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LOWER BOUNDS FOR THE HYPERBOLIC METRIC IN CONVEX REGIONS

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1. Introduction. Let $\lambda_\Omega(z)\,|dz|$ denote the hyperbolic metric on a hyperbolic region Ω in the complex plane $\mathbf{C}$. For convex regions we shall give sharp lower bounds for $\lambda_\Omega(z)$ in terms of the geometric quantity $\delta_\Omega(z)$, the distance from the point z to the boundary of Ω. In §3 we obtain a lower bound that applies to all hyperbolic convex regions. Then in §4 we derive a lower bound that is valid for any convex region with the property that δ_Ω is uniformly bounded in Ω. Each of these lower bounds leads to distortion and covering theorems for a certain family of possibly multiple-valued analytic functions defined in the unit disk. In particular, we obtain classical covering theorems for normalized convex univalent functions defined in the unit disk, including the fact that Bloch-Landau constant is $\pi/4$ for such functions. In order to obtain these distortion and covering theorems from the lower bounds for the hyperbolic metric, we require a generalization of the principle of hyperbolic metric which is given in §2. In this section we also present other results about the hyperbolic metric.

2. The hyperbolic metric. We begin this section with a brief introduction to the hyperbolic metric. For a general discussion of the hyperbolic metric we refer the reader to [2], [3] and [7]

Let Ω be a hyperbolic region in the complex plane; that is, the complement of Ω in $\mathbf{C}$ contains at least two points. Then there is an analytic universal covering projection φ of the open unit disk $\mathbf{B}$ onto Ω. If Ω is simply connected, then φ is just a conformal mapping of $\mathbf{B}$ onto Ω. The hyperbolic metric $\lambda_\Omega(z)|dz|$ on Ω is defined as follows: if $a \in \Omega$ and $b \in \varphi^{-1}(a)$, then

$$\lambda_\Omega(a) = 2/|\varphi'(b)|(1 - |b|^2).$$

The value of $\lambda_\Omega(a)$ is independent of both the choice of $b \in \varphi^{-1}(a)$ and the selection of the covering φ. The collection of all analytic coverings of $\mathbf{B}$ onto Ω consists of the functions $\varphi \circ T$, where T is any conformal automorphism of $\mathbf{B}$. Hence, for any fixed $a \in \Omega$, there is a unique analytic covering φ for which $\varphi(0) = a$ and $\varphi'(0) > 0$. In this case, $\lambda_\Omega(a) = 2/\varphi'(0)$.

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The function $\lambda_\Omega(z)$ is real-analytic on Ω and the metric $\lambda_\Omega(z)|dz|$ has constant (Gaussian) curvature -1. Recall that

$$\kappa(z) = -\Delta \log \lambda_\Omega(z)/\lambda_\Omega^2(z)$$

is the curvature of $\lambda_\Omega(z)\,|dz|$.

EXAMPLES. (i) $\lambda_{\mathbf{B}}(z) = 2/(1 - |z|^2)$.

(ii) $\lambda_{\mathbf{H}}(z) = 1/\mathrm{Re}(z)$, where $\mathbf{H} = \{z\colon \mathrm{Re}(z) > 0\}$.

(iii) $\lambda_{S(M)}(z) = \pi/2M \sin((\pi/2M)\mathrm{Re}(z))$, where $S(M) = \{z\colon 0 < \mathrm{Re}(z) < 2M\}$.

Note that the definition of the hyperbolic metric gives $\lambda_\Omega(\varphi(z))|\varphi'(z)| = \lambda_{\mathbf{B}}(z)$, whenever φ is an analytic covering of $\mathbf{B}$ onto Ω. We shall make use of the following two elementary properties of the hyperbolic metric which are stated without proof. Assume Ω and Δ are hyperbolic plane regions.

CONFORMAL INVARIANCE. *If f is a conformal mapping of Ω onto Δ, then $\lambda_\Delta(f(z))\,|f'(z)| = \lambda_\Omega(z)$ for $z \in \Omega$.*

MONOTONICITY. *If $\Omega \subset \Delta$, then $\lambda_\Delta(z) \leqq \lambda_\Omega(z)$ for $z \in \Omega$. If equality holds at a point, then $\Omega = \Delta$.*

For a hyperbolic region Ω and $z \in \Omega$ let $\delta_\Omega(z) = \min\{|z - c|\colon c \in \partial\Omega\}$. Thus, $\delta_\Omega(z)$ is just the distance between z and $\partial\Omega$; it is the radius of the largest open ball centered at z which is contained in Ω. It is elementary to show that $\lambda_\Omega(z) \leqq 2/\delta_\Omega(z)$. If equality holds at $a \in \Omega$, then Ω is a disk with center a. In subsequent sections we shall obtain sharp lower bounds for λ_Ω in terms of δ_Ω when Ω is a convex region.

The next two theorems generalize results of Nehari [6].

THEOREM 1. *Let Ω be a hyperbolic region in $\mathbf{C}$ Suppose f is analytic at $a \in \mathbf{B}$ and $f(a) \in \Omega$. If $\lambda_\Omega(f(z))\,|f'(z)| \leqq \lambda_{\mathbf{B}}(z)$ for all z in a neighborhood of a with equality at $z = a$, then f is an analytic universal covering of B onto Ω.*

PROOF. First, we show that we may specialize to consideration of the case in which $a = 0$, $\Omega = \mathbf{B}$ and $f(a) = 0$. Let $\varphi\colon \mathbf{B} \to \Omega$ be an analytic universal covering with $\varphi(0) = f(a)$ and let T be a conformal automorphism of $\mathbf{B}$ which sends 0 to a. Take φ^{-1} to denote the branch of the inverse of φ which is analytic at $f(a)$ and maps $f(a)$ to 0. Then $g = \varphi^{-1} \circ f \circ T$ is analytic at $0, g(0) = 0$ and $\lambda_{\mathbf{B}}(g(z))|g'(z)| \leqq \lambda_{\mathbf{B}}(z)$ for all z in a neighborhood of 0 with equality at $z = 0$. Now, we show that g is a conformal automorphism of $\mathbf{B}$. Note that $|g'(0)| = 1$. Select $r > 0$ so that g is analytic in $B(r) = \{z\colon |z| < r\}$ and maps $B(r)$ into $\mathbf{B}$. Let $d(0, z)$ denote the hyperbolic distance in $\mathbf{B}$ between the points $0, z$. Then $d(0, z) = \log(1 + |z|)/(1 - |z|)$. If γ denotes the radial path from 0 to $z \in B(r)$, then

$$d(0, z) = \int_\gamma \lambda_{\mathbf{B}}(z)\,|dz| \geqq \int_\gamma \lambda_{\mathbf{B}}(g(z))\,|g'(z)|\,|dz|$$
$$= \int_{g\circ\gamma} \lambda_{\mathbf{B}}(z)\,|dz| \geqq d(0, g(z)).$$

Since $t \to \log(1+t)/(1-t)$ is strictly increasing on $(0, 1)$, this implies that $|g(z)| \leqq |z|$ for $|z| < r$, so that g maps $B(r)$ into itself. Schwarz' lemma applied to g on $B(r)$ gives $|g'(0)| \leqq 1$ with equality only if $g(z) = e^{i\theta}z$, where $\theta \in \mathbf{R}$. Because $|g'(0)| = 1$, we conclude that $g(z) = e^{i\theta}z$ for some $\theta \in \mathbf{R}$. Then $f(z) = \varphi(e^{i\theta}T^{-1}(z))$ is an analytic universal covering of $\mathbf{B}$ onto Ω.

COROLLARY. *Suppose Ω and Δ are hyperbolic regions in $\mathbf{C}$ with $\Omega \cap \Delta \neq \emptyset$. Let $a \in \Omega \cap \Delta$ and assume that $\lambda_\Omega(z) \leqq \lambda_\Delta(z)$ for all z in a neighborhood of a with equality at $z = a$. Then $\Omega = \Delta$.*

PROOF. Let $\varphi: \mathbf{B} \to \Omega$ and $\psi: \mathbf{B} \to \Delta$ be analytic universal covering projections which send 0 to a and have positive derivative at the origin. Take φ^{-1} to be the branch of the inverse of φ at a which satisfies $\varphi^{-1}(a) = 0$. Then $f = \varphi^{-1} \circ \psi$ is analytic at 0, $f(0) = 0$ and satisfies

$$\lambda_{\mathbf{B}}(f(z))\,|f'(z)| = \lambda_\Omega(\psi(z))|\psi'(z)| \leqq \lambda_\Delta(\psi(z))\,|\psi'(z)| = \lambda_{\mathbf{B}}(z)$$

for z near 0 with equality at $z = 0$. The theorem implies that f is the identity function because $f'(0) > 0$. Hence, $\varphi = \psi$ so that $\Omega = \Delta$.

We shall establish a generalization of the principle of hyperbolic metric for a certain class of multiple-valued analytic functions. For this reason it is useful to define the following classes of functions.

DEFINITION. Let $\mathfrak{F}$ denote the family of functions f with the following property: there is a discrete subset E of $\mathbf{B}$, depending on f, such that f is analytic on $\mathbf{B}\backslash E$, f has an algebraic branch point at each point of E and $f'(z)$ is finite everywhere in $\mathbf{B}$. Let $\mathfrak{F}_0$ be the subfamily consisting of all $f \in \mathfrak{F}$ such that $f(0) = 0$ and $f'(0) = 1$ for some branch of f at the origin.

For a function $f \in \mathfrak{F}$ and $z \in \mathbf{B}$, we shall let $f(z)$, $f'(z)$ denote the value of the function f and its derivative at z using some fixed branch of the function. Also, $f(\mathbf{B})$ will denote the set of all values $f(z)$ as z ranges over $\mathbf{B}$ and we evaluate the function at all possible branches at z.

THEOREM 2. *Let Ω be a hyperbolic region in $\mathbf{C}$. Suppose $f \in \mathfrak{F}$ and $f(\mathbf{B}) \subset \Omega$. Then for any $z \in \mathbf{B}$ and any branch of f at z,*

$$\lambda_\Omega(f(z))\,|f'(z)| \leqq \lambda_{\mathbf{B}}(z). \tag{1}$$

If equality holds at a point, then f is an analytic universal covering of $\mathbf{B}$ onto Ω.

PROOF. In order to establish the inequality, it suffices to demonstrate that for $z \in B(r) = \{z: |z| < r\}$ and $0 < r < 1$ we have

(2) $$\lambda_\Omega(f(z))|f'(z)| \leqq \lambda_{B(r)}(z) = 2r/(r^2 - |z|^2).$$

We may then fix z and let r tend to 1 to obtain the desired inequality. If we restrict our attention to $B(r)$, then there are just finitely many branches of f determined at each point of $B(r)$. For $z \in B(r)$ let $\rho(z) = \max\{\lambda_\Omega(f(z))|f'(z)|\}$, where the maximum is taken over the finite number of branches of f at z determined via continuation in $B(r)$. We shall show that $\rho(z)|dz|$ is an ultrahyperbolic metric on $B(r)$. Then inequality (2) is a consequence of Ahlfors' extension of Schwarz' lemma ([1], [2, p. 13]). Clearly, ρ is continuous on $B(r)$. To show that $\rho(z)|dz|$ is ultrahyperbolic, we must demonstrate that there is a supporting metric at each point where ρ does not vanish. Fix $a \in B(r)$ with $\rho(a) \neq 0$. Select a branch of f at a such that $\rho(a) = \lambda_\Omega(f(a))|f'(a)|$. Then for z near a $\rho_a(z) = \lambda_\Omega(f(z))|f'(z)| \leqq \rho(z)$ with equality at $z = a$. Also, since the metric $\lambda_\Omega(z)|dz|$ has constant curvature -1, it follows that $\rho_a(z)|dz|$ has constant curvature -1 near a. Thus, $\rho_a(z)|dz|$ is a supporting metric for $\rho(z)\,|dz|$ at a, so (2) is established.

In order to complete the proof, we must investigate the situation when equality holds in (1). Assume that equality holds in (1) at some point $a \in \mathbf{B}$. This means that there is a branch of f defined in a neighborhood of a such that $\lambda_\Omega(f(z))|f'(z)| \leqq \lambda_{\mathbf{B}}(z)$ for z near a with equality at $z = a$. Then Theorem 1 implies that f is an analytic universal covering of $\mathbf{B}$ onto Ω.

THEOREM 3. *Let Ω be a hyperbolic region in $\mathbf{C}$. Suppose that λ_Ω has a local minimum at the point $a \in \Omega$. If $\varphi: \mathbf{B} \to \Omega$ is an analytic universal covering with $\varphi(0) = a$, then $\varphi''(a) = 0$.*

PROOF. Under the hypotheses of the theorem, we have $\lambda_\Omega(a) = 2/|\varphi'(0)|$, $\lambda_\Omega(\varphi(z)) = 2/|\varphi'(z)|(1 - |z|^2)$. Consequently, if λ_Ω has a local minimum at a, then for z near the origin

(3) $$0 \leqq \lambda_\Omega(\varphi(z)) - \lambda_\Omega(a).$$

Suppose that $\varphi(z) = a + a_1 z + a_2 z^2 + \cdots$, where we may assume $a_1 = \varphi'(0) > 0$ without loss of generality. Then

$$\frac{2}{|\varphi'(z)|(1 - |z|^2)} = \frac{2}{\varphi'(0)}\left\{1 - \frac{2}{\varphi'(0)}\operatorname{Re}(a_2 z) + O(|z|^2)\right\}$$

so that (3) yields $0 \leqq (-4/\varphi'(0)^2)\operatorname{Re}(a_2 z) + O(|z|^2)$ for z near the origin. Because the argument of z is unrestricted and we may let $z \to 0$, this inequality implies $0 = a_2 = \varphi''(0)/2$.

3. Lower bound in an arbitrary convex region. We now derive a sharp lower bound for the hyperbolic metric in a convex region by using elementary methods. The bound leads to a distortion theorem for functions in $\mathfrak{F}$ and a covering theorem for function in $\mathfrak{F}_0$.

THEOREM 4. *Let Ω be a convex region, $\Omega \neq \mathbf{C}$. Then for $z \in \Omega$, $1/\delta_\Omega(z) \leqq \lambda_\Omega(z)$. If equality holds at a point, then Ω is a half-plane.*

PROOF. Fix $a \in \Omega$. Select $c \in \partial\Omega$ with $|a - c| = \delta_\Omega(a)$. By performing a translation and a rotation, if necessary, we may assume without loss of generality that $c = 0$ and $a > 0$. Note that $a = \delta_\Omega(a)$ relative to this normalization. Then $\Omega \subset \mathbf{H}$ because Ω is convex. The monotonicity property of the hyperbolic metric gives $\lambda_\Omega(a) \geqq \lambda_{\mathbf{H}}(a) = 1/a = 1/\delta_\Omega(a)$, and if equality holds, then $\Omega = \mathbf{H}$; that is, Ω is a half-plane.

COROLLARY. *Let $f \in \mathfrak{F}$ and let Ω denote the convex hull of $f(\mathbf{B})$. Then for any $z \in \mathbf{B}$ and any branch of f at z,*

$$(1 - |z|^2)|f'(z)| \leqq 2\delta_\Omega(f(z)). \tag{4}$$

If equality holds at a point, then Ω is a half-plane and f is a conformal mapping of $\mathbf{B}$ onto Ω.

PROOF. If $\Omega = \mathbf{C}$, then $\delta_\Omega = \infty$ and there is nothing to prove. Hence, we may assume that $\Omega \neq \mathbf{C}$. From Theorem 2 we have $\lambda_\Omega(f(z))|f'(z)| \leqq \lambda_{\mathbf{B}}(z) = 2/(1 - |z|^2)$ and Theorem 4 gives $|f'(z)|/\delta_\Omega(f(z)) \leqq \lambda_\Omega(f(z))|f'(z)|$. By combining the two preceding inequalities, we obtain (4). If equality holds in (4), then equality must hold in both of the preceding inequalities. Equality in the latter implies that Ω is a half-plane and equality in the former requires that f be a conformal mapping of $\mathbf{B}$ onto Ω.

COROLLARY. *Let $f \in \mathfrak{F}_0$ and let Ω denote the convex hull of $f(\mathbf{B})$. Then either* $\operatorname{cl} B(1/2) = \{w: |w| \leqq 1/2\} \subset \Omega$ *or else $f(z) = z/(1 - e^{i\theta}z)$ for some $\theta \in \mathbf{R}$.*

PROOF. Let f denote the branch at the origin which satisfies $f(0) = 0$ and $f'(0) = 1$. If we use $z = 0$ and this branch of f in the preceding corollary, then we obtain $\delta_\Omega(0) \geqq 1/2$. Thus, either $\operatorname{cl} B(1/2) \subset \Omega$ or else $\delta_\Omega(0) = 1/2$. The latter case implies that f is a conformal mapping of $\mathbf{B}$ onto a half-plane whose edge has distance $1/2$ from the origin. Direct calculation, using the normalization of f, shows that f must have the specified form.

REMARK. In case f is analytic and univalent in $\mathbf{B}$, normalized by $f(0) = 0$, $f'(0) = 1$ and $f(\mathbf{B})$ is convex, then the conclusion of the second corollary is a classical result [**4**, p. 13].

4. Lower bound in restricted convex regions. This final section is devoted to establishing a sharp lower bound for the hyperbolic metric in any covex region Ω with the property that δ_Ω is bounded above. Of course, the lower bound for λ_Ω depends on the upper bound for δ_Ω. The techniques used in this section are not as elementary as those employed in §3. As applications of the lower bound, we again obtain a distortion theorem for $\mathfrak{F}$ and a covering theorem for $\mathfrak{F}_0$.

THEOREM 5. *Let Ω be a convex region in* $\mathbf{C}$. *If $\delta_\Omega(z) \leqq M$ for all $z \in \Omega$, then for $z \in \Omega$*

$$\pi/2M \sin((\pi/2M)\delta_\Omega(z)) \leqq \lambda_\Omega(z). \tag{5}$$

If equality holds at a point, then Ω is a strip of width $2M$.

PROOF. Initially, we establish the inequality (5) under the more restrictive hypothesis that $\delta_\Omega(z) < M$ for all $z \in \Omega$. Define

$$\rho(z) = \pi/2M \sin((\pi/2M)\delta_\Omega(z)).$$

We shall show that $\rho(z)|dz|$ is an ultrahyperbolic metric on Ω. Then the inequality is a direct consequence of Ahlfors' extension of Schwarz' lemma ([1], [2, p. 13]). Clearly, ρ is a continuous function on Ω. We will show that there is a supporting metric at each point of Ω. Fix $a \in \Omega$. Then pick $c \in \partial\Omega$ with $|a - c| = \delta_\Omega(a)$. Without loss of generality we may assume that $c = 0$ and $a > 0$. Note that $a = \delta_\Omega(a) < M$ in this special situation. Then $\Omega \subset \mathbf{H}$ because Ω is convex, but it need not be true that $\Omega \subset S(M)$. For z in Ω and near a, we have $\delta_\Omega(z) \leqq \mathrm{Re}(z) < M$, since the ball with center z and radius $\delta_\Omega(z)$ must be contained in $\mathbf{H}$. Because the function $\sin(\pi t/2M)$ is strictly increasing for $t \in (0, M)$, it follows that

$$\lambda_{S(M)}(z) = \pi/2M \sin((\pi/2M)\mathrm{Re}(z)) \leqq \rho(z)$$

for $z \in \Omega$ and near a. Note that equality holds for $z = a$. Since $\lambda_{S(M)}(z)|dz|$ has constant curvature -1, we conclude that $\lambda_{S(M)}(z)|dz|$ is a supporting metric for $\rho(z)|dz|$ at a. Thus, $\rho(z)|dz|$ is an ultrahyperbolic metric on Ω, so (5) is established in the special case $\delta_\Omega(z) < M$, $z \in \Omega$.

Now, we determine when equality can hold in (5) under our restricted hypothesis. Suppose $a \in \Omega$ and $\rho(a) = \lambda_\Omega(a)$. As in the first part of the proof, we may assume that $a > 0$ and $a = \delta_\Omega(a)$. Then $\lambda_{S(M)}(z)|dz|$ is a supporting metric for $\rho(z)|dz|$ at a, so we may conclude that $\lambda_{S(M)}(z) \leqq \rho(z) \leqq \lambda_\Omega(z)$ for $z \in \Omega$ and near a with equality when $z = a$. The corollary to Theorem 1 implies that $\Omega = S(M)$; that is, Ω is a strip of width $2M$.

Next, we turn to the general case $\delta_\Omega(z) \leqq M$, $z \in \Omega$. Set $M_n = M + 1/n$, where n is any positive integer, and

$$\rho_n(z) = \pi/2M_n \sin((\pi/2M_n)\delta_\Omega(z)).$$

Now, $\delta_\Omega(z) < M_n$ for all $z \in \Omega$, so the first part of the proof implies that $\rho_n(z)\,|dz|$ is an ultrahyperbolic metric on Ω. Thus, $\rho_n(z) \leqq \lambda_\Omega(z)$ for $z \in \Omega$. By allowing $n \to \infty$, we obtain the inequality (5) in the general case.

The determination of when equality holds in (5) is more involved in the general case. Assume $a \in \Omega$ and $\rho(a) = \lambda_\Omega(a)$. If $\delta_\Omega(a) < M$, then exactly as in the first part of the proof we may conclude that Ω is a strip of width $2M$. In order to treat the case $\delta_\Omega(a) = M$, we use a method of Pommerenke [8]. As usual, we may assume that $a > 0$ and $a = \delta_\Omega(a) = M$. In this situation we obtain

$$\lambda_{S(M)}(z) \leqq \rho(z) \leqq \lambda_\Omega(z) \tag{6}$$

for $z \in \Omega$ near $a = M$ with $\mathrm{Re}(z) < M$ and equality for $z = M$. Observe that we do not necessarily have the inequality (6) in a neighborhood of $a = M$; but rather only in a "half" neighborhood of $a = M$. Note that

$$\lambda_\Omega(M) = \pi/2M = \lambda_{S(M)}(M) \tag{7}$$

and both λ_Ω and $\lambda_{S(M)}$ have a minimum value at $z = M$. Let $\varphi\colon \mathbf{B} \to \Omega$ and $\psi\colon \mathbf{B} \to S(M)$ be conformal mappings with $\varphi(0) = M = \psi(0)$. From (7) we obtain

$$|\varphi'(0)| = 2M/\pi = |\psi'(0)|. \tag{8}$$

From Theorem 3 we conclude that

$$\varphi''(0) = 0 = \psi''(0). \tag{9}$$

Consider the function $v(z) = \lambda_{S(M)}(\varphi(z))/\lambda_\Omega(\varphi(z))$. If $\Omega \neq S(M)$, then we shall show that there is a point z near the origin with $v(z) > 1$ and $\mathrm{Re}\,\varphi(z) < M$. This would violate inequality (6) and will show that equality implies $\Omega = S(M)$. Define $f = \psi^{-1} \circ \varphi$ in a neighborhood of the origin. Then f is analytic at the origin, $f(0) = 0$ and

$$\begin{aligned} v(z) &= \lambda_{S(M)}(\varphi(z))|\varphi'(z)|/\lambda_\Omega(\varphi(z))|\varphi'(z)| = \lambda_{S(M)}(\varphi(z))|\varphi'(z)|/\lambda_{\mathbf{B}}(z) \\ &= \lambda_{\mathbf{B}}(f(z))|f'(z)|/\lambda_{\mathbf{B}}(z) = (1 - |z|^2)|f'(z)|/(1 - |f(z)|^2). \end{aligned}$$

From (8) and (9) we obtain

$$f(z) = e^{i\theta}(z + b_m z^m + \cdots), \tag{10}$$

for z near the origin, where $\theta \in \mathbf{R}$ and $m \geqq 3$. Notice that $b_m \neq 0$ since we are assuming $\Omega \neq S(M)$. By making use of (10), we obtain

$$v(z) = 1 + m\,\mathrm{Re}(b_m z^{m-1}) + O(|z|^m). \tag{11}$$

Since $\varphi(0) = M$ and $\varphi''(0) = 0$, we have

$$\mathrm{Re}\varphi(z) = M + \mathrm{Re}(\varphi'(0)z) + O(|z|^3). \tag{12}$$

Because $m \geqq 3$, the identities (11) and (12) show us that we can select z close to the origin so that $v(z) > 1$ and $\mathrm{Re}\,\varphi(z) < M$. This is a contradiction to (6). This completes the proof.

COROLLARY. *Let $f \in \mathfrak{F}$ and let Ω denote the convex hull of $f(\mathbf{B})$. Suppose $\delta_\Omega(z) \leqq M$ for all $z \in \Omega$. Then for any $z \in \mathbf{B}$ and any branch of f at z,*

$$(1 - |z|^2)|f'(z)| \leqq (4M/\pi)\sin((\pi/2M)\delta_\Omega(f(z))). \tag{13}$$

If equality holds at a point, then Ω is a strip of width $2M$ and f is a conformal mapping of $\mathbf{B}$ onto Ω.

PROOF. Theorem 2 gives $\lambda_\Omega(f(z))|f'(z)| \leqq \lambda_{\mathbf{B}}(z) = 2/(1-|z|^2)$, while Theorem 5 yields

$$\pi|f'(z)|/2M \sin((\pi/2M)\delta_\Omega(f(z))) \leqq \lambda_\Omega(f(z))|f'(z)|.$$

Inequality (13) results from combining these two inequalities. If equality holds in (13), then we must have equality in both of the above inequalities. Equality in the second inequality implies that Ω is a strip of width $2M$ and equality in the first inequality implies that f is a conformal mapping of $\mathbf{B}$ onto Ω.

COROLLARY. *Let $f \in \mathfrak{F}_0$ and let Ω be the convex hull of $f(\mathbf{B})$. Then either there exists a point $a \in \Omega$ with $\delta_\Omega(a) > \pi/4$ or else*

$$f(z) = (1/2e^{i\theta})\mathrm{Log}\,((1 + e^{i\theta}z)/(1 - e^{i\theta}z))$$

for some $\theta \in \mathbf{R}$.

PROOF. If there exists $a \in \Omega$ with $\delta_\Omega(a) > \pi/4$, then we are done, Otherwise, $\delta_\Omega(z) \leqq \pi/4$ for all $z \in \Omega$. If we use $z = 0$ and $M = \pi/4$ in the preceding corollary, then we obtain $1 \leqq \sin(2\,\delta_\Omega(0))$. Since $0 \leqq \delta_\Omega(0) \leqq \pi/4$, this implies that $\delta_\Omega(0) = \pi/4$ so equality holds in the preceding corollary. Hence, f is a conformal mapping of $\mathbf{B}$ onto a strip of width $\pi/2$ and 0 lies on the center line of the strip. Because f is normalized by $f(0) = 0$ and $f'(0) = 1$, direct calculation shows that f must have the specified form.

REMARK. The conclusion of the last corollary implies that the Bloch-Landau constant for normalized convex univalent functions is $\pi/4$, a classical result of Szegö [9]. Also, for $0 < \delta_\Omega(z) \leqq M$, we have

$$1/\delta_\Omega(z) < \pi/2M \sin((\pi/2M)\delta_\Omega(z)), \tag{14}$$

so that the lower bound given in Theorem 5 is a strict improvement of the one in Theorem 4. The limit, as $M \to \infty$, of the right-hand side of the inequality (14) is equal to the left-hand side. Finally, if Ω were contained in a strip of width $2M$, then the conclusion of Theorem 5 would follow immediately from the monotonicity property of the hyperbolic metric. However, the condition that $\delta_\Omega(z) \leqq M$ for all $z \in \Omega$ does not imply that Ω is contained in such a strip. This is easily demonstrated by considering an equilateral triangle such that the radius of the largest inscribed disk is M. The altitude of this triangle is $3M$ and the narrowest strip containing Ω has width $3M$.

The lower bound for the hyperbolic metric obtained in Theorem 5 can be used to derive a univalence criterion.

THEOREM 6. *Suppose Ω is a convex region in* $\mathbf{C}$ *and* $\delta_\Omega(z) \leqq \pi/4$ *for all* $z \in \Omega$. *If f is analytic in* $\mathbf{B}$, $f'(z) \neq 0$ *for all* $z \in \mathbf{B}$ *and* $\log f'(\mathbf{B}) \subset \Omega$, *then f is univalent in* $\mathbf{B}$.

PROOF. Let $\Lambda(\Omega) = \inf\{\lambda_\Omega(z)\colon z \in \Omega\}$. A result of Minda and Wright [5] implies that f is univalent provided $\Lambda(\Omega) \geqq 2$. Theorem 5 with $M = \pi/4$ gives $2 \leqq 2/\sin(2\delta_\Omega(z)) \leqq \lambda_\Omega(z)$. (Actually, the result in [5] requires that $\Lambda(\Omega) \geqq 1$. However, in [5] the hyperbolic metric was normalized to have constant curvature -4, so it is necessary to multiply by 2 to obtain constant curvature -1.)

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