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CONVERGENCE IN FUZZY TOPOLOGY

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Abstract. Convergence and weak fuzzy continuity are developed and applied in fuzzy topological spaces.

Introduction. Lowen has skillfully used lower semicontinuous functions [5] and convergence [6] to obtain significant results about a proper subclass of the fuzzy topological spaces of Chang [1]. Lowen introduced this subclass in [5], also called its members fuzzy topological spaces and has adhered to the concept in his subsequent work. The main thrust of this paper is to take Lowen's ideas and results into work [2, 8, 9] based on Chang's paper. Often this requires methods which differ from Lowen's, since his work rests upon the usual topology of the unit interval, whereas Chang's viewpoint does not require a topology on the unit interval.

In §1 open fuzzy sets are described in terms of generalized lower semicontinuous functions which are used to characterize fuzzy continuous maps. Convergence is developed in §2 and then used to characterize fuzzy limit point and fuzzy continuity. An example is given to show that one of the characterizations of fuzzy continuity is the best possible. In §3 weak fuzzy continuity is given six characterizations which interestingly show its relation to other concepts, and the question of a complement for it is examined.

This paper assumes that the reader is familiar with the results in [6] and [9]. In general, the terminology and notation follow [9], except that fuzzy sets are denoted by lower case Greek letters.

1. F-continuity. If X is a set, then a fuzzy topology on X is a family T of mappings from X into [0, 1] such that the constant maps 0 and 1 are in T, the supremum of any subcollection of T is in T, and the infimum of any finite subcollection of T is in T. The members of T are called open fuzzy sets. A mapping between fuzzy topological spaces is called F-continuous if the inverse image of each open fuzzy set is open.

Several of the results in this section are based on the fact that if $f: X \to Y$ and $A \subset Y$, then the function $f^{-1}(\mu_A)$ maps X into $\{0, 1\}$ and

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$$f^{-1}(\mu_A) = \mu_{f^{-1}(A)} = \mu_A \circ f$$

where μ_A and $\mu_{f^{-1}(A)}$ are characteristic functions. Let $J = \{0, 1\}$ and have the topology $\mathscr{S} = \{\varnothing, \{1\}, J\}$, i.e., $(J, \mathscr{S})$ is a Sierpinski space. For $E \subset X$, we have the following diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mu_E"] & Y \arrow[d, "\mu_A"] \\ J & J \end{tikzcd}$$

The well-known result that μ_E is lower semicontinuous if and only if E is open in X leads to the following statements:

(1) μ_E is continuous with respect to $(J, \mathscr{S})$ if and only if E is open in X.

(2) f is continuous if and only if whenever $A \subset Y$ and μ_A is continuous with respect to $(J, \mathscr{S})$, then $f^{-1}(\mu_A)$ is also.

Generalizing $(J, \mathscr{S})$ to fuzzy topological spaces, we consider $I = [0, 1]$ with the fuzzy topology $\mathscr{S} = \{0, 1, i\}$ where $i(x) = x$ for each $x \in I$. The diagram now is:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\nu"] & Y \arrow[d, "\tau"] \\ I & I \end{tikzcd}$$

The analogue of (1) is the following theorem.

THEOREM 1.1. *Let (X, T) be a fuzzy topological space. The fuzzy set ν in X is F-continuous with respect to $(I, \mathscr{S})$ if and only if $\nu \in T$.*

PROOF. If ν is open, then $\nu^{-1}(0) = 0$, $\nu^{-1}(1) = 1$ and $\nu^{-1}(i) = \nu$ are in T and so ν is F-continuous with respect to $(I, \mathscr{S})$. Conversely, if ν is F-continuous, then $\nu^{-1}(i) \in T$, which means $\nu \in T$.

The analogue of (2) is the following theorem.

THEOREM 1.2. *If f: $X \to Y$ where (X, T) and (Y, U) are fuzzy topological spaces, then the following are equivalent:*

(a) f is F-continuous, and

(b) whenever the fuzzy set τ in Y is F-continuous with respect to $(I, \mathscr{S})$, then $f^{-1}(\tau)$ is also.

PROOF. If (a) holds and if the fuzzy set τ in Y is F-continuous with respect to $(I, \mathscr{S})$, then by Theorem 1.1 $\tau \in U$. Thus $f^{-1}(\tau) \in T$. Again by Theorem 1.1, $f^{-1}(\tau)$ is F-continuous with respect to $(I, \mathscr{S})$.

If (b) holds and $\tau \in U$, then by Theorem 1.1, τ is F-continuous with respect to $(I, \mathscr{S})$. Thus $f^{-1}(\tau)$ is F-continuous with respect to $(I, \mathscr{S})$. Again by Theorem 1.1, $f^{-1}(\tau) \in T$.

Note that if $[0, 1]$ is replaced by a more general lattice used by Hutton [2], then Theorems 1.1 and 1.2 remain valid with the same fuzzy topology $\mathscr{S}$.

2. Applications of prefilters. The prefilter terminology and notation are from [6]. A prefilter is a filter on a lattice. Due to the role of a neighborhood in a fts [9], fuzzy topology differs from topology in that the set of all fuzzy neighborhoods of a point may be only a base for a prefilter and not a prefilter. This motivates the following:

A base $\mathscr{B}$ for a prefilter in I^X is said to *converge to a point* $x \in X$ if and only if every fuzzy neighborhood of x is in $\mathscr{B}$. If $f: X \to Y$ and if $\mathscr{B}$ is a base for a prefilter in I^X, then $\{f(\mu): \mu \in \mathscr{B}\}$ is a base for a prefilter in I^Y and is denoted by $f(\mathscr{B})$.

If $\mathscr{B}$ is a base for a prefilter $\langle\mathscr{B}\rangle$ in I^X and $x \in X$, then $\mathscr{B}_x$ denotes $\mathscr{B} \cup \{\eta_x: \eta_x$ is a neighborhood of x and $\eta_x \in \langle\mathscr{B}\rangle\}$. Expanding the definition of adherence [6, p. 153] to include a base, we say adh $\mathscr{B} = \bigwedge\{\bar{\mu}: \mu \in \mathscr{B}\}$. Clearly, adh $\mathscr{B}$ = adh $\langle\mathscr{B}\rangle$. Similarly, a base $\mathscr{B}$ is called *prime* if whenever $\mu \vee \nu \in \mathscr{B}$ there exists $\tau \in \mathscr{B}$ such that $\tau \leqq \mu$ or $\tau \leqq \nu$. Clearly, $\mathscr{B}$ is prime if and only if $\langle\mathscr{B}\rangle$ is prime.

THEOREM 2.1. *Let ν be a fuzzy set in the* fts (X, T) *and let* $A = \{x \in X: \nu(x) = 0\}$. *Then $y \in X$ is a fuzzy limit point of ν if and only if $\bar{\nu}(y) > 0$ and there exists a base $\mathscr{B}$ for a prefilter in I^X such that* (1) *if $\tau \in \mathscr{B}$ and $x \in A \cup \{y\}$, then $\tau(x) = 0$, and* (2) *$\langle\mathscr{B}\rangle$ converges to y.*

PROOF. If such a base exists and if η_y is a neighborhood of y, then $\eta_y \in \langle\mathscr{B}\rangle$, so that there exists $\tau \in \mathscr{B}$ for which $\eta_y \geqq \tau$. By (1) there exists $z \in X\backslash(A \cup \{y\})$ such that $\eta_y(z) > 0$. Thus $\eta_y(z) \wedge \nu(z) \neq 0$. By Definition 2.7 in [9], y is a fuzzy limit point of ν.

If y is a fuzzy limit point of ν, then from Definition 2.7 in [9], $\bar{\nu}(y) > 0$. Also $\{\eta_y \wedge \nu \wedge \mu_{x\backslash\{y\}}: \eta_y$ is an open neighborhood of $y\}$ is a base satisfying (1).

THEOREM 2.2. *Let (X, T) and (Y, U) be* fts *and let $f: X \to Y$ be* 1–1. *Then f is F-continuous if and only if for each $x \in X$ and for each base $\mathscr{B}$ (for a prefilter in I^X) which converges to x, the base $f(\mathscr{B})_{f(x)}$ converges to $f(x)$.*

PROOF. Suppose f is F-continuous, $x \in X$ and η is a neighborhood of $f(x)$. By Theorem 4.2(d) in [9], $f^{-1}(\eta)$ is a neighborhood of x and hence, if $\mathscr{B}$ converges to x, then $f^{-1}(\eta) \in \mathscr{B}$. Since $\eta \geqq f(f^{-1}(\eta))$ and $\eta(f(x)) = f(f^{-1}(\eta))(f(x))$ it follows that $\eta \in f(\mathscr{B})_{f(x)}$.

If f is not F-continuous, then there exists $x \in X$ and a neighborhood η

of $f(x)$ such that $f^{-1}(\eta)$ is not a neighborhood of x. Therefore if η_x is a neighborhood of x, then $\eta_x \leqq f^{-1}(\eta)$ fails, in which case $f(\eta_x) \leqq \eta$ fails; or $\eta_x \leqq f^{-1}(\eta)$ and $f^{-1}(\eta)(x) \neq \eta_x(x)$, in which case $\eta(f(x)) \neq f(\eta_x)(f(x))$ since f is 1–1. The conclusion is that $\eta \notin f(\mathscr{N})_{f(x)}$ where $\mathscr{N} = \{\eta_x : \eta_x$ is a neighborhood of $x\}$.

By considering the following example, it is easy to see that in Theorem 2.2 $\mathscr{B}$ cannot be replaced with any prefilter which converges to x, or $f(\mathscr{B})_{f(x)}$ cannot be replaced by $\langle f(\mathscr{B})\rangle$. Let $X = Y$ be a set with at least two elements, let f be the identity map, let T be the set of all constant maps, and let U contain all constant maps and a nonconstant map whose range is a finite subset of $(0, 1]$.

Furthermore, the hypothesis in Theorem 2.2 that f is one-to-one cannot be eliminated. Let $X = \{r, s, t\}$, $Y = \{y, z\}$, $f(r) = f(s) = y$, $f(t) = z$, $\tau(r) = 1/2 = \eta(y)$, $\tau(s) = 2/5$, $\tau(t) = \eta(z) = 1/3$, T has basis of all constant maps and τ, and U has basis of all constant maps and η. Then f is not F-continuous since $f^{-1}(\eta) \notin T$. However, the other conclusion of Theorem 2.2 holds for this example.

However, in Theorem 2.2 $f(\mathscr{B})_{f(x)}$ can be replaced by $f(\mathscr{B})$ if f is onto Y.

THEOREM 2.3. *Let* (X, T) *and* (Y, U) *be* fts *and* $f: X \to Y$. *Then* f *is* F-*continuous if and only if for each prefilter* $\mathscr{F}$ *in* I^X, adh $f(\mathscr{F}) \geqq f(\text{adh}\, \mathscr{F})$.

PROOF. The proof of Theorem 6.1 in [6] is valid for this more general situation.

Next we show that Theorem 6.2 from [6] is valid in the context of [1].

THEOREM 2.4. *Let* (X, T) *and* (Y, U) *be* fts *and let* $f: X \to Y$. *Then* f *is* F-*continuous if and only if for each prime prefilter* $\mathscr{G}$ *in* I^X, adh $f(\mathscr{G}) \geqq f(\text{adh}\, \mathscr{G})$.

PROOF. Necessity follows from Theorem 2.3. If f is not F-continuous, then from [9] there exists $\nu \in I^X$ and $y_0 \in Y$ such that $\overline{f(\nu)}(y_0) < f(\bar{\nu})(y_0)$. Let $\mathscr{F} = \langle \nu \rangle$. If $\mathscr{G} \supset \mathscr{F}$, then adh $f(\mathscr{G})(y_0) \leqq$ adh $f(\mathscr{G})(y_0) = \overline{f(\nu)}(y_0)$.

Let $E = f^{-1}(y_0)$ and $F = \{x : \nu(x) > 0\}$. If $A \subset F$, we define $\nu_A \in I^X$ by $\nu_A(x) = \nu(x)$ if $x \in A$ and $\nu_A(x) = 0$ otherwise. Let

$$\mathscr{P} = \{\nu_A : A \subset F \text{ and } \bigvee_{x \in E} \overline{\nu_A}(x) = \bigvee_{x \in E} \bar{\nu}(x)\}.$$

If $\nu_A \vee \nu_B \in \mathscr{P}$, then

$$\bigvee_{x \in E} \overline{\nu_A}(x) \vee \bigvee_{x \in E} \overline{\nu_B}(x) = \bigvee_{x \in E} \bar{\nu}(x),$$

and so ν_A or ν_B is in $\mathscr{P}$. By Zorn's lemma there exists a maximal subset $\mathscr{L}$ of $\mathscr{P}$ such that $\nu_F \in \mathscr{L}$ and $\mathscr{L}$ is a prime base for a prefilter. Let $\mathscr{G} = \langle \mathscr{L} \rangle$. Then $\mathscr{G} \supset \mathscr{F}$ and $f(\text{adh}\, \mathscr{G})(y_0) = f(\bar{\nu})(y_0)$.

If $F = \{x: \nu(x) > 0\}$ is finite, say $F = \{x_1, \ldots, x_n\}$, then Zorn's lemma is not needed in the above proof, since $\mathscr{G} \in \{\langle \nu_{\{x_1\}} \rangle, \ldots, \langle \nu_{\{x_n\}} \rangle\}$.

3. Weak F-continuity. Motivated by [4] and [7], if (X, T) and (Y, U) are fts, then a function $f: X \to Y$ is called *weakly F-continuous* if for each $x \in X$ and for each $\mu \in U$ such that $\mu(f(x)) > 0$, there exists $\nu \in T$ satisfying $\nu(x) \geqq f^{-1}(\mu)(x)$ and $f(\nu) \leqq \bar{\mu}$. When all fuzzy sets are restricted to be crisp, then weak F-continuity is equivalent to weak continuity in [4]. Also, F-continuity implies weak F-continuity.

THEOREM 3.1. *The following are equivalent.*

(i) *f is weakly F-continuous.*
(ii) *$f^{-1}(\mu) \leqq (f^{-1}(\bar{\mu}))^\circ$ for each $\mu \in U$.*
(iii) *$f^{-1}(\tau) \geqq \overline{f^{-1}(\tau^\circ)}$ for each closed fuzzy set τ in Y.*
(iv) *$f^{-1}(\gamma) \leqq (f^{-1}(\bar{\gamma}))^\circ$ for each γ in some basis for U.*
(v) *$\overline{f^{-1}(\mu)} \leqq f^{-1}(\bar{\mu})$ for each $\mu \in U$.*
(vi) *$f(\bar{\nu}) \leqq \bigwedge\{\bar{\mu}: \mu \geqq f(\nu)$ and $\mu \in U\}$ for each $\nu \in I^X$.*
(vii) *Θ-adh $f(\mathscr{B}) \geqq f(\text{adh } \mathscr{B})$ for each base $\mathscr{B}$ for a prefilter in I^X.*

PROOF. (i) $\Rightarrow$ (ii). If $x \in X$ and $f^{-1}(\mu)(x) > 0$, then $\mu(f(x)) > 0$ and there exists $\nu_x \in T$ satisfying $\nu_x(x) \geqq f^{-1}(\mu)(x)$ and $f(\nu_x) \leqq \bar{\mu}$. Let $\gamma = \bigvee\{\nu_x: x \in X$ and $f^{-1}(\mu)(x) > 0\}$. Then $f^{-1}(\mu) \leqq \gamma \leqq (f^{-1}(\bar{\mu}))^\circ$.

(ii) $\Rightarrow$ (iii). Follows from Theorem 2.13 in [9].

(iii) $\Rightarrow$ (vi). If $\mu \geqq f(\nu)$ and $\mu \in U$, the $\bar{\mu}^\circ \geqq \mu$ and so

$$\overline{f^{-1}(\bar{\mu}^\circ)} \geqq \overline{f^{-1}(\mu)} \geqq \overline{f^{-1}(f(\nu))} \geqq \bar{\nu}.$$

By (iii),

$$f^{-1}(\bar{\mu}) \geqq \overline{f^{-1}(\bar{\mu}^\circ)}.$$

Thus $f(\bar{\nu}) \leqq \bar{\mu}$.

(vi) $\Rightarrow$ (i). Let $\mu \in U$, $\tau = 1 - \mu$ and $\nu = (f^{-1}(\bar{\mu}))^\circ$.
Then

$$f(\overline{f^{-1}(\tau^\circ)}) \leqq \bigwedge\{\bar{\gamma}: \gamma \geqq f(f^{-1}(\tau^\circ)) \text{ and } \gamma \in U\}.$$

Since $\tau^\circ \geqq f(f^{-1}(\tau^\circ))$, it follows that $f(\overline{f^{-1}(\tau^\circ)}) \leqq \tau$. Therefore $\overline{f^{-1}(\tau^\circ)} \leqq f^{-1}(\tau)$ and the result follows from Theorem 2.13 in [9].

(vi) $\Rightarrow$ (vii). Since Θ-adh $f(\mathscr{B}) = \bigwedge\{\bar{\mu}: \mu \in U$ and $\mu \in \langle f(\mathscr{B})\rangle\}$ and $f(\bigwedge\{\bar{\tau}: \tau \in \mathscr{B}\}) \leqq \bigwedge\{f(\bar{\tau}): \tau \in \mathscr{B}\}$, the result follows from (vi).

(vii) $\Rightarrow$ (vi). For $\nu \in I^X$, let $\mathscr{B} = \langle \nu \rangle$. Then $f(\bar{\nu}) = f(\text{adh } \mathscr{B}) \leqq \Theta\text{-adh } f(\mathscr{B}) = \bigwedge\{\bar{\mu}: \mu \in U \text{ and } \mu \geqq f(\nu)\}$.

(ii) $\Rightarrow$ (iv). Clear.

(iv) $\Rightarrow$ (ii). Suppose B is such a basis and $\zeta_i \in B$. Then $f^{-1}(\bigvee \zeta_i) = \bigvee f^{-1}(\zeta_i) \leqq \bigvee(f^{-1}(\bar{\zeta}_i))^\circ \leqq (\bigvee f^{-1}(\bar{\zeta}_i))^\circ = (f^{-1}(\bigvee \bar{\zeta}_i))^\circ \leqq (f^{-1}(\overline{\bigvee \zeta_i}))^\circ$.

(ii) $\Rightarrow$ (v). Let $\mu \in U$ and $\gamma = 1 - \bar{\mu}$. Since γ is open, $f^{-1}(\gamma) \leqq (f^{-1}(\bar{\gamma}))^{\circ}$ by (ii). Thus

$$1 - f^{-1}(\bar{\mu}) = f^{-1}(1 - \bar{\mu}) \leqq (f^{-1}(\overline{1 - \bar{\mu}}))^{\circ}$$
$$= (f^{-1}(1 - \bar{\mu}^{\circ}))^{\circ} = (1 - f^{-1}(\bar{\mu}^{\circ}))^{\circ} = 1 - \overline{f^{-1}(\bar{\mu}^{\circ})}.$$

Hence $\overline{f^{-1}(\bar{\mu}^{\circ})} \leqq f^{-1}(\bar{\mu})$. Since μ is open and $\mu \leqq \bar{\mu}$, $\mu \leqq \bar{\mu}^{\circ}$, so that $f^{-1}(\mu) \leqq f^{-1}(\bar{\mu}^{\circ})$ and $\overline{f^{-1}(\mu)} \leqq \overline{f^{-1}(\bar{\mu}^{\circ})}$. Therefore $\overline{f^{-1}(\mu)} \leqq f^{-1}(\bar{\mu})$.

(v) $\Rightarrow$ (ii). Let $\mu \in U$ and $\gamma = 1 - \bar{\mu}$. Since γ is open, by hypothesis, $\overline{f^{-1}(\gamma)} \leqq f^{-1}(\bar{\gamma})$ so that $1 - (f^{-1}(\bar{\mu}))^{\circ} \leqq 1 - f^{-1}(\bar{\mu}^{\circ})$. Thus $(f^{-1}(\bar{\mu}))^{\circ} \geqq f^{-1}(\bar{\mu}^{\circ})$. Since μ is open and $\mu \leqq \bar{\mu}$, it follows that $\mu \leqq \bar{\mu}^{\circ}$, and therefore $f^{-1}(\mu) \leqq f^{-1}(\bar{\mu}^{\circ})$. Thus $f^{-1}(\mu) \leqq (f^{-1}(\bar{\mu}))^{\circ}$.

Is there a complement for weak F-continuity so that the complement and weak F-continuity are a decomposition of F-continuity? When all fuzzy sets are crisp, then **[4]** has a solution which has been fruitful in general topology. Although **[4]** is based on the boundary, the boundary in **[8]** is not part of a decomposition of a closed fuzzy set, and therefore cannot be expected to solve the question. It can be shown that the difference of a closed fuzzy set and its interior is not useful in answering this question.

References

1. C.L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl. **24** (1968), 182–190.

2. B.W. Hutton, *Normaility in fuzzy topological spaces*, J. Math. Anal. Appl. **50** (1975), 74–79.

3. J.E. Joseph, *Multifunctions and cluster sets*, Proc. Amer. Math. Soc. **74** (1979), 329–337.

4. N. Levine, *A decomposition of continuity in topological spaces*, Amer. Math. Monthly **68** (1961), 44–46.

5. R. Lowen, *Fuzzy topological spaces and fuzzy compactness*, J. Math. Anal. Appl. **56** (1976), 621–633.

6. ———, *Convergence in fuzzy topological spaces*. General Topology and Appl. **10** (1979), 147–160.

7. D.A. Rose, *Weak continuity and almost continuity*, preprint.

8. R.H. Warren, *Boundary of a fuzzy set*, Indiana Univ. Math. J. **26** (1977), 191–197.

9. ———, *Neighborhoods, bases and continuity in fuzzy topological spaces*, Rocky Mountain J. Math. **8** (1978), 459–470.

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