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ON THE STRUCTURE OF TORCH RINGS

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Torch rings appeared for the first time in the literature, under the name of ?-rings, when it was tried to characterize the commutative rings whose finitely generated modules are direct sums of cyclic modules ([**8**]; see [**4**] for an exposition of the history of the problem and the techniques with which it has been solved). Later torch rings have also appeared in [**10**] in the study of the commutative rings R with the property that the total ring of fractions of R/I is self-injective for all ideals I of R. In this paper we study the structure of torch rings and give an example of a torch ring which is not a trivial extension. This answers a question posed by T. Shores and R. Wiegand in [**8**].

A commutative ring R with identity is a *torch ring* if 1) R is not local, 2) the nilradical $N(R)$ of R is a prime ideal and is a non-zero uniserial R-module, 3) $R/N(R)$ is an h-local domain, and 4) R is a locally almost maximal Bézout domain (see [**4**] for the terminology.) Shores and Wiegand constructed a torch ring which was a *trivial extension*. Recall that an *extension* of the ring S by the S-module N is an exact sequence of abelian groups

$$0 \longrightarrow N \xrightarrow{i} R \xrightarrow{p} S \longrightarrow 0,$$

where R is a ring and p is a ring homomorphism such that $r \cdot i(x) = i(p(r) \cdot x)$ for all $r \in R$, $x \in N$. An extension

$$0 \longrightarrow N \xrightarrow{i} R \xrightarrow{p} S \longrightarrow 0$$

of the ring S by the S-module N is *trivial* if there exists a ring homomorphism $g \colon S \to R$ with $p \circ g = 1_S$ [**1**, Ch. 16].

Shores and Wiegand [**8**] have asked whether every torch ring R was a trivial extension of the ring $R/N(R)$ by its nilradical $N(R)$. In the first part of this paper we construct a torch ring R of characteristic p^2, where p is a prime; if R has characteristic p^2, the domain $R/N(R)$ must have characteristic p, so that there do not exist homomorphisms $R/N(R) \to R$.

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Thus R is not a trivial extension of $R/N(R)$ and this example settles Shores and Wiegand's question in the negative.

In the second part of this paper we study the torch rings of prime characteristic p. If R is a torch ring, then $R/N(R)$ is an almost maximal Bézout domain with a maximal ideal M such that $(R/N(R))_M$ is a maximal valuation domain. Let us therefore fix a domain S of characteristic p and let us suppose that S is an almost maximal Bézout domain with a maximal ideal M such that S_M is maximal. We prove that every torch ring R of characteristic p such that $R/N(R) \cong S$ is a trivial extension of $R/N(R)$ by $N(R)$ if and only if every element of S has a p-th root in S, that is if and only if the field of fractions of S is perfect. This is proved by making use of the cohomology theory for commutative algebras developed by André ([**1**]), Barr, Harrison and Quillen. The problem of the existence of torch rings of characteristic p which are not trivial extensions is therefore equivalent to the problem of the existence of a non-perfect field F with two independent valuations v_1 and v_2, where v_1 is maximal and v_2 is almost maximal. We do not know whether such a field can exist. Note that the completion of F with respect to v_2 must be an algebraically closed field [**2**, Chap. 6, §8, Ex. 15]. In particular every field with two independent maximal valuations must be algebraically closed [**9**].

1. Example of a torch ring of characteristic p². First of all let us prove a lemma that will often be useful in the sequel.

LEMMA. *Let S be a domain with field of fractions F and let N be an S-module. Let*

$$0 \longrightarrow N \xrightarrow{\ i\ } R \longrightarrow S \longrightarrow 0$$

be an extension of S by N. Then R is a torch ring if and only if

(i) *S is a non-local almost maximal Bézout domain, and*

(ii) *there is a maximal ideal M of S such that S_M is maximal and N is a non-zero homomorphic image of F/S_M.*

PROOF. Since R is an extension of S by N, if N is viewed as an ideal of R via i, $N^2 = 0$, so that $N \subseteq N(R)$, where $N(R)$ denotes the nilradical of R. Conversely, since S is a domain, $N(R) \subseteq \ker p = N$. Thus $N = N(R)$ and $S \cong R/N(R)$.

Now suppose R is a torch ring. Then (i) is trivial and by [**4**, Lemma 5.3], S has a maximal ideal M such that $N \cong N_M$. In particular S_M is maximal and N is a uniserial S_M-module. Thus $E_{S_M}(N)$, the S_M-injective envelope of N, is an indecomposable S_M-module. By [**7**, Prop. 1 and Th. 4] $E_{S_M}(N)$ is a S_M-homomorphic image of F. Hence N is an S_M-submodule of a homomorphic image of F. Since N is a torsion divisible S-module [**4**, Lemma 5.3], N is a homomorphic image of F/S_M. This proves (ii).

Let us show the converse. Since $N = N(R)$ and $S \cong R/N(R)$, R is not local and $N(R)$ is a prime ideal. The ring S is an almost maximal Bézout domain, and therefore it is h-local [3, Theorem 2.9], so that the S- and the S_M-submodules of F/S_M coincide [3, Lemma 2.7]. In particular N is a uniserial S-module. By localizing the exact sequence $0 \to N \to R \to S \to 0$ at the maximal ideals of R, it is easy to see that $R_{M'} \cong S_{M'}$ for every maximal ideal $M' \neq M$ of R. Moreover, by localizing at M, we obtain the exact sequence $0 \to N_M \to R_M \to S_M \to 0$. Since S_M is maximal and N is a homomorphic image of F/S_M, it follows that S_M and $N \cong N_M$ are linearly compact R_M-modules, so that R_M is a maximal valuation ring. Thus R is a locally almost maximal ring. Finally, since N is a homomorphic image of F/S_M, N is a divisible S-module, i.e., $sN = N$ for all $s \in S$, $s \neq 0$, so that $N = rN \subseteq rR$ for all $r \in R$, $r \notin N$. From this, the uniseriality of N and the fact that S is Bézout, it follows that every finitely generated ideal of R is principal, i.e., R is a Bézout ring. Thus R is a torch ring.

We are now ready to construct our example. Let p be a prime number and let ξ be a cardinal such that $\xi^{\aleph_0} = \xi$. Let F be the algebraically closed field of characteristic p and cardinality ξ. Then [9, Theorem A] F is multiply maximally complete, i.e., F has two subrings V_1 and V_2 which are maximal valuation domains with field of fractions F and no common non-zero prime ideal. Let M_1 and M_2 be the maximal ideals of V_1 and V_2 respectively and set $S = V_1 \cap V_2$. Then [5, Theorem 22.8] S is a Prüfer domain with field of fractions F and maximal ideals $M_1 \cap S$ and $M_2 \cap S$. Furthermore S is Bézout [4, Prop. 3.8] and h-local [5, Ex. 22.6]. The localizations of S at $M_1 \cap S$ and $M_2 \cap S$ are V_1 and V_2 respectively. Therefore S is a h-local locally almost maximal domain, i.e., an almost maximal Bézout domain [3, Theorem 2.9].

Let $W_2(F)$ be the ring of Witt vectors of length two over F [6, Ch. 3.4], and let T be the subring of $W_2(F)$ consisting of all Witt vectors with first component in S. Thus T is the cartesian product $S \times F$ with the following sum and product:

$$(x_1, y_1) + (x_2, y_2) = \left(x_1 + x_2, y_1 + y_2 - \sum_{i=1}^{p} \binom{p}{i} x_1^i x_2^{p-i}\right)$$

$$(x_1, y_1)(x_2, y_2) = (x_1 x_2, x_1^p y_2 + x_2^p y_1)$$

for all $x_1, x_2 \in S$, $y_1, y_2 \in F$. Then T is a commutative ring of characteristic p^2 and $I = \{(0, y) \mid y \in M_1\}$ is an ideal of T. Set $R = T/I$. Since $p = (0, 1)$ in T, R is a commutative ring of characteristic p^2. The nilradical $N(T)$ of T is $N(T) = \{(0, y) \mid y \in F\}$, so that $N(R) = N(T)/I$ and $R/N(R) \cong T/N(T) \cong S$. Thus R is an extension of S by $N(R) = N(T)/I$. As an

S-module, F is isomorphic to $N(T)$ via the isomorphism that maps $y \in F$ into $(0, y^p) \in N(T)$ (note that every element of F has a p-th root because F is algebraically closed.) This isomorphism induces an isomorphism $F/M_1 \cong N(T)/I$. Thus $N(R) \cong F/M_1$ is a homomorphic image of F/V_1. By the Lemma, R is a torch ring. Thus R is a torch ring of characteristic p^2; in particular R is not a trivial extension.

2. Torch rings of prime characteristic p. In the previous section we have shown that there exist torch rings R or characteristic p^2, where p is a prime, and those rings cannot be trivial extensions of $S = R/N(R)$. In that case the sequence $0 \to N(R) \to R \to S \to 0$ is never a split sequence of abelian groups. When the characteristic of R is p, the situation is different, because R contains the field with p elements, and therefore the exact sequence $0 \to N(R) \to R \to R/N(R) \to 0$ is always a split extension of abelian groups.

If R is a torch ring of characteristic p, by our Lemma $S = R/N(R)$ is a non-local almost maximal Bézout domain and has a maximal ideal M such that S_M is a maximal valuation domain. Conversely let us fix a Bézout domain S with these properties. By the Lemma if E is any proper S_M-homomorphic image of the field of fractions of S, the trivial extension of S by E is a torch ring of characteristic p. We want to know if there exist torch rings R of characteristic p with $R/N(R) \cong S$ which are not trivial extensions. The following theorem shows that this happens if and only if the field of fractions of S is not perfect.

THEOREM. *Let S be a non-local almost maximal Bézout domain and let F be the field of fractions of S. Suppose S has a maximal ideal M such that S_M is maximal. Then the following are equivalent:*

(i) *every torch ring R of characteristic p with $R/N(R) \cong S$ is a trivial extension of S, and*

(ii) *F is a perfect field.*

PROOF. (i) $\Rightarrow$ (ii). We shall make use of the cohomology theory for the commutative algebras. Suppose that every torch ring of characteristic p with $R/N(R) \cong S$ is a trivial extension. Let E be the S_M-module F/MS_M. Then by the Lemma of §1 every extension of S by E is a torch ring, because E is a homomorphic image of F/S_M. By [**1**, Prop. 16. 12] $H^1(S^p, S, E) = 0$, where S^p denotes the subring of S consisting of the p-th powers of all elements of S. Since $E \cong E_{S_M}(S_M/MS_M)$ is an S_M-module [**7**, Th. 4], $H^1(S^p, S, E) \cong H^1((S_M)^p, S_M, E)$ [**1**, Cor. 5.27], and since E is S_M-injective, $H^1((S_M)^p, S_M, E) \cong \mathrm{Hom}_{S_M}(H_1((S_M)^p, S_M, S_M), E)$ [**1**, Lemma 3.21]. But in our case E is a cogenerator in the category of all S_M-modules, and therefore $H_1((S_M)^p, S_M, S_M) = 0$. In particular

$$0 = H_1((S_M)^p, S_M, S_M) \otimes_{S_M} F \cong H_1((S_M)^p, S_M, F) \cong H_1(F^p, F, F)$$

[1, Lemma 3.20 and Th. 5.27]. But then F is a separable extension of its subfield F^p [1, Prop. 7.13], i.e., $F = F^p$ is perfect.

(ii) $\Rightarrow$ (i). If F is a perfect field, every element of S has a p-th root in S. Let R be a torch ring of characteristic p with $R/N(R) \cong S$. Let R^p be the subset of R consisting of the p-th powers of all elements of R. Then R^p is a subring of R and $R^p \cap N(R) = 0$ because $N(R)$ is prime and $N(R)^2 = 0$. Hence $R^p \oplus N(R)$ is a subring of R. Now if $r \in R$, there exists an element $r_1 \in R$ such that $r \equiv r_1^p \mod N(R)$ because in $R/N(R) \cong S$ every element has a p-th root. In particular $r \in R^p \oplus N(R)$. We have thus proved that $R = R^p \oplus N(R)$, i.e., R is a trivial extension of R^p by $N(R)$. Note that the surjective ring morphism $R \to S$ with kernel $N(R)$ induces a ring isomorphism $R^p \cong S$.

Let us conclude with some comments. If R is a torch ring of non-zero characteristic, then $R/N(R)$ must be a domain of characteristic p, i.e., $p \in N(R)$, so that $p^2 = 0$. Therefore every torch ring has characteristic either 0 or p or p^2, where p denotes a prime, and we have studied the cases of characteristic p and p^2. Let us briefly consider the case of the torch rings of characteristic zero. If we could prove that $H_1(\mathbf{Z}, V, V) = 0$ for all maximal valuation domains V of characteristic 0 (here $\mathbf{Z}$ is the ring of integers), then it would follow, as in the proof of the Theorem, that every torch ring of characteristic zero is a trivial extension. We are only able to prove that $H_1(\mathbf{Z}, V, V)$ is a torsion V-module ($H_1(\mathbf{Z}, V, V) \otimes F \cong H_1(\mathbf{Q}, F, F)$ where F is the field of fractions of V and $\mathbf{Q}$ is the field of rationals, and the last group is zero because F has characteristic zero [1, Prop. 7.13]). In the special case in which V is a discrete valuation ring (i.e., with divisibility group isomorphic to $\mathbf{Z}$) then $H_1(\mathbf{Z}, V, V) = 0$. This can be seen from the exact sequence $H_2(\mathbf{Z}, V, F/V) \to H_1(\mathbf{Z}, V, V) \to H_1(\mathbf{Z}, V, F)$ [1, Lemma 3.22]. Here the first group is zero by [1, Prop. 32, page 332], the second group is a torsion V-module and the third one is torsion-free. Hence $H_1(\mathbf{Z}, V, V) = 0$ in this case.

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