k-1}\sin\pi\mu n w^\ell)$ in the numerator of the resulting equation, we deduce, if k is odd

$$-\sum_{n=1}^{\infty} \frac{n^{k(2M+1)-1}(-1)^n \cos \pi\mu n \prod_{\ell=1}^{(k-1)/2}\left(\cosh\left\{2\pi\mu n \sin\frac{\pi\ell}{k}\right\} - \cos\left\{2\pi\mu n \cos\frac{\pi\ell}{k}\right\}\right)}{\prod_{\ell=1}^{(k-1)/2}(\cosh\{2\pi n \sin \pi\ell/k\} - \cos\{2\pi n \cos \pi\ell/k\})}$$

$$\begin{aligned} &= \sum_{n=1}^{\infty} n^{k(2M+1)-1}(-1)^n \cos \pi\mu n\left(\prod_{\ell=1}^{(k-1)/2}\left(\cosh\left\{2\pi\mu n \sin\frac{\pi\ell}{k}\right\}\right.\right. \\ &\quad \left. + \cos\left\{2\pi\mu n \cos\frac{\pi\ell}{k}\right\}\right) - \prod_{\ell=1}^{(k-1)/2}(\cosh\{2\pi\mu n \sin \pi\ell/k\} \\ &\quad \left.- \cos\{2\pi\mu n \cos \pi\ell/k\})\right) \times \left(\prod_{\ell=1}^{(k-1)/2}(\cosh\{2\pi n \sin \pi\ell/k\}\right. \\ &\quad \left.- \cos\{2\pi n \cos \pi\ell/k\})\right)^{-1}, \end{aligned} \tag{2.9}$$

while if k is even

$$-\sum_{n=1}^{\infty}$$

$$\frac{n^{k(2M+1)-1}(-1)^n \cos \pi\mu n \sinh \pi\mu n \prod_{\ell=1}^{(k/2)-1}\left(\cosh\left\{2\pi\mu n \sin\frac{\pi\ell}{k}\right\} - \cos\left\{2\pi\mu n \cos\frac{\pi\ell}{k}\right\}\right)}{\sinh n\pi \prod_{\ell=1}^{(k/2)-1}(\cosh\{2\pi n \sin \pi\ell/k\} - \cos\{2\pi n \cos \pi\ell/k\})}$$

$$\begin{aligned} &= \sum_{n=1}^{\infty} n^{k(2M+1)-1}(-1)^n \cos \pi\mu n\left(\cosh \pi\mu n \prod_{\ell=1}^{(k/2)-1}\left((\cosh\left\{2\pi\mu n \sin\frac{\pi\ell}{k}\right\}\right.\right. \\ &\quad \left. + \cos\left\{2\pi\mu n \cos\frac{\pi\ell}{k}\right\}\right) - \sinh \pi\mu n \prod_{\ell=1}^{(k/2)-1}(\cosh\{2\pi\mu n \sin \pi\ell/k\} \\ &\quad \left.- \cos\{2\pi\mu n \cos \pi\ell/k\})\right) \times \left(\sinh \pi n \prod_{\ell=1}^{(k/2)-1}(\cosh\{2\pi\mu n \sin \pi\ell/k\}\right. \\ &\quad \left.- \cos\{2\pi n n \cos \pi\ell/k\})\right)^{-1}. \end{aligned} \tag{2.10}$$

We propose to take the limit $\mu \to 1^-$ in both (2.9) and (2.10). Since the right hand sides of both equations converge uniformly in μ for $0 \leqq \mu \leqq 1$ (at least) we merely put $\mu = 1$ there. It remains to take the limit on the left hand sides. We do this using the following theorem.

THEOREM 2.2. *Let γ_j be arbitrary, $\alpha_0 > \alpha_j$, $\alpha_j > 0$ and β_j real ($j = 1, 2, \ldots, X$). Let*

$$B(\mu) = \sum_{n=1}^{\infty} n^{k(2N+1)-1}(-1)^n \cos \pi\mu n\, C(\mu),$$

where

$$C(\mu) = \frac{e^{-\alpha_0(1-\mu)n} + \sum_{j=1}^{X} \gamma_j e^{-\alpha_j n\mu} e^{\beta_j n\mu i}}{1 + \sum_{j=1}^{X} \gamma_j e^{-\alpha_j n} e^{\beta_j n i}}.$$

Then

$$\lim_{\mu\to 1^-} B(\mu) = \begin{cases} \zeta(1 - k(2N+1)), & \alpha_0 = \pi \cot \pi L/2k \\ \infty, & \textit{otherwise} \end{cases},$$

where it is assumed $k > 1$ and fixed (not necessarily an integer), $N = 0, 1, 2, \ldots,$ and L is any positive odd integer such that $L/k < 1$.

PROOF, Since $\alpha_j > 0$ for each j there exists an integer N_0 such that for $n \geqq N_0$, $|\sum_{j=1}^{X} \gamma_j e^{-\alpha_j n} e^{\beta_j n i}| < 1$. We can thus write for $n \geqq N_0$,

$$\begin{aligned} C(\mu) &= e^{-\alpha_0(1-\mu)n} + \sum_{j=1}^{\infty} (-1)^j e^{-\alpha_0(1-\mu)n} \Big(\sum_{k=1}^{X} \gamma_k e^{-\alpha_k n\mu} e^{\beta_k n\mu i}\Big)^j \\ &\quad + \sum_{j=0}^{\infty} (-1)^j \Big(\sum_{k=1}^{X} \gamma_k e^{-\alpha_k n} e^{\beta_k n i}\Big)^j \Big(\sum_{k=1}^{X} \gamma_k e^{-\alpha_k n\mu} e^{\beta_k n\mu i}\Big) \\ &\equiv e^{-\alpha_0(1-\mu)n} + C_1(\mu) + C_2(\mu), \end{aligned}$$

say.

Hence

$$\begin{aligned} \lim_{\mu\to 1^-} B(\mu) &= \sum_{n=1}^{N_0-1} n^{k(2N+1)-1} \\ &\quad + \lim_{\mu\to 1^-} \sum_{n=N_0}^{\infty} n^{k(2N+1)-1}(-1)^n \cos \pi\mu n\, e^{-\alpha_0(1-\mu)n} \\ &\quad + \lim_{\mu\to 1^-} \sum_{n=N_0}^{\infty} n^{k(2N+1)-1} (-1)^n \cos \pi\mu n (C_1(\mu) + C_2(\mu)). \end{aligned} \tag{2.11}$$

But the last term in (2.11) converges uniformly for $0 \leqq \mu \leqq 1$ (at least), and since $C_1(1) = -C_2(1)$, we have

$$\lim_{\mu\to 1^-} B(\mu) = \lim_{\mu\to 1^-} \sum_{n=1}^{\infty} n^{k(2N+)-1}(-1)^n \cos \pi\mu n\, e^{-\alpha_0(1-\mu)n}. \tag{2.12}$$

Applying the Poisson summation formula to (2.12), which is valid for $\mu < 1$, we have

$$\begin{aligned} &\lim_{\mu\to 1^-} B(\mu) \\ &= \Gamma(k(2N+1)) \lim_{\mu\to 1^-} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\cos\Big\{k(2N+1)\operatorname{artan}\Big(\dfrac{2\pi n + \pi(1-\mu)}{\alpha_0(1-\mu)}\Big)\Big\}}{(\alpha_0^2(1-\mu)^2 + (2\pi n + \pi(1-\mu))^2)^{(k(2N+1))/2}} \\ &\quad + \Gamma(k(2N+1)) \lim_{\mu\to 1^-} \frac{\cos\{k(2N+1) \operatorname{artan} \pi/\alpha_0\}}{(\alpha_0^2(1-\mu)^2 + \pi^2(1-\mu)^2)^{(k(2N+1))/2}}, \end{aligned} \tag{2.13}$$

where we have separated off the $n = 0$ term. Thus for the limit to exist we require $\cos\{k(2N+1) \operatorname{artan} \pi/\alpha_0\} = 0$, i.e., $\alpha_0 = \pi \cot \pi L/2k$. Assuming this condition, and using the uniform convergence with respect to μ of the first term in (2.13), we have

$$\lim_{\mu\to 1^-} B(\mu) = \frac{\Gamma(k(2N+1))}{(2\pi)^{k(2N+1)}} 2\cos\left(\frac{k(2N+1)\pi}{2}\right)\zeta(k(2N+1))$$
$$= \zeta(1 - k(2N+1)),$$

where to obtain the last line we have used the functional equation of the Riemann zeta function [9, p. 269].

Return to (2.9) and (2.10) we see both can be written in the form of theorem 2.2 with $\alpha_0 = 2\pi \sum_{\ell=1}^{(k-1)/2} \sin \pi\ell/k$ if k is odd and $\alpha_0 = \pi + 2\pi \sum_{\ell=1}^{(k/2)-1} \sin \pi\ell/k$ if k is even. It is a simple exercise in summing geometric series to show that in both cases $\alpha_0 = \pi \cot \pi/2k$. Hence applying theorem 2.2 and recalling that if k is odd $\zeta(1 - k(2N+1)) = 0$ [9, p. 268] we deduce from (2.9)

$$\sum_{n=1}^{\infty} n^{k(2M+1)-1} \left(\frac{\prod_{\ell=1}^{(k-1)/2}\left(\cosh\left\{2\pi n \sin\frac{\pi\ell}{k}\right) + \cos\left\{2\pi n \cos\frac{\pi\ell}{k}\right\}\right)}{\prod_{\ell=1}^{(k-1)/2}\left(\cosh\left\{2\pi n \sin\frac{\pi\ell}{k}\right\} - \cos\left\{2\pi n \cos\frac{\pi\ell}{k}\right\}\right)} - 1\right) = 0, \tag{2.14}$$

while if k is even, $\zeta(1 - k(2M+1)) = -B_{k(2M+1)}/k(2M+1)$ [9, p. 268], so from (2.10) we have

$$\sum_{n=1}^{\infty} n^{k(2M+1)-1} \left(\frac{\cosh \pi n \prod_{\ell=1}^{(k/2)-1}\left(\cosh\left\{2\pi n \sin\frac{\pi\ell}{k}\right\} + \cos\left\{2\pi n \cos\frac{\pi\ell}{k}\right\}\right)}{\sinh \pi n \prod_{\ell=1}^{(k/2)-1}\left(\cosh\left\{2\pi n \sin\frac{\pi\ell}{k}\right\} - \cos\left\{2\pi n \cos\frac{\pi\ell}{k}\right\}\right)} - 1\right) \tag{2.15}$$
$$= B_{k(2M+1)}/k(2M+1).$$

As commented in §1 the case $k = 2$ in (2.15) is the summation theorem generally attributed to Ramanujan (it was pointed out by Berndt [1] that (1.1) was in fact discovered by Glaisher [5]).

Summation formulae similar to (2.14) and (2.15) can be deduced from theorem 2.1 by choosing $f(z) = \cos \pi z/z \sin \pi\mu z$, $0 < \mu < 1$, and then establishing the analogues of (2.9) and (2.10). To take the limit $\mu \to 1^-$ on the left hand side we require the following theorem:

THEOREM 2.3. *Let γ_j be arbitrary, $\alpha_0 > \alpha_j$, $\alpha_j > 0$ and β_j real* ($j = 1, 2, \ldots, X$). *Then*

$$\lim_{\mu\to 1^-} \sum_{n=1}^{\infty} (-1)^n (n - 1/2)^{k(2N+1)-1} \sin \pi\mu(n - 1/2)$$

$$\times \left(\frac{e^{-\alpha_0(1-\mu)(n-1/2)} + \sum_{j=1}^{X} \gamma_j e^{-\alpha_j(n-1/2)\mu} e^{\beta_j(n-1/2)\mu i}}{1 + \sum_{j=1}^{X} \gamma_j e^{-\alpha_j(n-1/2)} e^{\beta_j(n-1/2)i}} \right)$$

$$= \begin{cases} (-2^{1-k(2N+1)} + 1)\zeta(1 - k(2N+1)), & \alpha_0 = \pi \cot \pi L/2k \\ \infty, & \textit{otherwise} \end{cases},$$

where it is assumed $k > 1$ and fixed, $N = 0, 1, 2, \ldots$, and L is any positive odd integer such that $L/k < 1$.

Since the proof of theorem 2.3 is substantially similar to that of theorem 2.2 it will not be given. Employing theorem 2.3 then shows, if k is odd

$$\sum_{n=1}^{\infty} \left(n - \frac{1}{2}\right)^{k(2M+1)-1}$$

$$(2.16)\quad \left(\frac{\prod_{\ell=1}^{(k-1)/2} \left(\cosh\left\{2\pi\left(n - \frac{1}{2}\right)\sin\frac{\pi\ell}{k}\right\} - \cos\left\{2\pi\left(n - \frac{1}{2}\right)\cos\frac{\pi\ell}{k}\right\}\right)}{\prod_{\ell=1}^{(k-1)/2} \left(\cosh\left\{2\pi\left(n - \frac{1}{2}\right)\sin\frac{\pi\ell}{k}\right\} + \cos\left(2\pi\left(n - \frac{1}{2}\right)\cos\frac{\pi\ell}{k}\right\}\right)} - 1 \right) = 0,$$

while if k is even

$$(2.17)\quad \sum_{n=1}^{\infty} \left(n - \frac{1}{2}\right)^{k(2M+1)-1}$$

$$\left(1 - \frac{\sinh\pi\left(n - \frac{1}{2}\right) \prod_{\ell=1}^{(k/2)-1} \left(\cosh\left\{2\pi\left(n - \frac{1}{2}\right)\sin\frac{\pi\ell}{k}\right\} - \cos\left\{2\pi\left(n - \frac{1}{2}\right)\cos\frac{\pi\ell}{k}\right\}\right)}{\cosh\pi\left(n - \frac{1}{2}\right) \prod_{\ell=1}^{(k/2)-1} \left(\cosh\left(2\pi\left(n - \frac{1}{2}\right)\sin\frac{\pi\ell}{k}\right\} + \cos\left\{2\pi\left(n - \frac{1}{2}\right)\cos\frac{\pi\ell}{k}\right\}\right)}\right)$$

$$= (1 - 2^{1-k(2M+1)})\, B_{k(2M+1)}/k(2M+1).$$

The case $k = 2$ of (2.17) was first derived by Glaisher [5, p. 82].

3. Some formulae relating infinite series to definite integrals. In this section we will derive the cases $k = 2$ of (2.15) and (2.17) from theorem 2.1 in another way which leads us to formulae relating definite integrals to series. By selecting $k = 2$ and $f(z) = z^{-\nu}J_\nu(z)/z^\nu J_{-\nu}(\mu z)$, $0 < \mu < 1$, $\nu > -1$ in theorem 2.1 and then subtracting and adding $z_n^{4(M+\nu)-1} J_{-\nu}(\mu z_n)$ $I_\nu(\mu z_n)$ in the numerator of the resulting expression (where I_ν denotes the Bessel function of order ν of pure imaginary argument, and z_n denotes the n^{th} positive zero of J_ν), we conclude

$$
(3.1)\quad \begin{aligned}
&-\sum_{n=1}^{\infty} z_n^{4(M+\nu)-1} J_{-\nu}(\mu z_n) I_\nu(\mu z_n)/J_{\nu+1}(z_n) I_\nu(z_n) \\
&\quad = \sum_{n=1}^{\infty} z_n^{4(M+\nu)-1} J_{-\nu}(\mu z_n)(I_{-\nu}(\mu z_n) - I_\nu(\mu z_n))/J_{\nu+1}(z_n) I_\nu(z_n).
\end{aligned}
$$

From the asymptotic expansion of $I_{\pm\nu}$ [9, p. 373] $I_{-\nu}(x) - I_\nu(x) \sim (2/\pi x)^{1/2} e^{-x} \sin \pi\nu$ as $x \to \infty$. Hence the right hand side of (3.1) is convergent for $\mu > 0$ while the left hand side diverges for $\mu \geqq 1$. However, we can express the left hand side as a contour integral which is convergent for all $\mu > 0$. Consider the contour integral

$$
K = \frac{1}{2\pi i} \int_{i\varepsilon-\infty}^{i\varepsilon+\infty} \frac{z^{4(M+\nu)-1} I_{-\nu}(\mu z) J_\nu(\mu z)}{I_\nu(z) J_\nu(z)} dz,
$$

where $0 < \varepsilon < z_1$, z_1 denoting the first positive zero of J_ν, and the many-valued function $z^{4(M+\nu)-1}$ is made definite by selecting $\arg(z)$ to assume its principal value. Let τ_N denote the semi-circle with centre $i\varepsilon$, radius R_N, beginning at $i\varepsilon + R_N$ and being enscribed in the positive direction, R_N being chosen so that $z_N < R_N - \varepsilon < z_{N+1}$. Then for $0 < \mu < 1$ we can add to the contour of integration the contour $\lim_{N\to\infty} \tau_N$ without changing the value of K, since the integrand is $O(e^{-c|z|})$, $c > 0$ on τ_N. We now have a closed contour and further, the integrand is analytic within this region apart from simple poles at the zeros of $I_\nu(z)$. Evaluating the residues at these poles, we have by Cauchy's theorem

$$
(3.2)\qquad K = -e^{\pi i\nu} \sum_{n=1}^{\infty} z_n^{4(M+\nu)-1} J_{-\nu}(\mu z_n) I_\nu(\mu z_n)/J_{\nu+1}(z_n) I_\nu(z_n).
$$

On the other hand, deforming the path of integration so that it touches the origin, and then on the path from $i\varepsilon - \infty$ to 0 changing variables $z = e^{\pi i} z'$ (which is permissible since the integrand is one-valued in the cut plane from $-\infty$ to 0) we have

$$
(3.3)\quad \begin{aligned}
2\pi i K &= \left(\int_0^{i\varepsilon+\infty} - e^{2\pi i\nu} \int_0^{-i\varepsilon+\infty}\right) \frac{z^{4(M+\nu)-1} J_\nu(\mu z)(I_{-\nu}(\mu z) - I_\nu(\mu z))}{J_\nu(z) I_\nu(z)} dz \\
&\quad + \left(\int_0^{i\varepsilon+\infty} - e^{2\pi i\nu} \int_0^{-i\varepsilon+\infty}\right) \frac{z^{4(M+\nu)-1} J_\nu(\mu z) I_\nu(\mu z)}{J_\nu(z) I_\nu(z)} dz,
\end{aligned}
$$

where we have subtracted and added $z^{4(M+\nu)-1} J_\nu(\mu z) I_\nu(\mu z)$ in the numerator of the integrand. Consider the last integral in (3.3). Changing variables $z = e^{-\pi i/2} z'$, we see

$$
\begin{aligned}
&\int_0^{-i\varepsilon+\infty} \frac{z^{4(M+\nu)-1} J_\nu(\mu z) I_\nu(\mu z)}{J_\nu(z) I_\nu(z)} dz \\
&\quad = e^{-2\pi i\nu} \int_0^{\varepsilon+i\infty} \frac{z^{4(M+\nu)-1} J_\nu(\mu z) I_\nu(\mu z)}{J_\nu(z) I_\nu(z)} dz.
\end{aligned}
$$

Hence the last term in (3.3) can be written

$$\left(\int_0^{i\varepsilon+\infty} + \int_{\varepsilon+i\infty}^0\right) \frac{z^{4(M+\nu)-1} J_\nu(\mu z) I_\nu(\mu z)}{J_\nu(z) I_\nu(z)} dz \equiv J$$

On the path from $i\varepsilon + \infty$ to $\varepsilon + i\infty$ the integrand is $O(e^{-c|z|})$, $c > 0$, so we can add this path of integration to J without changing its value. But we then have a closed contour, and since the integrand is analytic within the enclosed region, we have by Cauchy's theorem

$$J = 0. \tag{3.4}$$

Substituting (3.4) into (3.3) and then equating (3.3) and (3.2), we see we have the desired contour integral representation convergent for $\mu > 0$. Substituting (3.1) into the resulting equation, we then have

$$\begin{aligned} &2\pi i\, e^{\pi i\nu} \sum_{n=1}^{\infty} \frac{z_n^{4(M+\nu)-1} J_{-\nu}(\mu z_n)(I_{-\nu}(\mu z_n) - I_\nu(\mu z_n))}{J_{\nu+1}(z_n) I_\nu(z_n)} \\ &= \left(\int_0^{i\varepsilon+\infty} - e^{2\pi i\nu} \int_0^{-i\varepsilon+\infty}\right) \frac{z^{4(M+\nu)-1} J_\nu(\mu z)(I_{-\nu}(\mu z) - I_\nu(\mu z))}{J_\nu(z) I_\nu(z)} dz. \end{aligned} \tag{3.5}$$

In particular, when $\mu = 1$, we can collapse the contours of integration onto the real axis. Equating real or imaginary parts shows

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{z_n^{4(M+\nu)-1} J_{-\nu}(z_n)(I_{-\nu}(z_n) - I_\nu(z_n))}{J_{\nu+1}(z_n) I_\nu(z_n)} \\ &\qquad = -\frac{\sin \pi\nu}{\pi} \int_0^\infty \frac{x^{4(M+\nu)-1}\,(I_{-\nu}(x) - I_\nu(x))}{I_\nu(x)} dx. \end{aligned} \tag{3.6}$$

We can also collapse the path of integration onto the real axis when $\nu = -1/2$ or $1/2$ and μ is a positive odd or positive integer respectively. In these cases we conclude from (3.5) after some simple manipulation

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{(n - 1/2)^{4M-3} \sin(\pi p(n - 1/2))\, e^{-\pi p(n-1/2)}}{\sin(\pi(n - 1/2)) \cosh \pi(n - 1/2)} \\ &\qquad = \int_0^\infty \frac{x^{4M-3} \cos(\pi p x)\, e^{-\pi p x}}{\cos(\pi x) \cosh \pi x} dx \end{aligned} \tag{3.7}$$

and

$$\sum_{n=1}^{\infty} \frac{n^{4M+1} \cos(\pi \ell n)\, e^{-\pi \ell n}}{\cos(\pi n) \sinh \pi n} = \int_0^\infty \frac{x^{4M+1} \sin(\pi \ell x)\, e^{-\pi \ell x}}{\sin(\pi x) \sinh \pi x} dx, \tag{3.8}$$

where p denotes an odd positive integer, and ℓ a positive integer. We note that in the cases $p = 1$ and $\ell = 1$ we can evaluate the integrals, reclaiming summation formulae (2.17) and (2.15) in the case $k = 2$.

4. A second class of one-dimensional summation formulae. When $M = 0$, summation formula (1.1) assumes the modified form

$$\sum_{n=1}^{\infty} n/(e^{2\pi n} - 1) = 1/24 - 1/8\pi. \tag{4.1}$$

This can be derived from a modified form of theorem 2.1.

THEOREM 4.1. *Suppose the three hypotheses of theorem* 2.1 *are satisfied, and let N_0 be defined as in the statement of that theorem. Then*

$$\sum_{n=1}^{\infty} \frac{z_n^{2k(N_0-M)-1}}{f'(z_n)\left(\prod_{\ell=1}^{k-1} f(w^\ell z_n)\right)} = -\frac{1}{2k}\left(\text{residue at the origin of } \frac{z^{2k(N_0-M)-1}}{\prod_{\ell=0}^{k-1} f(w^\ell z_n)}\right), \tag{4.2}$$

where k is any integer > 1 and, as always, M denotes any positive integer.

PROOF. Apply Cauchy's theorem to the integral

$$\frac{1}{2\pi i}\int_{\Gamma_K}\left(z^{2k(N_0-M)-1}\Big/\prod_{\ell=0}^{k-1} f(w^\ell z)\right)dz$$

in the limit $K \to \infty$.

We will restrict our attention to the cases in which $z^{2k(N_0-1)-1}/\prod_{\ell=0}^{k-1} f(w^\ell z)$ has a simple pole at $z = 0$. Thus (4.2) assumes the form

$$\sum_{n=1}^{\infty} \frac{z_n^{2k(N_0-1)-1}}{f'(z_n)\left(\prod_{\ell=1}^{k-1} f(w^\ell z_n)\right)} = -\frac{1}{2k}\lim_{z\to 0}\frac{z^{2k(N_0-1)}}{\prod_{\ell=0}^{k-1} f(w^\ell z)}. \tag{4.3}$$

By choosing $f(z) = z^{-\nu}J_\nu(z)$, $\nu > -1$, we can apply (4.3) with $N_0 = 1$ since $z^{-\nu}J_\nu(z) = 1/2^\nu\Gamma(\nu+1) \neq 0$ for $\nu > -1$. Hence with this choice of f, (4.3) reads

$$\sum_{1=n}^{\infty} z_n^{k\nu-1}/J_{\nu+1}(z_n)\left(\prod_{\ell=1}^{k-1} J_\nu(w^\ell z_n)\right) = \frac{1}{2k}(2^\nu\Gamma(\nu+1))^k, \tag{4.4}$$

where z_n denotes the n^{th} positive zero of J_ν. If we further specialize, selecting $k = 2$, $\nu = 1/2$ in (4.4), we see

$$\sum_{n=1}^{\infty} (-1)^n\, n/\sinh \pi n = -1/4\pi \tag{4.5}$$

while $\nu = 1/2$ and $k = 3$ shows

$$\sum_{n=1}^{\infty} n^2/(1-(-1)^n \cosh \pi\sqrt{3}\, n) = 1/12\pi^2. \tag{4.6}$$

Summation formula (4.5) was first proved by Cauchy [**3**, p. 361], while (4.6) can be found in Berndt [**1**, p. 163].

To establish (4.1) we choose $f(z) = \sin \pi z/ z \cos \pi\mu z$, $0 < \mu < 1$, then after deducing the analogues of (2.9) and (2.10) and taking the limit on the left hand side using theorem 2.2 with $N = 0$, we conclude, for k odd

$$\sum_{n=1}^{\infty} n^{k-1}\left(\frac{\prod_{\ell=1}^{(k-1)/2}\left(\cosh\left\{2\pi n \sin\frac{\pi\ell}{k}\right\}+\cos\left\{2\pi n\cos\frac{\pi\ell}{k}\right\}\right)}{\prod_{\ell=1}^{(k-1)/2}\left(\cosh\left\{2\pi n \sin\frac{\pi\ell}{k}\right\}-\cos\left\{2\pi n\cos\frac{\pi\ell}{k}\right\}\right)}-1\right) = -\frac{1}{2k}\left(\frac{1}{\pi}\right)^{k-1} \tag{4.7}$$

and for k even

$$\sum_{n=1}^{\infty} n^{k-1}\left(\frac{\cosh \pi n \prod_{\ell=1}^{(k/2)-1}\left(\cosh\left(2\pi n \sin\frac{\pi\ell}{k}\right)+\cos\left(2\pi n\cos\frac{\pi\ell}{k}\right)\right)}{\sinh \pi n \prod_{\ell=1}^{(k/2)-1}\left(\cosh\left(2\pi n \sin\frac{\pi\ell}{k}\right)-\cos\left(2\pi n\cos\frac{\pi\ell}{k}\right)\right)}-1\right) = \frac{B_k}{k}-\frac{1}{2k}\left(\frac{1}{\pi}\right)^{k-1}. \tag{4.8}$$

On choosing $k = 2$ in (4.8) and recalling $B_2 = 1/6$, we reclaim (4.1). As our final conclusion from (4.3) we choose $k = 2$ and $f(z) = z^{-\nu}J_\nu(z)/(\mu z)^\nu J_{-\nu}(\mu z)$, $0 < \mu < 1$. We then have, analogous to (3.1)

$$\begin{aligned} -\mu^{2\nu}\sum_{n=1}^{\infty}\frac{z_n^{4\nu-1}J_{-\nu}(\mu z_n)I_\nu(z_n)}{J_{\nu+1}(z_n)I_\nu(z_n)} &+ \frac{1}{4}\left(\frac{2^{2\nu}\Gamma(\nu+1)}{\Gamma(-\nu+1)}\right) \\ &= \mu^{2\nu}\sum_{n=1}^{\infty}\frac{z_n^{4\nu-1}J_{-\nu}(\mu z_n)(I_{-\nu}(\mu z_n)-I_\nu(\mu z_n))}{J_{\nu+1}(z_n)I_\nu(z_n)}. \end{aligned} \tag{4.9}$$

Recalling our analysis of the integral denoted by K in section 3, we note (3.2), (3.3) and (3.4) are valid for $M = 0$ providing $\nu > 0$ (this ensures the validity of deforming the contour to touch the origin). We thus have the contour integral representation

$$\begin{aligned} -2\pi i e^{\pi i\nu}&\sum_{n=1}^{\infty}\frac{z_n^{4(\nu-1)}J_{-\nu}(\mu z_n)\,I_\nu(\mu z_n)}{J_{\nu+1}(z_n)I_\nu(z_n)} \\ &= \left(\int_0^{i\varepsilon+\infty} - e^{2\pi i\nu}\int_0^{-i\varepsilon+\infty}\right)\frac{z^{4\nu-1}J_\nu(z\mu)(I_{-\nu}(\mu z)-I_\nu(\mu z))}{J_\nu(z)I_\nu(z)}\,dz. \end{aligned} \tag{4.10}$$

Substituting (4.10) in (4.9) and then choosing $\mu = 1$ we can collapse the contours of integration onto the real axis, with the result

$$\begin{aligned} -\frac{\sin\pi\nu}{\pi}\int_0^\infty \frac{x^{4\nu-1}(I_{-\nu}(x)-I_\nu(x))}{I_\nu(x)}\,dx &+ \frac{1}{4}\left(\frac{2^{2\nu}\Gamma(\nu+1)}{\Gamma(-\nu+1)}\right)^2 \\ &= \sum_{n=1}^{\infty}\frac{z_n^{4\nu-1}J_{-\nu}(z_n)(I_{-\nu}(z_n)-I_\nu(z_n))}{J_{\nu+1}(z_n)I_\nu(z_n)} \end{aligned} \tag{4.11}$$

valid for $\nu > 0$, and where z_n denotes the n^{th} positive zero of J_ν. We can also collapse the path of integration onto the real axis when $\nu = 1/2$ and μ is a positive integer. In this case we conclude from (4.9) and (4.10) the identity

$$\int_0^\infty \frac{x \sin(\pi\ell/x)\, e^{-\pi\ell/x}}{\sin(\pi x) \sinh \pi x}\, dx - \frac{1}{4\pi} = \sum_{n=1}^{\infty} \frac{n \cos(\pi\ell/n)\, e^{-\pi\ell/x}}{\cos(\pi n) \sinh \pi n}, \tag{4.12}$$

where ℓ denotes any positive integer. Notice that when $\ell = 1$ we can evaluate the integral to deduce (4.1).

5. Multidimensional summation formulae. We will now formulate some d-dimensional summation formulae using Cauchy's theorem. This approach to evaluating multiple series is due to Glasser and Zucker [**6**, p. 132], who proved theorem 5.1(b) below in the special case $f(z) = \sin z$, $k = 1$, $d = 2$.

THEOREM 5.1. *Let f be and odd function with a countably infinite number of zeros, all real, and suppose $1/f$ is analytic except at the zeros of f. Label the zeros z_0 (= 0 if appropriate), $\pm z_1$, $\pm z_2$, ... Further, suppose hypothesis* (iii) *of theorem* 2.1 *is satisfied. Then*

(a) *if f does not have a zero at the origin,*

$$\begin{aligned} S_{d,k} \equiv \sum_{Y=1}^{k} \sum_{\ell_1,\ell_2,\ldots,\ell_d=1}^{\infty} & e^{((2Y-1)\pi i)/2k} \\ \times & \frac{(z_{\ell_1}^{2k} + z_{\ell_2}^{2k} + \cdots + z_{\ell_d}^{2k})^{1/2k}}{\prod_{j=1}^{d} f'(z_{\ell_j}))\, f(e^{((2Y-1)\pi i/2k}(z_{\ell_1}^{2k} + z_{\ell_2}^{2k} + \cdots + z_{\ell_d}^{2k})^{1/2k})} = 0, \end{aligned} \tag{5.1}$$

(b) *if f has a first order zero at the origin,*

$$S_{d,k} = \frac{1}{f'(0)} \left(\frac{-1}{2f'(0)}\right)^d \left(\prod_{j=0}^{d-2} \frac{1}{\left(\frac{d+1-j}{d-j} + k - 1\right)}\right) \frac{k}{(2+k-1)}, \tag{5.2}$$

where d and k are any positive integers.

PROOF. Consider the integral

$$L_K = \frac{1}{2\pi i} \int_{\Gamma_K} \frac{z^{2k}}{(z^{2k} + a^{2k})^{(2k-1)/2k} f(e^{((2X-1)\pi i)/2k}(z^{2k} + a^{2k})^{1/2k})\, f(z)}\, dz,$$

where k and X are positive integers such that $X \leqq k$ and it is assumed $a > 0$. By hypothesis (iii) of theorem 2.1,

$$\lim_{K\to\infty} L_K = 0. \tag{5.3}$$

Case (a). The integrand is analytic apart from poles which occur at $z = \pm z_n$, $\pm e^{((2Y-1)/2k)\pi i}(z_n^{2k} + a^{2k})^{1/2k}$, $Y = 1, 2, \ldots, k$.

The sum of the residues from the poles at $z = \pm z_n$ is, in the limit $K \to \infty$

$$2 \sum_{n=1}^{\infty} \frac{z_n^{2k}}{(z_n^{2k} + a^{2k})^{(2k-1)/2k} f(e^{((2X-1)/2k)\pi i}(z_n^{2k} + a^{2k})^{1/2k})\, f'(z_n)}, \tag{5.4}$$

while the sum of the residues from the poles at $z = \pm\, e^{((2Y-1)\pi i)/2k}(z_n^{2k} + a^{2k})^{1/2k}$ equals, in the limit $K \to \infty$

$$(5.5)\qquad 2\sum_{Y=1}^{k} e^{((Y-X)\pi i)/k} \sum_{n=1}^{\infty} \frac{(z_n^{2k} + a^{2k})^{1/2k}}{f'(z_n)\, f(e^{((2Y-1)\pi i)/2k}(z_n^{2k} + a^{2k})^{1/2k})}.$$

Equating the sum of (5.4) and (5.5) to (5.3), we have

$$\sum_{n=1}^{\infty} \frac{2z_n^{2k} + a^{2k}}{(z_n^{2k} + a^{2k})^{(2k-1)/2k} f'(z_n)\, f(e^{((2X-1)\pi i)/2k}(z_n^{2k} + a^{2k})^{1/2k})} + \sum_{\substack{Y=1\\ Y\neq X}}^{k} e^{((Y-X)\pi i)/k} \sum_{n=1}^{\infty} \frac{(z_n^{2k} + a^{2k})^{1/2k}}{f'(z_n)\, f(e^{((2Y-1)\pi i)/2k}(z_n^{2k} + a^{2k})^{1/2k})} = 0.$$

By choosing $a = (z_{\ell_2}^{2k} + z_{\ell_3}^{2k} + \cdots + z_{\ell_d}^{2k})^{1/2k}$, multiplying through by $f'(z_{\ell_2})\, f'(z_{\ell_3}) \cdots f'(z_{\ell_d})$ and then summing over ℓ_i from 1 to ∞, we have after a little manipulation of the first sum

$$(5.6)\qquad \begin{aligned} &\frac{d+1}{d} \sum_{\ell_1, \ell_2, \ldots, \ell_d = 1}^{\infty} \frac{(z_{\ell_1}^{2k} + z_{\ell_2}^{2k} + \cdots + z_{\ell_d}^{2k})^{1/2k}}{\left(\prod_{j=1}^{d} f'(z_{\ell_j})\right) f(e^{((2X-1)\pi i)/2k}(z_{\ell_1}^{2k} + z_{\ell_1}^{2k} + \cdots + z_{\ell_d}^{2k})^{1/2k})} \\ &\quad + \sum_{\substack{Y=1\\ Y\neq X}}^{k} \sum_{\ell_1, \ell_2, \ldots, \ell_d = 1}^{\infty} e^{((Y-X)\pi i)/k} \\ &\quad \times \frac{(z_{\ell_1}^{2k} + z_{\ell_2}^{2k} + \cdots + z_{\ell_d}^{2k})^{1/2k}}{\left(\prod_{j=1}^{d} f'(z_{\ell_j})\right) f(e^{((2Y-1)\pi i)/2k}(z_{\ell_1}^{2k} + z_{\ell_2}^{2k} + \cdots + z_{\ell_d}^{2k})^{1/2k})} = 0. \end{aligned}$$

To obtain (5.1) from (5.6), multiply through by $e^{((2X-1)\pi i)/2k}$ and then sum over X from 1 to k. This shows

$$(5.7)\qquad \left(\frac{d+1}{d} + k - 1\right) S_{d,k} = 0$$

from which (5.1) is immediate.

Case (b). We must now consider the pole of the integrand occurring at $z = \pm\, e^{((2Y-1)\pi i)/2k}\, a$, $Y = 1, 2, \ldots, k$. The sum of the residues at these points is

$$(1/kf'(0)) \sum_{Y=1}^{k} e^{((Y-X)\pi i)/k}\, a/f(e^{((2Y-1)\pi i)2k}\, a).$$

If we include this extra term in the steps leading to (5.7), we see the term $-(1/2f'(0))S_{d-1,k}$ must replace zero on the right hand side of (5.7). Solving the resulting difference equation, we conclude

$$(5.8)\qquad S_{d,k} = \left(\frac{-1}{2f'(0)}\right)^{d-1} \left(\prod_{j=0}^{d-2} \frac{1}{\left(\dfrac{d+1-j}{d-j} + k - 1\right)}\right) S_{1,k}.$$

It is straightforward to evaluate $S_{1,k}$ by considering the contour integral

$$\frac{1}{2\pi i}\int_{\Gamma_K}\frac{z}{f(z)f(e^{((2X-1)\pi i)/2k}\,z)}\,dz.$$

We find

$$S_{1,k} = -(1/2(f'(0))^2)\,(k/(2+k-1)). \tag{5.9}$$

Substituting (5.9) in (5.8) gives (5.2).

We will restrict our applications of theorem 5.1 to the case $k = 1$. Since $f(z) = \cos \pi z/\sin \pi\mu z$, $0 < \mu < 1$, satisfies the hypotheses for the validity of (5.1), we conclude

$$\begin{aligned} &\sum_{\ell}(-1)^{\ell_1+\ell_2+\cdots+\ell_d}\left(\prod_{j=1}^{d}\sin\pi\mu\left(\ell_j-\frac{1}{2}\right)\right)|\ell+\mathbf{g}_{-1/2}|\,\frac{\cosh\pi\mu|\ell+\mathbf{g}_{-1/2}|}{\cosh\pi|\ell+\mathbf{g}_{-1/2}|} \\ &= 2\sum_{\ell}(-1)^{\ell_1+\ell_2+\cdots+\ell_d}\left(\prod_{j=1}^{d}\sin\pi\mu\left(\ell_j-\frac{1}{2}\right)\right)\frac{|\ell+\mathbf{g}_{-1/2}|}{e^{\pi(1+\mu|\ell+\mathbf{g}_{-1/2}|}+1}, \end{aligned} \tag{5.10}$$

where the sum is over the d-dimensional integer lattice, and $\mathbf{g}_x$ denotes the d-dimensional vector with all components x. Taking the limit $\mu \to 1^-$ on both sides of (5.10), we will obtain the d-dimensional analogue of (2.17) in the case $k = 2$, $M = 1$. To take the limit on the left hand side we first require some notation. Let

$$Z\left|\begin{matrix}\mathbf{g}\\ \mathbf{h}\end{matrix}\right|(T;s) = \sum_{\ell}\frac{e^{-2\pi i\mathbf{h}\cdot\ell}}{((\ell+\mathbf{g}),\,T(\ell+\mathbf{g}))^{s/2}}, \quad \mathrm{Re}\,(s) > d$$

denote the Epstein zeta function and its analytic continuation, where T is a positive definite matrix, and the sum is over the d-dimensional integer lattice, omitting $\ell = -\mathbf{g}$ if $\mathbf{g}$ is a liattce vector. We then have the following result.

THEOREM 5.2.

$$\begin{aligned} &\lim_{\mu\to 1^-}\sum_{\ell}(-1)^{\ell_1+\ell_2+\cdots+\ell_d}\left(\prod_{i=1}^{d}\sin\pi\mu\left(\ell_i-\frac{1}{2}\right)\right)|\ell+\mathbf{g}_{-1/2}|\,\frac{\cosh\pi\mu|\ell+\mathbf{g}_{1/2}|}{\cosh\pi|\ell+\mathbf{g}^{-}_{1/2}|} \\ &\quad = (-1)^d\,Z\left|\begin{matrix}\mathbf{g}_{1/2}\\ \mathbf{0}\end{matrix}\right|(I;1), \end{aligned}$$

where I denotes the identity matrix.

PROOF. Denote

$$\begin{aligned} D(\mu) = (-1)^d &\sum_{\ell}(-1)^{\ell_1+\ell_2+\cdots+\ell_d} \\ &\left(\prod_{j=1}^{d}\sin\pi\mu\left(\ell_j-\frac{1}{2}\right)\right)|\ell+\mathbf{g}_{-1/2}|\,\frac{\cosh\pi\mu|\ell+\mathbf{g}_{-1/2}|}{\cosh\pi|\ell+\mathbf{g}_{-1/2}|}. \end{aligned}$$

Then proceeding as in (2.11) and (2.12) of the proof of theorem 2.2 we have

$$\lim_{\mu\to 1^-} D(\mu) = \lim_{\mu\to 1^-} \sum_{\ell} |\ell + \mathbf{g}_{-1/2}|\, e^{-\pi(1-\mu)|\ell+\mathbf{g}_{-1/2}|} e^{\pi i \mathbf{g}(1-\mu)\cdot(\ell+\mathbf{g}_{-1/2})}. \tag{5.11}$$

By using the d-dimensional Poisson summation formula and then adopting a simple change of variables, (5.11) shows

$$\lim_{\mu\to 1^-} D(\mu) = \lim_{\mu\to 1^-} \sum_{\ell} (-1)^{\ell_1+\ell_2+\cdots\ell_d} \int_{\mathbf{R}^d} |\mathbf{h}|\, e^{-\pi(1-\mu)|\mathbf{h}|}\, e^{\pi i \mathbf{h}\cdot(2\ell+\mathbf{g}(1-\mu))}\, d\mathbf{h}. \tag{5.12}$$

If we denote the integral in (5.12) by K and rewrite the factor $e^{-\pi(1-\mu)|\mathbf{h}|}$ using the integral identity

$$e^{-\alpha R} = (2/\sqrt{\pi})R \int_0^\infty e^{-R^2t^2} e^{-\alpha^2/4t^2}\, dt, \tag{5.13}$$

we have

$$K = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{(-\pi^2(1-\mu)^2)/4t^2}\, dt \int_{\mathbf{R}^d} \mathbf{h}^2\, e^{-\mathbf{h}^2t^2+\pi i \mathbf{h}\cdot(\ell+1/2\mathbf{g}(1-\mu))}\, d\mathbf{h}.$$

The integral over $\mathbf{h}$ is now essentially of the Gaussian type, and can be computed immediately. We then have

$$\begin{aligned} K = {} & d\pi^{(d-1)/2} \int_0^\infty t^{-d-2}\, e^{-\pi^2((\ell+1/2\mathbf{g}(1-\mu))^2+1/4(1-\mu)^2)t^{-2}}\, dt \\ & - 2\pi^{(d+3)/2}\Big(\ell + \frac{1}{2}\mathbf{g}_{(1-\mu)}\Big)^2 \int_0^\infty t^{-d-4}\, e^{-\pi^2((\ell+1/2\mathbf{g}(1-\mu))^2+1/4(1-\mu)^2)t^{-2}}\, dt. \end{aligned} \tag{5.14}$$

The integrals in (5.14) are straightforward. We find

$$K = \frac{1}{2}\pi^{-(d+3)/2}\, \Gamma\Big(\frac{d+1}{2}\Big)\Big(\frac{(d/4)(1-\mu)^2 - (\ell + (1/2)\mathbf{g}_{(1-\mu)})^2}{((\ell+(1/2)\mathbf{g}_{(1-\mu)})^2+(1/4)(1-\mu)^2)^{(d+3)/2}}\Big) \tag{5.15}$$

We note that since $((1/2)\mathbf{g}_{(1-\mu)})^2 = (d/4)(1-\mu)^2$, $K = 0$ when $\ell = \mathbf{0}$. Hence, we can substitute (5.15) into (5.12), exclude the $\ell = \mathbf{0}$ term from the sum, and then take the limit by putting $\mu = 1$. This shows

$$\begin{aligned} \lim_{\mu\to 1^-} D(\mu) &= \frac{-1}{2}\pi^{-(d+3)/2}\, \Gamma\Big(\frac{d+1}{2}\Big) Z\left|\begin{matrix}\mathbf{0}\\ -\mathbf{g}_{-1/2}\end{matrix}\right|(I;\, d+1) \\ &= Z\left|\begin{matrix}\mathbf{g}_{-1/2}\\ \mathbf{0}\end{matrix}\right|(I;\, -1), \end{aligned}$$

where to obtain the last line we have used the functional equation of the Epstein zeta function [4, p. 625].

Applying theorem 5.2 to (5.10), we conclude

$$\sum_{\ell} \frac{|\ell + \mathbf{g}_{-1/2}|}{\exp\{2\pi|\ell + \mathbf{g}_{-1/2}|\} + 1} = (1/2)\, Z\left|\begin{matrix}\mathbf{g}_{-1/2}\\ \mathbf{0}\end{matrix}\right|(I;\, 1). \tag{5.16}$$

Closed form evaluations of

$$Z\left|\begin{matrix} \mathbf{g}_{-1/2} \\ \mathbf{0} \end{matrix}\right|(I; s)$$

are known in 2, 4, 6 and 8 dimensions **[10]**. Thus, denoting the sum in (5.16) by A_d, d indicating the dimension, we have the following results:

$$\begin{aligned} &A_1 = 1/24, \qquad A_2 = (1/2\pi^2)\,\eta(3/2)\beta(3/2), \\ &A_4 = (3/2\pi^3)\eta(3/2)\eta(5/2),\ A_6 = (15/8\pi^4)(4\eta(3/2)\beta(7/2) - \beta(3/2)\eta(7/2)), \\ &A_8 = (105/4\pi^5)\eta(3/2)\zeta(9/2), \end{aligned} \tag{5.17}$$

where $\beta(s) = \sum_{n=0}^{\infty}(-1)^n/(2n+1)^s$ and $\eta(s) = \sum_{n=0}^{\infty}(-1)^n/(n+1)^s = (1 - 2^{1-s})\zeta(s)$.

Finally, to illustrate (5.2), we take $f(z) = J_1(z)$ and $k = 1$ to obtain the d-dimensional summation formula

$$\sum_{\ell_1, \ell_2, \ldots, \ell_d = 1}^{\infty} \frac{(z_{\ell_1}^2 + z_{\ell_2}^2 + \cdots + z_{\ell_d}^2)^{1/2}}{I_1((z_{\ell_1}^2 + z_{\ell_2}^2 + \cdots + z_{\ell_d}^2)^{1/2})\left(\prod_{j=1}^{d} J_0(z_{\ell_j})\right)} = \frac{2(-1)^d}{d+1}, \tag{5.18}$$

where z_{ℓ_i} denotes the ℓ^{th} positive zero of J_1.

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References

1. B. C. Berndt, *Modular transformations and generalisations of several formulae of Ramanujan*, Rocky Mountain J. Math. **7** (1977), 147–189.

2. ———, *Analytic Eisenstein series, theta-functions, and series relations in the spirit of Ramanujan*, J. reine angew. Math. **303** (1978), 333–365.

3. A. Cauchy, *Oeuvres*, *Serie* II.*t*. VII Paris 1889.

4. P. Epstein, *Zur Theorie allgemeiner Zetafunctionen*, Math. Ann. **56** (1902), 615–644.

5. J.W.L. Glaisher, *On the series which represent the twelve elliptic and the four zeta functions*, Mess. Math. **18** (1889), 1–84.

6. M.L. Glasser and I.J. Zucker, *Lattice Sums*, in: Theoretical Chemistry: Advances and Perspectives vol. 5, ed. D. Henderson, New York, 1980, 67–139.

7. S. Ramanujan, *Question* **358**, J. Indian Math. Society **4** (1912), 78.

8. ———, *Collected Papers*, New York 1972.

9. E.T. Whittaker and G.N. Watson, *Modern Analysis* **4***th ed.*, Cambridge, 1962.

10. I.J. Zucker, *Exact results for some lattice sums in* 2, 4, 6 *and* 8 *dimensions*, J. Phys. A **7** (1974), 1568–1575.

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