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TYPICALLY-REAL FUNCTIONS OF ORDER p

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1. Introduction. A number of authors have considered generalizations of Rogosinski's class T [9] of typically-real functions. Primary attention has been given to the class of functions $f \in H(\mathbf{B})$ (i.e., holomorphic in the unit disk, $\mathbf{B}$) which have real coefficients in their Taylor expansions about zero and, for some positive integer p, satisfy one of the following two conditions:

(I) $f \in H(\bar{\mathbf{B}})$ and $\operatorname{Im} f$ changes sign exactly $2p$ times as z traverses $\partial\mathbf{B}$, and

(II) $\exists \rho \in (0, 1)$ such that $\operatorname{Im} f$ changes sign exactly $2p$ times as z traverses $|z| = r$ for each $r \in (\rho, 1)$.

Functions satisfying (I) or (II) have at most p zeros in $\mathbf{B}$, counting multiplicity. Robertson [7] showed that members, f, of this class which have a zero of order p at $z = 0$ and normalization $f(z) = z^p + \cdots$ may be represented in the form

$$f(z) = \frac{z}{1 - z^2} \prod_{k=1}^{p-1} \frac{z}{1 - 2z \cos s_k + z^2} u(z), \tag{1}$$

where $s_k \in \mathbf{R}$ (the real numbers), $k = 1, \ldots, p - 1$, and $u \in H(\mathbf{B})$, $\operatorname{Re} u > 0$. Extremal problems for the class have been studied by various authors, including Goodman, Robertson, and Umezawa. In particular, the coefficient problem is treated in [3], [4], [10].

Conditions (I) and (II), however, are not completely satisfactory for defining a class of functions to be called typically-real of order p. There are functions of the form (1) which satisfy neither (I) nor (II). Moreover, there are sequences in this class which converge (uniformly on compact subsets of $\mathbf{B}$) to limits which do not satisfy (I) or (II). Examples are given in §5 of this paper.

In §2 we define an argument function for the boundary values of suitably restricted members of $H(\mathbf{B})$. We use this boundary argument to formulate a less restrictive condition than (I) or (II) which generates a class, $T(p)$, of functions which we call typically-real of order p. Functions in $T(p)$ will be required to have exactly p zeros, counting multiplicity. We show that $T(p)$ is characterized by a product representation like (1) which accounts for zeros other than $z = 0$. Furthermore, $T(p)$ is essentially

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closed in the topology of locally uniform convergence. More precisely, if $f_n \in T(p)$, $f_n \to f$, and f has p zeros in $\mathbf{B}$, then $f \in T(p)$. We also show that if $f \in T(p)$, then f' has at most $p-1$ zeros in $f^{-1}(\mathbf{R})$ and, for each $x \in \mathbf{R}$, $f(z) - x = 0$ has at most p roots in $\mathbf{R}$, counting multiplicity. These results complement Hummel's work [5] on weakly starlike multivalent functions. The weakly starlike p-valent functions with real Taylor coefficients are contained in $T(p)$.

Robertson [7] also studied functions satisfying (I) or (II) that do not necessarily have real coefficients. He referred to such functions as being starlike in the direction of the real axis. We consider this more general situation first and treat $T(p)$ as a special subclass.

2. Preliminaries. Suppose $f \in H(\mathbf{B})$ is of bounded Nevanlinna characteristic (i.e., $f \in N$) and has exactly p zeros in $\mathbf{B}$, $z_1, \ldots, z_p$, repeated according to multiplicity. Let

$$L(z) = \log\left(f(z)/z^p \prod_{j=1}^{p} \phi(z, z_j)\right), \tag{2}$$

where $\phi(z, z_j) = (z - z_j)(1 - \bar{z}_j z)/z$. Now, $f \in N$ if and only if Re $L \in h^1$[**2**, p. 29]. Thus, Im $L \in h^q$ [**2**, p. 35], and consequently $L \in H^q$ for $q < 1$. Since $\phi(e^{it}, z_j) = |e^{it} - z_j|^2 > 0$, the boundary values of f and of $w = z^p \exp(L(z))$ have the same arguments. Thus, for almost all $t \in \mathbf{R}$

$$A(t) = pt + \lim_{r \to 1} \operatorname{Im} L(re^{it}) \tag{3}$$

serves as an argument for $f(e^{it})$. Note that

$$A(t + 2\pi) = A(t) + 2p\pi \tag{4}$$

a.e. in $\mathbf{R}$, and $A \in L^q[0, 2\pi]$ for $0 < q < 1$. In order to recover f from A it will be necessary to require that $A \in L^1[0, 2\pi]$, i.e., Im $L \in h^1$. Since Re $L \in h^1$, this amounts to assuming $L \in H^1$. In that case, Im L is the Poisson intergral of its boundary values and we have

$$\operatorname{Im} L(re^{i\theta}) = \operatorname{Im}\left\{\frac{i}{2\pi}\int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}[A(t) - pt]dt\right\},$$

and, for some $c \in \mathbf{R}$,

$$f(z) = e^c z^p \prod_{j=1}^{p} \phi(z, z_j) \exp\left\{\frac{i}{2\pi}\int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}[A(t) - pt]dt\right\}. \tag{5}$$

DEFINITION. If f is holomorphic and has a finite number of zeros in $\mathbf{B}$ and if $L \in H^1$, then we say that f has boundary argument A defined by (3). Of course different branches of the logarithm in (2) generate different boundary arguments. When clarity requires specific reference to f we shall use the notation L_f and A_f.

LEMMA 1. *Let f have boundary argument A and zeros $z_1, \ldots, z_p$. If A*

is of bounded variation on $[0, 2\pi]$, *then* $\exists d \in \mathbf{C}$ (*the complex numbers*) *such that*

$$(6) \qquad f(z) = dz^p \prod_{j=1}^{p} \phi(z, z_j)\exp\left\{-\frac{1}{\pi}\int_0^{2\pi} \log(1 - ze^{-it})dA(t)\right\}.$$

PROOF. Writing $(1 + ze^{-it})/(1 - ze^{-it}) = 1 + (2/i)(d\log(1 - ze^{-it})/dt)$, $\log 1 = 0$, an integration by parts gives

$$\frac{i}{2\pi}\int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}[A(t) - pt]dt$$
$$= ib - \frac{1}{\pi}\int_0^{2\pi} \log(1 - ze^{-it})d[A(t) - pt],$$

where $b = \int_0^{2\pi}[A(t) - pt]dt/2\pi \in \mathbf{R}$. Since $\int_0^{2\pi} \log(1 - ze^{-it})d(pt) = 2p\pi \log 1 = 0$, (6) follows from (5) with $d = e^{c+ib}$.

Hummel's class, $S_w(p)$, of weakly starlike p-valent functions [5] has an expected characterization in terms of the boundary argument.

LEMMA 2. *Assume* $f \in H(\mathbf{B})$ *has exactly* p *zeros*, $z_1, \ldots, z_p$ *in* $\mathbf{B}$. *Then* $f \in S_w(p)$ *if and only if* f *has an increasing boundary argument.*

PROOF. By [5, Theorem 1], $f \in S_w(p)$ if and only if

$$(7) \qquad f(z) = [g(z)]^p \prod_{j=1}^{p} \phi(z, z_j),$$

where $g(z) = a_1 z + \cdots$ is a univalent starlike function. If $f \in S_w(p)$, then $L_f(z) = \log[g(z)/z]^p = pL_g(z)$. Since $\operatorname{Im} L_g \in h^\infty$, we have $\operatorname{Re} L_g \in h^q$ and $L_g \in H^q$ for $q < \infty$. Thus, f and g have boundary arguments satisfying $A_f = pA_g$. Since A_g is increasing, so is A_f. On the other hand, if f has increasing boundary argument A, then by Lemma 1, f is of the form (7) with

$$g(z) = d_1 z \exp\left\{-\frac{1}{\pi}\int_0^{2\pi} \log(1 - ze^{-it})d(A(t)/p)\right\}, \quad d_1^p = d.$$

Thus,

$$\operatorname{Re}(zg'(z)/g(z)) = \operatorname{Re}\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d(A(t)/p)\right\} \geqq 0,$$

and g is univalent starlike.

Conditions (I) and (II) involve changes of sign of $\operatorname{Im} f$. Such a change of sign corresponds to A_f "crossing" an integer multiple of π. In the remainder of this section we develop this notion of "crossing", which will be used in our discussion of $T(p)$.

DEFINITION. Let $h : \mathbf{R} \to \mathbf{R}$ be measurable, $r, a, b \in \mathbf{R}$, with $a \leqq b$, and let $S(\eta) = (a - \eta, a) \times (b, b + \eta)$, $\eta > 0$. We say that h crosses r in $[a, b]$ if $h(x) = r$ a.e. in $[a, b]$ and

$$\operatorname*{ess\,inf}_{(x,y)\in S(\eta)} [h(x) - r]\,[h(y) - r] < 0$$

for all $\eta > 0$. If in addition $\exists\eta > 0$ such that $h(x) \leqq r$ a.e. in $(a-\eta, a)$ and $h(y) \geqq r$ a.e. in $(b, b + \eta)$, then h crosses r in the increasing direction. A crossing in the decreasing direction is defined similarly. If $a = b$, we say that h crosses r at $x = a$.

A function, h, may cross more than one value at a given point. For example, if h has a jump discontinuity at $x = a$, then at $x = a$ h crosses each r which lies strictly between $h(a^-)$ and $h(a^+)$.

LEMMA 3. *Assume h: $\mathbf{R} \to \mathbf{R}$ is measurable, α, β, $r \in \mathbf{R}$, and $\alpha \neq \beta$. If*

$$\operatorname*{ess\,inf}_{|x-\alpha|<\eta} h(x) < r < \operatorname*{ess\,sup}_{|x-\beta|<\eta} h(x) \tag{8}$$

for all $\eta > 0$, then h crosses r in some interval between α and β. If $\alpha < \beta (> \beta)$, then the crossing is not in the decreasing (increasing) direction.

PROOF. Assume $\alpha < \beta$. We claim that $\exists$ $\alpha' \in [\alpha, \beta)$ such that $\operatorname{ess\,inf}_{\alpha'-\eta<x<\alpha'} h(x) < r$ for all $\eta > 0$. The alternative is that for each $y \in [\alpha, \beta)$, $\exists\eta(y) > 0$ such that $h(x) \geqq r$ a.e. in $(y - \eta(y), y)$. The collection $\{[y-\delta, y]: y \in [\alpha, \beta), 0 < \delta < \eta(y)\}$ forms a Vitali cover of $(\alpha-\eta(\alpha), \beta)$. By Vitali's Theorem, $h(x) \geqq r$ a.e. in $(\alpha - \eta(\alpha), \beta)$, which is contrary to (8). Similarly, $\exists\beta' \in (\alpha', \beta]$ such that $\operatorname{ess\,sup}_{\beta'<x<\beta'+\eta} h(x) > r$ for all $\eta > 0$. Let

$$a = \sup\{t \in [\alpha', \beta']: \operatorname*{ess\,inf}_{t-\eta<x<t} h(x) < r \text{ for all } \eta > 0\}$$

and

$$b = \inf\{t \in [a, \beta']: \operatorname*{ess\,sup}_{t<x<t+\eta} h(x) > r \text{ for all } \eta > 0\}\,.$$

For each $\eta > 0$,

$$\operatorname*{ess\,inf}_{a-\eta<x<a} h(x) < r < \operatorname*{ess\,sup}_{b<x<b+\eta} h(x)\,.$$

If $a = b$, then h crosses r at $x = a$. Suppose $a < b$ and let $y \in (a, b)$. By the definition of a, $\exists\eta(y) > 0$ such that $h(x) \geqq r$ a.e. in $(y - \eta(y), y)$. By Vitali's Theorem, $h(x) \geqq r$ a.e. in (a, b). Similarly, $h(x) \leqq r$ a.e. in (a, b), so we conclude that h crosses r in $[a, b]$. The crossing is clearly not in the decreasing direction. The case $\alpha > \beta$ is treated similarly.

LEMMA 4. *Let h: $\mathbf{R} \to \mathbf{R}$ be measurable, r, s, $t \in \mathbf{R}$, with $r \neq t$, and suppose that s lies strictly between r and t. Assume h crosses r and t in the intervals $[a, b]$ and $[c, d]$, respectively, and suppose $b \leqq c$.*

(i) *If $b < c$, then h crosses s in an interval between b and c.*

(ii) *If* $b = c$ *and either* $[a, b]$ *or* $[c, d]$ *is non-degenerate, then* h *crosses* s *at* $x = b$.

(iii) *If* $a = b = c = d$ *and one crossing is in a given direction, then* h *crosses* s *at* $x = a$.

PROOF. We consider only certain representative cases. Assume $r < s < t$ and consider (i). If $a = b$, then by the definition of crossing, $\operatorname{ess\,inf}_{|x-b|<\eta} h(x) < r < s$ for all $\eta > 0$. If $a < b$, then $\operatorname{ess\,inf}_{b-\eta<x<b} h(x) = r < s$ for all $0 < \eta < b - a$. Similarly, $\operatorname{ess\,sup}_{|x-c|<\eta} h(x) > s$ for all $\eta > 0$, and so the conclusion follows from Lemma 3.

As for (ii), assume $b = c$ and $a < b$. For each $0 < \eta < b - a$, $h(x) = r < s$ a.e. in $(b - \eta, b)$. Thus, since h crosses t at $x = b$, $\operatorname{ess\,sup}_{b<x<b+\eta} h(x) > t > s$ for all $\eta > 0$. It follows from the definition that h crosses s at $x = b$.

For (iii), assume h crosses r in the increasing direction at $x = a$. Then $h(x) \leqq r < s$ a.e. in $(a - \eta, a)$ for some $\eta > 0$, and the conclusion follows from the definition of crossing.

3. The class T(p). Let $\mathbf{Z}$ denote the integers and assume $p \in \mathbf{Z}$, $p \geqq 1$. Suppose $f \in H(\mathbf{B})$ has exactly p zeros in $\mathbf{B}$ and boundary argument A. Given $r \in \mathbf{R}$, it follows from (4) that $\{t: A(t) \neq r\}$ has positive measure. Assume $m(\{t: A(t) > r\}) > 0$ and let F be a subset of $\{t: A(t) > r\}$ of positive measure on which A is bounded, say by M. If for each $t \in F$, $\exists \eta(t) > 0$ such that $m(F \cap (t - \eta(t), t + \eta(t))) = 0$, then by Vitali's Theorem $m(F) = 0$. Thus, $\exists \beta \in F$ such that $\operatorname{ess\,sup}_{|t-\beta|<\eta} A(t) > r$ for all $\eta > 0$. Let $n \in \mathbf{Z}$ such that $2np\pi > M - r$ and let $\alpha = \beta - 2n\pi$. If $t \in F - 2n\pi$, then by (4) $A(t) = A(t + 2n\pi) - 2n\pi p \leqq M - 2n\pi p < r$. Thus, $\operatorname{ess\,inf}_{|t-\alpha|<\eta} A(t) < r$ for all $\eta > 0$, and by Lemma 3, A crosses r in some interval between α and β. We have shown that A crosses each $r \in \mathbf{R}$ at least once.

DEFINITION. Let $SR(p)$ denote the class of functions $f \in H(\mathbf{B})$ which have exactly p zeros in $\mathbf{B}$ and have boundary arguments which cross each value $k\pi$, $k \in \mathbf{Z}$, exactly once. Let $T(p)$ denote the class of functions $f \in SR(p)$ whose Taylor coefficients about zero are real.

LEMMA 5. *Let* $f \in H(\mathbf{B})$ *have boundary argument* A *and exactly* p *zeros. The following are equivalent*:

(i) $f \in SR(p)$,

(ii) *for each* $k \in \mathbf{Z}$, *if* A *crosses* $k\pi$ *in* $[a, b]$, *then* $A(t) \geqq k\pi$ a.e. *in* $(b, +\infty)$ *and* $A(t) \leqq k\pi$ a.e. *in* $(-\infty, a)$, *and*

(iii) *each crossing by* A *of an integral multiple of* π *is in the increasing direction.*

PROOF. (i) $\Rightarrow$ (ii). Assume A crosses $k\pi$ in $[a, b]$. With either $c = a$

or $c = b$ we have ess $\sup_{|t-c|<\eta} A(t) > k\pi$ for all $\eta > 0$. Now, suppose $\exists x \in (b, +\infty)$ such that ess $\inf_{|t-x|<\eta} A(t) < k\pi$ for all $\eta > 0$. Let $n \in \mathbf{Z}$ such that $d = c + 2n\pi > x$. By (4), ess $\sup_{|t-d|<\eta} A(t) = 2n\pi p +$ ess $\sup_{|t-c|<\eta} A(t) > k\pi$ for all $\eta > 0$, so by Lemma 3, A crosses $k\pi$ in some interval between x and d. This is contrary to (i), so for each $x \in (b, +\infty)$, $\exists\eta(x) > 0$ such that $A(t) \geqq k\pi$ a.e. in $|t - x| < \eta(x)$. Another application of Vitali's Theorem gives $A(t) \geqq k\pi$ a.e. in $(b, +\infty)$. Similarly, $A(t) \leqq k\pi$ a.e. in $(-\infty, a)$.

That (ii) $\Rightarrow$ (iii) is clear. Now, assume (iii) and suppose that for some $k \in \mathbf{Z}$, A crosses $k\pi$ in both $[a, b]$ and $[a', b']$. Since each crossing is in the increasing direction, ess $\sup_{b<t<b+\eta} A(t) > k\pi$ and ess $\inf_{a'-\eta<t<a'} A(t) < k\pi$ for all $\eta > 0$. Suppose $a < a'$. Since $A(t) = k\pi$ a.e. in $[a, b]$, $b < a'$. Then by Lemma 3, A crosses $k\pi$ in some interval between b and a' and the crossing is not in the increasing direction. This is contrary to (iii), so $a \geqq a'$. Interchanging a and a' we conclude $a = a'$. Similarly $b = b'$, so A crosses each $k\pi$, $k \in \mathbf{Z}$, in exactly one interval.

Suppose now that $f \in SR(p)$ and let $[a_k, b_k]$ denote the unique interval in which A crosses $k\pi$, $k \in \mathbf{Z}$. By Lemma 5, (ii), $b_k \leqq a_{k+1}$ for each $k \in \mathbf{Z}$. Also, (4) implies $[a_{k+2p}, b_{k+2p}] = [a_k + 2\pi, b_k + 2\pi]$, $k \in \mathbf{Z}$.

THEOREM 1. *Let $f \in H(\mathbf{B})$ have exactly p zeros, $z_1, \ldots, z_p$, repeated according to multiplicity. If $f \in SR(p)$, then $\exists u \in H(\mathbf{B})$ with* Re $u > 0$, *real numbers $s_0 \leqq s_1 \leqq \cdots \leqq s_{2p-1} \leqq s_0 + 2\pi$, and $d \in \mathbf{C}\backslash\{0\}$ such that*

$$f(z) = dz^p \prod_{k=0}^{2p-1} \frac{1}{1 - ze^{-is_k}} \prod_{j=1}^{p} \phi(z, z_j)u(z)\,. \tag{9}$$

If f is of the form (9) then $\sigma f \in SR(p)$, where $|\sigma| = 1$ and arg $\sigma = -\arg d - (\pi + \sum_{k=0}^{2p-1} s_k)/2$.

PROOF. Assume $f \in SR(p)$ and let $s_k = (a_k + b_k)/2$, $k \in \mathbf{Z}$. Then $\{s_k\}$ is an increasing sequence, and $s_{k+2p} = s_k + 2\pi$. Define $B: \mathbf{R} \to \mathbf{R}$ as follows: if $s_k < s_{k+1}$ and $t \in (s_k, s_{k+1})$, let $B(t) = (k + 1/2)\pi$; at jump discontinuities, s_k, assign the value $[B(s_k^+) + B(s_k^-)]/2$. Then B is increasing on $(-\infty, \infty)$. If t is a point of continuity of B, i.e., $s_k < t < s_{k+1}$ for some $k \in \mathbf{Z}$, then $t + 2\pi \in (s_k + 2\pi, s_{k+1} + 2\pi) = (s_{k+2p}, s_{k+2p+1})$. Thus, $B(t + 2\pi) = (k + 2p + 1/2)\pi = B(t) + 2p\pi$, so B satisfies (4) a.e. in $\mathbf{R}$. Referring to the proof of Lemma 1, $\exists\ d \in \mathbf{C}\backslash\{0\}$ with arg $d = \int_0^{2\pi}[B(t) - pt]dt/2\pi$ such that

$$g(z) = dz^p \prod_{j=1}^{p} \phi(z, z_k)\exp\left\{-\frac{1}{\pi}\int_0^{2\pi} \log(1 - ze^{-it})dB(t)\right\}$$

has boundary argument B. Furthermore,

$$\int_0^{2\pi} \log(1 - ze^{-it})dB(t) = \int_{a_0}^{a_0+2\pi} \log(1 - ze^{-i(t-a_0)})dB(t - a_0)$$
$$= \pi \sum_{k=0}^{2p-1} \log(1 - ze^{-is_k}),$$

so

$$g(z) = dz^p \prod_{j=1}^{p} \psi(z, z_j) \prod_{k=0}^{2p-1} (1/(1 - ze^{-is_k})).$$

If $u = f/g$, then $\log u \in H^1$, and by proper choice of branch, $A(t) = B(t) + \lim_{r\to 1}\mathrm{Im}\{\log u(re^{it})\}$ a.e. in $\mathbf{R}$. If $s_k < s_{k+1}$ then by Lemma 5, (ii), $k\pi \leqq A(t) \leqq (k + 1)\pi$ a.e. in (s_k, s_{k+1}). Thus, by construction of B, $|\arg u(e^{it})| = |A(t) - B(t)| \leqq \pi/2$ a.e. in $\mathbf{R}$. By the Poisson integral formula, $\mathrm{Re}\, u > 0$, and thus f has the desired form.

Assume now that f is of the form (9). It suffices to consider the case $d = 1$. Let $F(z) = z^p \prod_{k=0}^{2p-1}(1 - ze^{-is_k})^{-1}$ and consider

$$A_F(t) = pt - \sum_{k=0}^{2p-1} \arg(1 - e^{i(t-s_k)}).$$

If $k \in \{0, \ldots, 2p - 1\}$ and $s_k < t < s_{k+1}$, then $-2\pi < t - s_j < 0$ for $j = 0, \ldots, k$ and $0 < t - s_j < 2\pi$ for $j = k + 1, \ldots, 2p$. Choose the branch of $\arg(1 - e^{i\theta})$ which has value $(\theta - \pi)/2$ for $-2\pi < \theta < 0$ and $(\theta + \pi)/2$ for $0 < \theta < 2\pi$. Then for $s_k < t < s_{k+1}$ and $0 \leqq k \leqq 2p - 1$, we have

$$A_F(t) = pt - \frac{1}{2}\sum_{j=0}^{k}(t - s_j - \pi) + \frac{1}{2}\sum_{j=k+1}^{2p-1}(t - s_j + \pi) \tag{10}$$
$$= (k + 1 - p)\pi + s,$$

where $s = \sum_{j=0}^{2p-1} s_j/2$. Extending $\{s_k\}_0^{2p-1}$ to $\{s_k\}_{-\infty}^{\infty}$ so that $s_{k+2p} = s_k + 2\pi$, $k \in \mathbf{Z}$, (10) continues to hold when $s_k < t < s_{k+1}$, $k \in \mathbf{Z}$. For each $k \in \mathbf{Z}$, A_F has a jump discontinuity at s_k, and $A_F(s_k^+) - A_F(s_k^-) = q\pi$, where q is the number of repetitions of the value s_k in $\{s_k\}_{-\infty}^{\infty}$. Let $\sigma = e^{-i(s+\pi/2)}$. Then $A_{\sigma F} = A_F - s - \pi/2$ is increasing on $\mathbf{R}$, and if $t \neq s_k$, $k \in \mathbf{Z}$, then $A_{\sigma F}(t)$ is an odd multiple of $\pi/2$. Thus, for each $\ell \in \mathbf{Z}$, $A_{\sigma F}$ crosses $\ell\pi$ at a unique point, say c, in $\{s_k\}_{-\infty}^{\infty}$, and in fact, $A_{\sigma F}(t) \leqq \ell\pi - \pi/2$ a.e. in $(-\infty, c)$ and $A_{\sigma F}(t) \geqq \ell\pi + \pi/2$ a.e. in $(c, +\infty)$. Finally, $A_{\sigma f}(t) = A_{\sigma F}(t) + \arg u(e^{it})$ and $|\arg u(e^{it})| \leqq \pi/2$ a.e. in $\mathbf{R}$, so $A_{\sigma f}(t) \leqq \ell\pi$ a.e. in $(-\infty, c)$ and $A_{\sigma f}(t) \geqq \ell\pi$ a.e. in $(c, +\infty)$. If $A_{\sigma f}$ crosses $\ell\pi$ in $[a, b]$, then $a \leqq c \leqq b$. By Lemma 5, (ii), $\sigma f \in SR(p)$.

THEOREM 2. *Suppose $f \in H(\mathbf{B})$ has exactly p zeros, $z_1, \ldots, z_p$, repeated according to multiplicity and real coefficients in its Taylor expansion about* *zero. Then* $f \in T(p)$ *if and only if* $\exists u \in H(\mathbf{B})$ *with* $\operatorname{Re} u > 0$, *real numbers* $0 \leqq s_1 \leqq \cdots \leqq s_{p-1} \leqq \pi$, *and* $d \in \mathbf{R}\backslash\{0\}$ *such that*

$$(11) \qquad f(z) = d\frac{z}{1-z^2}\prod_{k=1}^{p-1}\frac{z}{1-2z\cos s_k + z^2}\prod_{j=1}^{p}\phi(z, z_j)u(z).$$

Proof. Suppose $f \in T(p)$. Since $f(\bar{z}) = \overline{f(z)}$, $z \in \mathbf{B}$, the non-real zeros of f occur in conjugate pairs. Thus, $\prod_{j=1}^{p}\phi(z, z_j)$ is real-valued for $z \in (-1, 1)$, and $f(z)/z^p \prod_{j=1}^{p}\phi(z, z_j)$ is of constant sign on $(-1, 1)$. Assuming the sign is positive, (otherwise consider $-f$) there is a branch of L with real Taylor coefficients and hence a branch of $A(t) = pt + \lim_{r\to 1}\operatorname{Im}L(re^{it})$ which satisfies $A(0) = 0$, $A(\pi) = p\pi$ and $A(-t) = -A(t)$. Also, from (4),

$$(12) \qquad A(\pi + t) - p\pi = p\pi - A(\pi - t)$$

a.e. in $\mathbf{R}$. Let $a = \sup\{y \geqq 0\colon A(t) = 0$ a.e. in $[0, y]\}$. Since A is odd, A crosses 0 in $[-a, a]$ i.e., $[a_0, b_0] = [-a, a]$. If $b = \inf\{y \leqq \pi\colon A(t) = p\pi$ a.e. in $[y, \pi]\}$, then (12) implies that $[a_p, b_p] = [b, 2\pi - b]$. Finally, for $1 \leqq k \leqq p - 1$, A crosses $k\pi$ in $[a_k, b_k] \subset [0, \pi]$, and by (12), $[a_{2p-k}, b_{2p-k}] = [2\pi - b_k, 2\pi - a_k]$. Setting $s_k = (a_k + b_k)/2$, we have $s_0 = 0$, $s_p = \pi$, $s_{2p-k} = 2\pi - s_k$, $1 \leqq k \leqq p - 1$, and

$$\prod_{k=0}^{2p-1}\frac{1}{(1 - ze^{-is_k})} = \frac{1}{1-z^2}\prod_{k=1}^{p-1}\frac{1}{(1-ze^{-is_k})(1-ze^{-i(2\pi - s_k)})}$$
$$= \frac{1}{1-z^2}\prod_{k=1}^{p-1}\frac{1}{(1-2z\cos s_k + z^2)}.$$

As in the proof of Theorem 1 we conclude that f is of the form (11) for some $d \in \mathbf{C}\backslash\{0\}$ with $\arg d = \int_0^{2\pi}[B(t) - pt]dt/2\pi$. But A being odd implies B is odd, and $B(t) - pt$ is periodic of period 2π, so $\arg d = 0$, i.e., $d \in \mathbf{R}\backslash\{0\}$.

If f is given by (11), then f has the form (9) with $s_0 = 0$, $s_p = \pi$, and $s_{2p-k} = 2\pi - s_k$, $1 \leqq k \leqq p - 1$. From Theorem 2, $\sigma f \in SR(p)$, where $\sigma = e^{-i(s+\pi/2)}$ and $s = \sum_{j=1}^{2p-1}s_j/2 = \pi p - \pi/2$. But $\sigma f = (-1)^p f \in SR(p)$ implies $f \in SR(p)$, and since f has real coefficients, $f \in T(p)$.

Theorem 3. *If* $f_n \in T(p)$, $n = 1, 2, \ldots, f_n \to f$ *locally uniformly in* $\mathbf{B}$, *and* f *has* p *zeros in* $\mathbf{B}$, *then* $f \in T(p)$.

Proof. Suppose f has zeros $z_1, \ldots, z_p$. By Hurwitz's Theorem, we can index the zeros $z_{n,1}, \ldots, z_{n,p}$ of f_n so that $\lim_{n\to\infty} z_{n,j} = z_j$, $j = 1, \ldots, p$. By Theorem 2, $\exists u_n \in H(\mathbf{B})$ with $\operatorname{Re} u_n > 0$, real numbers $0 \leqq s_{n,1} \leqq \cdots \leqq s_{n,p-1} \leqq \pi$, and $d_n \in R\backslash\{0\}$ such that f_n has representation (11). By adjusting d_n we may assume $u_n(0) = 1$. Extracting subsequences, if necessary, we may assume that $u_n \to u$ locally uniformly in $\mathbf{B}$ and that

$s_{n,k} \to s_k$, $k = 1, 2, \ldots, p-1$. Furthermore, $u(z) \neq 0$, $z \in \mathbf{B}$, so $\{d_n\}_1^\infty$ is bounded, and we may assume $d_n \to d \neq 0$. Thus,

$$f(z) = \lim f_n(z) = d\frac{z}{1-z^2}\prod_{k=1}^{p-1}\frac{z}{1-2z\cos s_k + z^2}\prod_{j=1}^{p}\phi(z, z_j)u(z),$$

and by Theorem 2, $f \in T(p)$.

Now, suppose f has exactly p zeros, $z_1, \ldots, z_p$, in $\mathbf{B}$ and real Taylor coefficients. Assume f satisfies condition (I) and let $h(z) = f(z)/\prod_{j=1}^{p}\phi(z, z_j) = a_p z^p + \cdots$, where $a_p \in \mathbf{R}\backslash\{0\}$. If $z \in \partial\mathbf{B}$, then $\prod_{j=1}^{p}\phi(z, z_j) > 0$, so $\operatorname{Im} f(z)$ and $\operatorname{Im} h(z)$ have the same sign. Thus, h satisfies condition (I). By Robertson's work [**8**], h/a_p is of the form (1), and by Theorem 2, $f \in T(p)$. If, instead, f satisfies (II), then $\exists \rho \in (0, 1)$ such that $f(rz)$ satisfies (I) for $\rho < r < 1$, and by Theorem 3, $f \in T(p)$. We have established the following result.

COROLLARY. *Suppose $f \in H(\mathbf{B})$ has exactly p zeros in* $\mathbf{B}$ *and real Taylor coefficients. If f satisfies* (I) *or* (II), *then $f \in T(p)$.*

4. Valence and critical points on $f^{-1}(\mathbf{R})$. In the present section we prove the following theorem.

THEOREM 4. *If $f \in T(p)$, then*

(i) *for each $x \in \mathbf{R}$, $f(z) - x = 0$ has at most p roots in* $\mathbf{B}$, *counting multiplicity, and*

(ii) *f' has at most $p - 1$ zeros in f^{-1}* $(\mathbf{R})$, *counting multiplicity.*

We begin by considering a dense subclass of $T(p)$ consisting of certain functions which are meromorphic on $\bar{\mathbf{B}}$. For such functions, g, we shall view $\overline{g^{-1}(\mathbf{R})}$ as a directed planar graph and obtain (i) and (ii) by counting arguments. Conclusion (i) then follows for the full class, $T(p)$, by Hurwitz's Theorem. To prove (ii) for $T(p)$ we shall use a modification of the argument for the special subclass. The reader is referred to [**1**] for the graph theoretic definitions and results which will be used.

DEFINITION. Let $T_m(p)$ denote the class of functions $g \in T(p)$ which are meromorphic on $\bar{\mathbf{B}}$ and for which $g(z) \notin \mathbf{R}$ for $z \in \partial\mathbf{B}$.

LEMMA 6. *$T_m(p)$ is dense in $T(p)$.*

PROOF. If $f \in T(p)$, then by Theorem 2,

$$f(z) = d\frac{z}{1-z^2}\prod_{k=1}^{p-1}\frac{z}{1-2z\cos s_k + z^2}\prod_{j=1}^{p}\phi(z, z_j)u(z),$$

where $d \in \mathbf{R}\backslash\{0\}$, $0 \leqq s_1 \leqq \cdots \leqq s_{p-1} \leqq \pi$, and $\operatorname{Re} u > 0$. For $0 < r < 1$, let $g_r(z) = f(z)u(rz)/u(z)$. Clearly $g_r \to f$ as $r \to 1$. By Theorem 2,

$g_r \in T(p)$. If $z_0 \in \partial\mathbf{B}$ and $g_r(z_0) \neq \infty$, then $g_r(z_0)/u(rz_0)$ is pure imaginary, but $u(rz_0)$ is not. Thus, $g_r(z_0) \notin \mathbf{R}$, and $g_r \in T_m(p)$.

To each $g \in T_m(p)$ we associate a finite, directed, planar graph, G, in the following fashion. The vertices of G are

$$V = \{v \in \overline{g^{-1}(\mathbf{R})}\text{: either } g(v) = 0,\ g'(v) = 0, \text{ or } v \in \partial\mathbf{B}\}\,.$$

Each v in $V \cap \partial\mathbf{B}$ is a pole of g. The edges of G are the components of $g^{-1}(\mathbf{R})\backslash V$. If e is an edge of G, then $g(e) \subset \mathbf{R}\backslash\{0\}$ and g' has no zeros on e. Thus, e may be so directed that $|g(z)|$ is strictly increasing as z traverses e from initial to terminal vertex. The number of edges for which v is the initial or terminal vertex is denoted by $\delta^+(v)$ or $\delta^-(v)$ respectively. The degree of v is $\delta(v) = \delta^+(v) + \delta^-(v)$. Let $P = V \cap \partial\mathbf{B}$, $Z = \{v \in V\colon g(v) = 0\}$, $C = \{v \in V\colon g'(v) = 0\}$ and $Q = V\backslash(P \cup C)$. Although $Z \cap C$ may be non-empty, P, C and Q partition V. Q consists of simple zeros of g.

LEMMA 7. *Assume* $g \in T_m(p)$.
(i) *If* $v \in P$, *then* $\delta^+(v) = 0$.
(ii) *If* v *is a zero of order* m *of* g, *then* $\delta^-(v) = 0$ *and* $\delta(v) = \delta^+(v) = 2m$.
(iii) *If* $v \in C$ *is a critical point of order* m, *then* $\delta(v) = 2(m + 1)$.
(iv) $\sum_P \delta(v) = \sum_Z \delta(v) = 2p$.

PROOF. Items (i), (ii) and (iii) follow directly from local analysis. From (i), (ii) and [1, p. 37] we have

$$\sum_P \delta(v) = \sum_P [\delta^-(v) - \delta^+(v)] = \sum_Z [\delta^+(v) - \delta^-(v)] = \sum_Z \delta^+(v) = 2p\,.$$

A directed graph is said to be pseudosymmetric if $\delta^+(v) = \delta^-(v)$ for each vertex, v. For the graph, G, of $g \in T_m(p)$, $\delta^+(v) = \delta^-(v)$ only if $v \in C\backslash Z$. Each $v \in C\backslash Z$ is the terminal vertex of some edge, e, and the initial vertex of some other edge, e'. The initial vertex of e is either in Z of is the terminal vertex of some other edge. Similarly, if the terminal vertex of e' is not in P, then it is the initial vertex of another edge. With a finite number of such steps one constructs a simple path in G which begins at a zero of g, ends at a pole of g, and along which $|g|$ is strictly increasing. Thus, each component of G contains both a zero and a pole of g, and no component of G is pseudosymmetric.

Let ν_c, ν_f, ν_e and ν_v denote the number of components, faces, edges and vertices of G, respectively. By Euler's formula [6, p. 48], $1 = \nu_f - \nu_e + \nu_v - \nu_c$.

LEMMA 8. *If* $g \in T_m(p)$ *and* $x \in \mathbf{R}$, *then* $g(z) - x = 0$ *has at most* p *roots in* $\mathbf{B}$, *counting multiplicity. Furthermore,* g' *has at most* $p - \nu_c$ *zeros in* $g^{-1}(\mathbf{R})$, *counting multiplicity.*

PROOF. From the remarks above, we may apply Theorem 3 – 5 in [1, p. 37] to each component of G. We conclude that the number of paths in a minimal covering of G (see [1, p. 34, 36]) is $\sum_Z[\delta^+(v) - \delta^-(v)] = \sum_Z\delta(v) = 2p$ (by Lemma 7), and each path in such a covering joins a vertex in Z to one in P. The edges are so directed that $|g|$ is strictly increasing along each of these paths, and local analysis at each $v \in Z$ shows that g is positive on p of these paths and negative on the others. Thus, if $x \in \mathbf{R}\backslash\{0\}$, then there are at most p of these paths which contain a root of $g(z) - x = 0$. Moreover, if z_0 is a multiple root of order m, then exactly m of these paths pass through z_0. This establishes the first conclusion.

Suppose g' has zeros $\zeta_1, \ldots, \zeta_k$ in $g^{-1}(\mathbf{R})$ with multiplicities $m_1, \ldots, m_k$, respectively. By Lemma 7, (iii), $\delta(\zeta_j) = 2(m_j + 1)$, $1 \leqq j \leqq k$. Also, $\delta(v) = 2$ if $v \in Q$. Thus,

$$\sum_1^k m_j = \frac{1}{2}\sum_1^k(\delta(\zeta_j)-2) = \frac{1}{2}\sum_{C\cup Q}(\delta(v)-2)$$
$$= \frac{1}{2}(\sum_V - \sum_P)(\delta(v) - 2)\,.$$

Now, $2\nu_e = \sum_V\delta(v)$, [1, p. 9], and by Lemma 7, (iv), $2p = \sum_P\delta(v)$. Thus,

$$\sum_1^k m_j = \nu_e - \nu_v - \sum_P[(1/2)\delta(v)-1] = p + \nu_e - \nu_v - \sum_P(\delta(v)-1)\,,$$

and by Euler's formula,

$$\sum_1^k m_j = p - \nu_c + [\nu_f - 1 - \sum_P(\delta(v)-1)]\,. \tag{13}$$

Each vertex, v, lies on the boundary of at most $\delta(v) - 1$ bounded faces of G. If F is a bounded face of G, then $\partial F \cap P \neq \phi$, for otherwise, $g \in H(\bar{F})$ and $\operatorname{Im} g = 0$ on ∂F would imply g is constant. Thus, the number of bounded faces of G, i.e., $\nu_f - 1$, is at most $\sum_p(\delta(v) - 1)$, which together with (13) gives the desired conclusion.

PROOF OF THEOREM 4. As remarked earlier, (i) follows from Lemmas 6 and 8. As for (ii), suppose $v_1, \ldots, v_\ell$ are zeros of f' in $f^{-1}(\mathbf{R})$ with orders $n_1, \ldots, n_\ell$, respectively. We shall show that $\sum_1^\ell n_i \leqq p - 1$. Let $B(a, r) = \{z: |z - a| < r\}$. For $1 \leqq i \leqq \ell$, let $B_i = B(v_i, r_i)$, where r_i is sufficiently small that $\bar{B}_i \subset \mathbf{B}$, the functions f' and $f - f(v_i)$ have no zeros in $\bar{B}_i\backslash\{v_i\}$, and $\bar{B}_i \cap \bar{B}_j = \phi$ for $i \neq j$. Let $\varepsilon_i = \operatorname{dist}(f(\partial B_i), f(v_i))$ and let $\varepsilon = \min\{\varepsilon_i: 1 \leqq i \leqq \ell\}$. By Lemma, 6, $\exists g \in T_m(p)$ such that $|f(z) - g(z)| < \varepsilon/2$ for $z \in \bigcup_1^\ell \bar{B}_i$. We may assume, by Hurwitz's Theorem, that g' has exactly n_i zeros in B_i and that their images under g lie in $B(f(v_i), \varepsilon/2)$, $1 \leqq i \leqq \ell$. Let $\zeta_1, \cdots, \zeta_k$ denote the zeros of g' in $\bigcup_1^\ell B_i$ with respective orders $m_1, \cdots,$

m_k. Then $\sum_1^\ell n_i = \sum_1^k m_j$, and (ii) would follow directly from Lemma 8 if $\zeta_j \in g^{-1}(\mathbf{R})$ for each $1 \leqq j \leqq k$. Suppose $1 < q \leqq k$, $g(\zeta_j) \notin \mathbf{R}$ for $1 \leqq j \leqq q$, and $g(\zeta_j) \in \mathbf{R}$ otherwise. As in the proof of Lemma 8,

$$\sum_1^k m_j \leqq \sum_1^q m_j + p + \nu_e - \nu_v - \sum_P (\delta(v) - 1)\,. \tag{14}$$

We shall enlarge the graph, G, of g so as to include $\zeta_1, \cdots, \zeta_q$ as vertices. By the choice of ε, $f(z) - w = 0$ has exactly $n_i + 1$ roots in B_i for each $w \in B(f(v_i), \varepsilon)$, $1 \leqq i \leqq \ell$. Since $|f(z) - g(z)| < \varepsilon/2$ for $z \in \partial B_i$, $g(z) - w = 0$ has exactly $n_i + 1$ roots in B_i for each $w \in B(f(v_i), \varepsilon/2)$, $1 \leqq i \leqq \ell$. Suppose $1 \leqq j \leqq q$ and $\zeta_j \in B_i$. Let Γ_j be a Jordan curve in $B(f(v_i), \varepsilon/2)$ with initial point $g(\zeta_j)$ and terminal point $w_j \in \mathbf{R}\backslash\{0\}$. We may assume that $\Gamma_j\backslash\{g(\zeta_j)\}$ contains no branch point of g and that $(\Gamma_j\backslash\{w_j\}) \cap \mathbf{R} = \phi$. Corresponding to Γ_j under g there are Jordan curves $\gamma_{j,s}$, $1 \leqq s \leqq m_j + 1$ in B_i which have ζ_j as initial point and are otherwise pairwise disjoint. If $z_{j,s}$ denotes the terminal point of $\gamma_{j,s}$, then $z_{j,s} \in g^{-1}(\mathbf{R})\backslash V$ and $(\gamma_{j,s}\backslash\{z_{j,s}\}) \cap g^{-1}(\mathbf{R}) = \phi$, $1 \leqq s \leqq m_j + 1$. We may require also that $\Gamma_i = \Gamma_j$ when $g(\zeta_i) = g(\zeta_j)$ and $\Gamma_i \cap \Gamma_j = \phi$ otherwise. Let $\hat{G}$ denote the finite, planar graph whose vertices are

$$\hat{V} = V \cup \{\zeta_j\colon 1 \leqq j \leqq q\} \cup \{z_{j,s}\colon 1 \leqq j \leqq q,\ 1 \leqq s \leqq m_j + 1\}$$

and whose edges are the components of

$$\{g^{-1}(\mathbf{R}) \cup (\bigcup[\gamma_{j,s}\colon 1 \leqq j \leqq q,\ 1 \leqq s \leqq m_j + 1])\}\backslash\hat{V}\,.$$

Each new vertex, $z_{j,s}$, lies on an edge of G. The addition of a new vertex to a previous edge increases the edge count by 1. There are $\sum_1^q(m_j + 1)$ new vertices, $z_{j,s}$, and $\sum_1^q(m_j + 1)$ edges $\gamma_{j,s}$. Thus, $\hat{\nu}_e = \nu_e + 2\sum_1^q(m_j + 1)$. Since $\hat{\nu}_v = \nu_v + q + \sum_1^q(m_j + 1)$, we have $\nu_e - \nu_v = \hat{\nu}_e - \hat{\nu}_v - \sum_1^q m_j$. Thus, (14) becomes $\sum_1^k m_j \leqq p + \hat{\nu}_e - \hat{\nu}_v - \sum_P(\delta(v) - 1)$, and Euler's formula, applied to $\hat{G}$, yields

$$\sum_1^k m_j \leqq p - \hat{\nu}_c + [\hat{\nu}_f - 1 - \sum_P(\delta(v) - 1)]\,. \tag{15}$$

Let $\hat{F}$ be a bounded face of $\hat{G}$. If $\partial\hat{F} \cap P = \phi$, then $g(\hat{F})$ is a bounded open set and $\partial g(\hat{F})$ disconnects the plane. But $\partial g(\hat{F}) \subset g(\partial\hat{F})$, which is a bounded subset of $\mathbf{R} \cup (\bigcup_1^q\Gamma_j)$. The curves Γ_j, $1 \leqq j \leqq q$, were so chosen that no bounded subset of $\mathbf{R} \cup (\bigcup_1^q\Gamma_j)$ disconnects the plane. Thus, $\partial\hat{F} \cap P \neq \phi$ for each bounded face $\hat{F}$ of $\hat{G}$. Also, the order in $\hat{G}$ of each vertex in P is the same as that in G. As in the proof of Lemma 8, the quantity in braces in (15) is nonpositive, and $\sum_1^k m_j \leqq p - \hat{\nu}_c \leqq p - 1$.

5. Examples. The first example shows that $T(p)$ properly contains the class of functions satisfying (I) or (II). In the second example we show that each member of $T_m(b)$ satisfies (II). It then follows from Lemma 6 and Example 1 that sequences of functions satisfying (I) or (II) may converge to a limit function which does not satisfy (I) or (II).

EXAMPLE 1. Assume $p > 1$. Let $\Omega = \mathbf{B}\backslash K$, where

$$K = \{re^{i\theta}: |\theta| = 3\pi/2p,\quad 1/2 \leqq r < 1\} \cup \left(\bigcup_{n=1}^{\infty}\left\{\frac{ne^{i\theta}}{n+1} : \pi/p \leqq |\theta| \leqq 3\pi/2p\right\}\right),$$

and let $\phi \in H(\mathbf{B})$ be the univalent map of $\mathbf{B}$ onto Ω satisfying $\phi(0) = 0$, $\phi'(0) > 0$. Since Ω is symmetric about $\mathbf{R}$, $\phi \in T(1)$. Let $e^{i\alpha}$, $0 < \alpha < \pi$, be the point of $\partial\mathbf{B}$ which corresponds to the prime end of the second kind of Ω in the upper half-plane, and let β be the unique number in (α, π) such that $\phi(e^{i\beta}) = e^{3\pi i/2p}$. If A_ϕ is the boundary argument for ϕ satisfying $A_\phi(0) = 0$, then A_ϕ is strictly increasing on $[0, \alpha) \cup (\beta, \pi]$, $\pi/p \leqq A_\phi(t) \leqq 3\pi/2p$ for $\alpha \leqq t \leqq \beta$, and for each $k \in \{1, \cdots, p-1\}$, A_ϕ crosses $k\pi/p$ eactly once between 0 and π. Since A_ϕ is odd and satisfies $A_\phi(t + 2\pi) = A_\phi(t) + 2\pi$, we conclude that A_ϕ crosses each value $k\pi/p$, $k \in \mathbf{Z}$, exactly once. If $f = \phi^p$ and $A_f = pA_\phi$, then A_f crosses $k\pi$ exactly once for each $k \in \mathbf{Z}$, and consequently, $f \in T(p)$. Clearly, $f \notin H(\bar{\mathbf{B}})$. For each positive $N \in Z$ there is an $r \in (0, 1)$ sufficiently near 1 that the image of $|z| = r$ under ϕ crosses the ray $\arg w = \pi/p$ more than N times. Thus, Im f changes sign more than N times, and f does not satisfy (II).

EXAMPLE 2. Let $g \in T_m(p)$ and $\rho \in (0, 1)$ such that all edges of G except those which terminate at a pole of g lie in $|z| < \rho$. By local analysis at each $v \in P$ we may choose ρ so large that each edge which terminates at a pole of g crosses $|z| = r$, $\rho < r < 1$, exactly once. By Lemma 7, (iv), $\sum_P \delta(v) = 2p$, so Im g changes sign $2p$ times on $|z| = r$, $\rho < r < 1$, and g satisfies (II).

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