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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO PERTURBED LINEAR DIFFERENTIAL EQUATIONS

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Dedicated to Professor Lloyd K. Jackson
on the occasion of his sixtieth birthday.

Introduction. The purpose of this paper is to investigate the asymptotic relationship between solutions of the n-th order linear homogeneous equation

$$L_n y = 0 \tag{1}$$

and those of the perturbed equation

$$L_n y + B(x, y, y', \ldots, y^{(n-1)}) = 0. \tag{2}$$

The results will involve certain smallness conditions on the function $B(x, y, y', \ldots, y^{(n-1)})$ which will be made more precise in later sections. In the first section we will consider the general case where $L_n y$ admits a Mammana factorization [6]. In the second section we shall consider the case where $L_n y$ is a constant coefficient operator. In the third section we shall consider the specific operator $L_n y = y^{(n)}$. This section also contains examples to show the results obtained here generalize those of Svec [7], [8], and Belohorec [3].

I. Perturbed linear equations. Mammana [6] has shown that, under certain conditions, an n-th order linear differential operator with leading coefficient one admits a factorization of the form

$$L_n[y] = \left(\prod_{j=1}^{n} (D - \eta_j(x))\right)[y] \tag{3}$$

where $\eta_j(x) = D[\ln W_j/W_{j-1}]$, $1 \leqq j \leqq n$, and W_j is the Wronskian of the solutions $\xi_1, \xi_2, \ldots, \xi_j (W_0 \equiv 1)$ of (1). The solutions $\xi_1, \xi_2, \ldots, \xi_n$ have the property that for every j, W_j is different from zero, which requires, in general, that the ξ_j be complex and hence the $\eta_i(x)$ will be complex. Levin [5] has observed the interval on which this holds may be half-line of the form $[a, \infty)$.

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We shall assume this factorization (3) holds so that (2) takes the form

$$L_n y = \left(\prod_{j=1}^{n} (D - \eta_j)\right)[y] = -B(x, \vec{y}) \tag{4}$$

where $\vec{y} = (y, y', \ldots, y^{(n-1)})$. For convenience we let $L_k = \prod_{j=1}^{k}(D - \eta_k)$, $1 \leqq k \leqq n$, and $L_0 = I$. If $z = L_k[y]$, $0 \leqq k \leqq n - 1$, then (4) becomes

$$\left(\prod_{j=k+1}^{n} (D - \eta_j)\right)[z] = -B(x, \vec{y}). \tag{5}$$

Proceeding formally to solve (5) for a particular solution we find, using variation of parameters,

$$z = L_k[y] = \int_x^\infty g_k(x, t)B(t, \vec{y})dt \tag{6}$$

where

$$g_k(x, t) = \sum_{j=k+1}^{n} [\xi_{k,j}(x)w_{k,j}(t)/w_k(t)],$$

$\xi_{k,j}(j = k + 1, \ldots, n)$ are $n - k$ independent solutions of $(\prod_{j=k+1}^{n}(D - \eta_j))[y] = 0$, $w_k = W(\xi_{k,k+1}, \xi_{k,k+2}, \ldots, \xi_{k,n})$ and $w_{k,j}$ is w_k with the $(j - k)$-th column replaced by $(0, 0, \ldots, 0, 1)$.

If $k = 0$, $z = y$ and (6) becomes

$$y = \int_x^\infty g_0(x, t)B(t, y)dt. \tag{7}$$

If $1 \leqq k \leqq n$, then (6) can again be solved in the same way yielding

$$y = \int_b^x G_k(x, t)\left(\int_t^\infty g_k(t, s)B(s, \vec{y})ds\right)dt \text{ for } 1 \leqq k \leqq n-1 \tag{8}$$

and

$$y = -\int_b^x G_n(x, g)B(t, \vec{y})dt \text{ for } k = n \tag{9}$$

where

$$G_k(x, t) = \sum_{j=1}^{k} [\phi_{k,j}(x)W_{k,j}(t)/W_k(t)],$$

$\phi_{k,j}$ are k linearly independent solutions of $L_k[y] = 0$, $W_k(t) = W(\phi_{k,1}, \ldots, \phi_{k,k})$ and $W_{k,j}$ is W_k with the j-th column replaced by $(0, 0, \ldots, 0, 1)$.

To show the expressions (7), (8), and (9) are indeed solutions of (4), we first note that $(\partial^i/\partial x^i)G_k(x, t) = \sum_{j=1}^{k}[\phi_{k,j}^{(i)}(x)W_{k,j}(t)/W_k(t)]$, $1 \leqq k \leqq n$, hence $(\partial^i/\partial x^i)G_k(x, x) = 0$, $0 \leqq i \leqq k-2$ and $(\partial^{k-1}/\partial x^{k-1})G(x, x) = 1$. Thus, using Leibniz's formula,

$$\frac{d^k}{dx^k}\int_b^x G_k(x, t)\left(\int_t^\infty g_k(t, s)B(s, \vec{y})ds\right)dt$$
$$= \int_x^\infty g_k(x, t)B(t, \vec{y})dt + \int_b^x \frac{\partial^k}{\partial x^k} G_k(x, t)\left(\int_t^\infty g_k(t, s)B(s, \vec{y})ds\right)dt.$$

Thus it follows that

$$L_k\left[\int_b^x G_k(x, t)\left(\int_t^\infty g_k(t, s)B(s, \vec{y})ds\right)dt\right] = \int_x^\infty g_k(x, t)B(t, \vec{y})dt.$$

If $k = n$, we may use Leibniz' rule to show that

$$\frac{d^n}{dx^n}\int_b^x G_n(x, t)B(t, \vec{y})dt = B(x, \vec{y}) + \int_b^x \frac{\partial^n}{\partial x^n} G_n(x, t)B(t, \vec{y})dt.$$

Thus it follows that $L_n[-\int_b^x G_n(x, t)B(t, \vec{y})dt] = -B(x, \vec{y})$. For $0 \leqq k \leqq n-1$ we have ([1], p. 443)

$$\frac{d^{n-k}}{dx^{n-k}}\int_x^\infty g_k(x, t)B(t, \vec{y})dt = -B(x, \vec{y}) + \int_x^\infty \frac{\partial^{n-k}}{\partial x^{n-k}} g_k(x, t)B(t, \vec{y})dt.$$

From these equations we see that

$$\left(\prod_{j=k+1}^{n}(D - \eta_j)\right)\left[\int_x^\infty g_k(x, t)B(t, \vec{y})dt\right] = -B(x, \vec{y})$$

and so

$$L_n\left[\int_b^x G_k(x, t)\left(\int_t^\infty g_k(t, s)B(s, \vec{y})ds\right)dt\right] = -B(x, \vec{y}).$$

For convenience we define

$$I_0^0(x, B(\vec{y})) = \int_x^\infty g_0(x, t)B(t, \vec{y})dt,$$
$$I_k^0(x, B(\vec{y})) = \int_b^x G_k(x, t)\left(\int_t^\infty g_k(t, s)B(s, \vec{y})ds\right)dt, \ 1 \leqq k \leqq n-1,$$
$$I_n^0(x, B(\vec{y})) = -\int_b^x G_n(x, t)B(t, \vec{y})dt,$$

and

$$I_k^i(x, B(\vec{y})) = L_i[I_k^0(x, B(\vec{y}))], \ 0 \leqq k \leqq n, \ 1 \leqq i \leqq n.$$
$$\ell_k^i[y] = \left(\prod_{j=k+1}^{i}(D - \eta_j)\right)[y], \ 0 \leqq k \leqq n-1, \ k+1 \leqq i \leqq n,$$

and

$$L_0[y] = \ell_k^k[y] = y, \ 0 \leqq k \leqq n-1.$$

Thus $I_k^n(x, B(\vec{y})) = -B(x, \vec{y})$, $0 \leqq k \leqq n$, and we have shown that if $y(x)$ is a solution of $y(x) = I_k^0(x, B(\vec{y}))$, $0 \leqq k \leqq n$, then $y(x)$ is a solution of (4).

In the following we shall suppose the Mammana factorization is valid on the appropriate half-line (see [5]) and that the coefficients of the homogeneous equation are continuous there.

THEOREM 1. *Let* $B(x, u_0, u_1, \ldots, u_{n-1})$ *be continuous on* D: $a \leqq x < \infty$, $-\infty < u_i < \infty$, $0 \leqq i \leqq n-1$. *Let* $F(x)$ *be continuous on* $[a, \infty)$ *such that* $|B(x, \vec{u})| \leqq F(x)$ *for each* $(x, \vec{u}) \in D$. *Let* ϕ *be an arbitrary solution of* $L_n[y] = 0$.

Then for all $b \geqq a$, *equation* (4) *has at least one solution* $y_n(x)$ *defined at least on* $[b, \infty)$ *satisfyjng*

$$(S) \qquad y_n^{(i)}(b) = \phi^{(i)}(b), \ 0 \leqq i \leqq n-1,$$

and

$$(S_1) \qquad L_i[y_n] = L_i[\phi] + O\left(\int_b^x \left| L_i[G_n(x, t)] \right| \cdot F(t)dt\right), \ 0 \leqq i \leqq n-1.$$

Further, if for some k, $0 \leqq k \leqq n-1$, $\int_a^\infty |g_k(x, t)| \cdot F(t)dt < \infty$, *then for all* $b \geqq a$ *the equation* (4) *has at least one solution* $y_k(x)$ *defined at least on* $[b, \infty)$ *satisfying*

$$(S') \qquad y_k^{(i)}(b) = \phi^{(i)}(b), \ 0 \leqq i \leqq k-1,$$

$$(S_1') \qquad L_i[y] = L_i[\phi] + O\left(\int_b^x |L_i[G_k(x, t)]|\left(\int_t^\infty |g_k(t, s)| \cdot F(s)ds\right)dt\right), \quad 0 \leqq i \leqq k-1,$$

$$(S_2') \qquad L_i[y] = L_i[\phi] + O\left(\int_x^\infty |\mathscr{L}_k^i[g_k(x, t)]| \cdot F(t)dt\right), \ k \leqq i \leqq n-1.$$

PROOF. Consider the equations

$$(10_k^i) \qquad L_i[y_k] = L_i[\phi] + I_k^i(x, B(\vec{y}_k)), \ 0 \leqq i, k \leqq n.$$

If $y_k(x)$ is a solution of (10_k^0), then $y_k(x)$ is a solution of (10_k^n), which is (2). Thus it will suffice to show that (10_k^0) has a solution with the stated properties.

We set $L_i[y_{k,1}] = L_i[\phi]$ and use equations (10_k^i) for the successive approximations

$$(11_k^i) \qquad L_i[y_{k,m+1}] = L_i[\phi] + I_k^i(x, B(\vec{y}_{k,m})), \ m = 1, 2, 3, \ldots, \quad 0 \leqq i, k \leqq n.$$

Since $L_i[y_{k,m+1}]$ is a linear combination of $y_{k,m+1}^{(j)}$, $0 \leqq j \leqq i$, in which the coefficient of $y_{k,m+1}^{(i)}$ is 1, we can solve equations (11_k^i) for $y_{k,m+1}^{(i)}$ in terms of $y_{k,m+1}^{(j)}$, $0 \leqq j \leqq i-1$, and $\vec{y}_{k,m}$. Thus the successive approximations $\vec{y}_{k,m}$ are well defined.

Let N be the least integer such that $N > b \geqq a$ and set $I_q = [b, N+q]$, $q = 0, 1, 2, \ldots$. We now show that the sequences $\{I_k^i(x, B(\vec{y}_{k,m}))\}_{m=1}^{\infty}$, $0 \leqq i \leqq n$, are uniformly bounded on I_q by some number D_q. First we observe that

$$(12) \qquad |I_k^i(x, B(\vec{y}_{k,m}))| \leqq \begin{cases} \displaystyle\int_b^x |L_i[G_k(x,t)]| \left(\int_t^\infty |g_k(t,s)| \cdot F(s)ds\right)dt, \\ \qquad\qquad 0 \leqq i \leqq k-1 < n-1, \\ \displaystyle\int_x^\infty |\mathscr{L}_k^i[g_k(x,t)]| \cdot F(t)dt, \ 0 \leqq k \leqq i \leqq n-1, \\ \displaystyle\int_b^x |L_i[G_n(x,t)]| \cdot F(t)dt, \ 0 \leqq i \leqq n-1, k = n, \\ F(x), \ 0 \leqq k \leqq n, i = n. \end{cases}$$

Thus the right hand side of (12) is bounded by a continuous function which is independent of m, and the left hand side can therefore be bounded on I_q by some number D_q which is independent of m.

If the sequences $\{I_k^i(x, B(\vec{y}_{k,m}))\}$ are uniformly bounded on I_q, it can be shown by induction on i that the sequences $\{y_{k,m}^{(i)}\}$, $0 \leqq i \leqq n$, are likewise uniformly bounded on each interval I_q by some number B_q independent of m. To see this, first note that $y_{k,m+1} = \phi + I_k^0(x, B(\vec{y}_{k,m}))$. Since ϕ is continuous on each I_q and $\{I_k^0(x, B(\vec{y}_{k,m}))\}$ is uniformly bounded there, the result follows for $i = 0$. Suppose now that it is true for some $i-1 < n$. Now $L_i[y_{k,m+1}]$ is a linear combination of $y_{k,m+1}^{(j)}$, $0 \leqq j \leqq i$, in which the coefficient of $y_{k,m+1}^{(i)}$ is 1 and the coefficients of $y_{k,m+1}^{(j)}$, $0 \leqq j \leqq i-1$, call them $\xi_k^j(x)$, $0 \leqq j \leqq i-1$, are continuous. Thus we may write

$$|y_{k,m+1}^{(i)}| \leqq \sum_{j=0}^{i-1} |\xi_k^j y_{k,m+1}^{(i)}| + |L_i[\phi]| + |I_k^i(x, B(\vec{y}_{k,m}))|.$$

Since the ξ_k^j and ϕ are continuous on I_q and the sequence $\{I_k^i(x, B(\vec{y}_{k,m}))\}$ is uniformly bounded there, the assertion follows from the induction assumption.

Since the sequences $\{y_{k,m}^{(i)}\}$, $1 \leqq i \leqq n$, are uniformly bounded on I_q, the sequences $\{y_{k,m}^{(i)}\}$, $0 \leqq i \leqq n-1$, are equicontinuous there. Hence we can extract from these latter sequences subsequences $\{y_{0,k,m}^{(i)}\}$ which converge uniformly on I_0 to a limit function y_k^i. For the same reason we can extract subsequences $\{y_{1,k,m}^{(i)}\}$ of $\{y_{0,k,m}^{(i)}\}$ which converge uniformly on I_1 to a limit function which we may also call y_k^i because it agrees with y_k^i on I_0, since $\{y_{1,k,m}^{(i)}\}$ is a subsequence of $\{y_{0,k,m}^{(i)}\}$. Inductively, we extract subsequences $\{y^{(i)}_{q+1,k,m}\}$ of $\{y^{(i)}_{q,k,m}\}$ which converge uniformly on I_{q+1} to a limit function which we may again call y^i_k, since it agrees with the limit of $\{y^{(i)}_{q,k,m}\}$ on I_q. This defines y^i_k on $[b, \infty)$. The diagonal sequence $\{y^{(i)}_{j-1,k,j}\}^{\infty}_{j=1}$ converges uniformly on every compact subinterval of $[b, \infty)$ to y^i_k. It follows that $y^i_k = (d^i/dx^i)(y^0_k)$, $0 \leqq i \leqq n-1$ (see R.G. Bartle [2], p. 217). We write $y^0_k = y_k$ and show that y_k is the required solution.

We first consider the case in which $0 \leqq k \leqq n-1$. Now B is continuous on D so that $\lim_{j\to\infty} B(x, \vec{y}_{j-1,k,j}) = B(x, \vec{y}_k)$ for all $x \in [b, \infty)$. Moreover, $|B(x, \vec{y}_k)| \leqq F(x)$ for all $x \in [b, \infty)$ and $\int_b^\infty |g_k(x, t)| \cdot F(t)dt < \infty$. It now follows from Lebesgue's Dominated Convergence Theorem that for all $x \in [b, \infty)$

$$(13^i_k) \qquad \begin{aligned} L_i[y_k] &= \lim_{j\to\infty}\{L_i[\psi] + I^i_k(x, B(\vec{y}_{j-1,k,j}))\} \\ &= L_i[\psi] + I^i_k(x, B(\vec{y}_k)),\ 0 \leqq i \leqq n-1, \end{aligned}$$

whence $y_k(x)$ is a solution of (10^0_k) and thus of (4) also. If $k = n$ then, without appeal to Lebesgue's Theorem,

$$(14^i_n) \qquad \begin{aligned} L_i[y_n] &= \lim_{j\to\infty} L_i[\psi] + I^i_n(x, B(\vec{y}_{j-1,n,j}))\} \\ &= L_i[\psi] + I^i_n(x, B(\vec{y}_n)),\ 0 \leqq i \leqq n-1, \end{aligned}$$

so that $y_n(x)$ is a solution of (10^0_n) and thus of (4).

As for the properties (S) and (S'), we first observe that $y_k(b) = \psi_k(b)$, since $I^0_k(b, B(\vec{y}_k)) = 0$, $1 \leqq k \leqq n$. Then, since $(D - \eta_1)[y_k] = (D - \eta_1)[\psi] + I^1_k(x, B(\vec{y}_k))$ and $I^1_k(b, B(\vec{y})) = 0$, $2 \leqq k \leqq n$, it follows that $y'_k(b) = \psi'(b)$. Likewise, since $L_2[y_k] = L_2[\psi] + I^2_k(x, B(\vec{y}_k))$ and $I^2_k(b, B(\vec{y}_k)) = 0$, $3 \leqq k \leqq n$, we have $y''_k(b) = \psi''(b)$. Since the coefficient of $y^{(i)}_k$ in $L_i[y_k]$ is not zero, this reasoning may be continued so long as $I^i_k(b, B(\vec{y}_k)) = 0$, namely for $0 \leqq i \leqq k-1$.

The proof is complete when we observe that, in view of equations (14^i_k), $0 \leqq k \leqq n$, $0 \leqq i \leqq n-1$, properties (S_1), (S'_1) and (S'_2) follow immediately from the inequality (12) with $\vec{y}_{k,m}$ replaced by $\vec{y}_k$.

Note that Theorem 1 offers an asymptotic comparison between solutions of $L_n y + B(x, \vec{y}) = 0$ and those of $L_n y = 0$.

II. Perturbed constant cofficient equations.

2.1. An integral condition for the nonlinear terms. In this section we shall restrict $L_n y$ to the constant coefficient operator $y^{(n)} + \sum_{r=1}^n a_i y^{(n-i)}$ where the a_i are complex constants. The solutions of the homogeneous equation (1) are linear combinations of functions of the form $\zeta(x) = x^j e^{\lambda x}$, where λ is a root of the characteristic polynomial, $r^n + \sum_{r=1}^n a_i r^{n-i}$ and j is a non-negative integer less than the multiplicity of λ. We shall refer to these solutions as standard form solutions. The Green's function in this case can be written as a linear combination of such ζ, evaluated at the argument $x - t$ [4]. We shall begin by defining a partial order and equivalence on these functions.

Let $\zeta_1 = x^j e^{\lambda x}$ and $\zeta_2 = x^k e^{\beta x}$ be two such functions. Then, if (i) $\operatorname{Re}(\lambda - \beta) > 0$ or (ii) $\operatorname{Re}(\lambda - \beta) = 0$ and $j-k < 0$, we write $\zeta_1 \ll \zeta_2$, while if (iii) $\operatorname{Re}(\lambda - \beta) < 0$ or (iv) $\operatorname{Re}(\lambda - \beta) = 0$ and $j-k > 0$, we write $\zeta_1 \gg \zeta_2$. Finally, if $\operatorname{Re}(\lambda - \beta) = 0$ and $j-k = 0$, we write $\zeta_1 \equiv \zeta_2$. Then $\zeta_1 \lll \zeta_2$ means $\zeta_1 \ll \zeta_2$ or $\zeta_1 \equiv \zeta_2$.

Let $F(t)$ be a non-negative function and set $I_1 = \int_x^\infty |\zeta_1(x - t)| \cdot F(t) dt$ and $I_2 = \int_x^\infty |\zeta_2(x - t)| \cdot F(t) dt$. If $\zeta_1 \lll \zeta_2$, then, if I_2 exists, so does I_1, while if I_1 diverges, so does I_2. If $\zeta_1 \equiv \zeta_2$, then I_1 exists if and only if I_2 exists.

We will use the same notation here as in Theorem 1, except that, because k will be uniquely determined, we drop the subscript k from the notation for some functions.

Order the n independent standard form solutions of $L_n[y] = 0$ so that, calling them ζ_i, $\zeta_i \ggg \zeta_{i+1}$, $1 \leqq i \leqq n-1$. Thus, if $\int_x^\infty |\zeta_i(x - t)| \cdot F(t) dt$ exists so does $\int_x^\infty |\zeta_{i+1}(x - t)| \cdot F(t) dt$, $1 \leqq i \leqq n-1$. Let n_i and λ_i be such that

$$\zeta_i = x^{n_i} e^{\lambda_i x}, \; 1 \leqq i \leqq n.$$

Note that $\zeta_{k+1}, \ldots, \zeta_n$ are the standard form solutions of the equation

$$\ell_k^n[y] = \Big(\prod_{j=k+1}^{n} (D - \lambda_j)\Big)[y] = 0.$$

This is so because, if

$$x^{n_i} e^{\lambda_i x} \in \{\zeta_{k+1}, \ldots, \zeta_n\},$$

then

$$x^{n_i - 1} e^{\lambda_i x} \in \{\zeta_{k+1}, \ldots, \zeta_n\},$$

since

$$x^{n_i} e^{\lambda_i x} \gg x^{n_i - 1} e^{\lambda_i x}.$$

DEFINITION 1. We define k to be the smallest integer for which there exist a non-negative integer N and real number c such that both $\zeta_{k+1} \lll x^N e^{cx}$ and

$$(A) \qquad \int_a^\infty t^N e^{-ct} F(t) dt < \infty,$$

provided such a k exists. In this case, $\int_x^\infty \zeta_i(x - t) F(t) dt$ exists, where $k+1 \leqq i \leqq n$.

Denoting the k solutions of $(\prod_{j=1}^{k}(D - \lambda_j))[y] = 0$ by

$$\phi_i = x^{n_i}e^{\lambda_i x},\ 1 \leqq i \leqq k,$$

we order the set $\{\phi_1, \ldots, \phi_k\}$ so that if $\mathrm{Re}(\lambda_i - \lambda_j) > 0$, then $i < j$; and if $\mathrm{Re}(\lambda_i - \lambda_j) = 0$ and $n_i > n_j$, then $i < j$. Note that $\mathrm{Re}\,(c - \lambda_i) \geqq 0$, $1 \leqq i \leqq k$. Further, if $\mathrm{Re}(c - \lambda_i) = 0$, then $\mathrm{Re}(c - \lambda_j) = 0$, $1 \leqq j \leqq i$, while if $\mathrm{Re}(c - \lambda_i) > 0$, then $\mathrm{Re}(c - \lambda_j) > 0$, $i \leqq j \leqq k$.

For $k+1 \leqq i \leqq n$, $\mathrm{Re}(c - \lambda_i) \leqq 0$. In this case, if $\mathrm{Re}(c - \lambda_i) = 0$, then $\mathrm{Re}(c - \lambda_j) = 0$, $k+1 \leqq j \leqq i$, while if $\mathrm{Re}(c - \lambda_i) < 0$, then $\mathrm{Re}(c - \lambda_j) < 0$, $i \leqq j \leqq n$.

Let $m_{i+1}(\lambda)$ be the multiplicity of λ as a root of $\prod_{j=i+1}^{n}(r - \lambda_j)$, $0 \leqq i \leqq n-1$. Then let $m_{i+1} = \max_{\mathrm{Re}\,\lambda = c}\{m_{i+1}(\lambda)\}$. Thus if all the roots of $\prod_{j=1}^{n}(r - \lambda_j)$, the characteristic polynomial of L_n, are real, then $m_{i+1} = m_{i+1}(c)$, $0 \leqq i \leqq n-1$.

Suppose first that $k \leqq i \leqq n-1$. If $\mathrm{Re}(c - \lambda_{i+1}) < 0$, then $m_{i+1} = 0$, because no λ with real part c appears in the list $\{\lambda_{i+1}, \ldots, \lambda_n\}$, while if $\mathrm{Re}(c - \lambda_{i+1}) = 0$, then $m_{i+1} = n_{i+1} + 1$, because no λ with real part c can occur more times in the list $\{\lambda_{i+1}, \ldots, \lambda_n\}$ than λ_{i+1} occurs.

Suppose now that $0 \leqq i \leqq k-1$. If $\mathrm{Re}(c - \lambda_{i+1}) > 0$, then $m_{i+1} = m_{k+1}$, because no λ with real part c occurs in the list $\{\lambda_{i+1}, \ldots, \lambda_k\}$. Suppose, on the other hand, that $\mathrm{Re}(c - \lambda_{i+1}) = 0$. Then λ_{i+1} appears exactly m_{k+1} times in the list $\{\lambda_{k+1}, \ldots, \lambda_n\}$, because some λ with real part c appears m_{k+1} times in this list so that

$$x^{m_{k+1}-1}e^{\lambda_{i+1}x} \ll x^N e^{cx}.$$

Then $m_{i+1} = n_{i+1} + 1 + m_{k+1}$ because no λ with real part c can occur more times in the list $\{\lambda_{i+1}, \ldots, \lambda_k\}$ than λ_{i+1} occurs. Thus we see that for $0 \leqq i \leqq n-1$, if $\mathrm{Re}(c - \lambda_{i+1}) = 0$, then $m_{i+1} = m_{i+1}(\lambda_{i+1})$.

We now state three lemmas that we will find useful in estimating the asymptotic size of the integrals I_k^i. All three can be proved by induction on n and N, integration by parts and l'Hôpital's rule.

LEMMA 1. *Let n and N be non-negative integers and α and λ complex numbers. Let*

$$J_n^N(x) = \int_b^x (x - t)^n\, e^{\lambda(x-t)}\, t^N e^{\alpha t} dt.$$

Then, (a) *if $\alpha - \lambda \neq 0$, then $J_n^N(x) = P_N(x)e^{\alpha x} + Q_n(x)e^{\lambda x}$, where $P_N(x)$ and $Q_n(x)$ are polynomials of degree at most N and n, respectively, and* (b) *if $\alpha - \lambda = 0$, then $J_n^N(x) = P_{N+n+1}(x)e^{\alpha x}$ where $P_{N+n+1}(x)$ is a polynomial of degree $N + n + 1$.*

LEMMA 2. *Let n be a non-negative integer, N a positive integer and α and λ real numbers. Let*

$$J_n^N(x) = \int_b^x ((x-t)^n/t^N)e^{\lambda(x-t)}e^{\alpha t}dt,\ b < 0.$$

Then (a) *if* $\alpha - \lambda > 0$, *then* $\lim_{x\to\infty} J_n^N(x)/x^{-N}e^{\alpha x} = n!/(\alpha-\lambda)^{n+1}$, *and* (b) *if* $\alpha - \lambda = 0$, *then* $\lim_{x\to\infty} J_n^1(x)/x^n(\log x)e^{\alpha x} = 1$.

LEMMA 3. *Let n be a non-negative integer and let (A) be satisfied. Let c and N be as in (A) and λ a complex number. With $F(t)$ the same as in (A), put*

$$J_n(x) = \int_x^\infty (x-t)^n e^{\lambda(x-t)}F(t)dt.$$

Then, (a) *if* $\mathrm{Re}(c-\lambda) < 0$, *then* $\lim_{x\to\infty} J_n(x)/x^{-N}e^{cx} = 0$, *and* (b) *if* $\mathrm{Re}(c-\lambda) = 0$ *and* $n \leqq N$, *then* $\lim_{x\to\infty} J_n(x)/x^{n-N}e^{cx} = 0$.

We are now prepared to prove the following theorem.

THEOREM 2. *Let $B(x, u_0, u_1, \ldots, u_{n-1})$ be continuous on D: $a \leqq x < \infty$, $-\infty < u_i < \infty$, $0 \leqq i \leqq n-1$. Let $F(x)$ be continuous on $[a, \infty)$ such that $|B(x, \vec{u})| \leqq F(x)$ for each $(x, \vec{u}) \in D$. Let c, N and k be as in Definition* 1, *and we suppose such a k exists.*

If ψ is any solution of $L_n[y] = 0$, then for all $b \geqq a$ the equation (4) *has a solution $y(x)$ defined at least on $[b, \infty)$ satisfying*

$$(S) \qquad y^{(i)}(b) = \psi^{(i)}(b),\ 0 \leqq i \leqq k-1,$$

and, for each i for which $m_{i+1} = 0$, $0 \leqq i \leqq n-1$,

$$(S_1) \qquad L_i[y] = L_i[\psi] + o(x^{-N}e^{cx}),$$

while for each i for which $m_{i+1} \neq 0$, $0 \leqq i \leqq n-1$,

$$(S_2) \qquad L_i[y] = L_i[\psi] + o(x^{m_{i+1}-1-N}e^{cx}).$$

PROOF. It follows from the definition of k that $\int_x^\infty |\zeta_i(x-t)|\cdot F(t)dt < \infty$, $k+1 \leqq i \leqq n$, so that $\int_x^\infty |g_k(x, y)|\cdot F(t)dt < \infty$, whence the proof of Theorem 1 applies. Thus it remains only to show validity of properties (S_1) and (S_2). Thus we must estimate the asymptotic size of the $I_k^i(x, B(x, \vec{y}))$, $0 \leqq i \leqq n-1$.

For this purpose, we will show by induction that a linear combination of $\{L_i[\phi_1], \ldots, L_i[\phi_k]\}$ is a linear combination of $\{\phi_{i+1}, \ldots, \phi_k\}$, $0 \leqq i \leqq k-1$. The case $i = 0$ is trivial. Suppose now that it is true for some j, $0 \leqq j < k-1$, and let z be a linear combination of $\{L_{j+1}[\phi_1], \ldots, L_{j+1}[\phi_k]\}$. Then it follows from the induction hypothesis that z is a linear combination of $\{(D-\lambda_{j+1})[\phi_{j+1}], \ldots, (D-\lambda_{j+1})[\phi_k]\}$. Since

$$(D-\lambda_{j+1})[\phi_{j+1}] = n_{j+1}x^{n_{j+1}-1}e^{\lambda_{j+1}x},$$

it is now clear that z is a linear combination of $\{\phi_{j+2}, \ldots, \phi_k\}$. This completes the induction. In like manner, we may show that every linear combination of $\{\ell_k^i[\zeta_{k+1}], \ldots, \ell_k^i[\zeta_n]\}$ is a linear combination of $\{\zeta_{i+1}, \ldots, \zeta_n\}$, $k \leqq i \leqq n-1$.

It now follows that for $k \leqq i \leqq n-1$, $I_k^i(x, B(x, \vec{y})) = \int_x^\infty \ell_k^i[g_k(x, t)] B(t, \vec{y})dt$ is a linear combination of $\int_x^\infty \zeta_j(x - t)B(t, \vec{y})dt$, $i+1 \leqq j \leqq n$. Thus it suffices to examine

$$H_j(x) = \int_x^\infty |x - t|^{n_j} e^{\operatorname{Re} \lambda_j(x-t)} F(t)dt, \; i+1 \leqq j \leqq n.$$

According to Lemma 3 the largest, asymptotically, of the H_j, $i+1 \leqq j \leqq n$, is H_{i+1}. If $\operatorname{Re}(c - \lambda_{i+1}) < 0$, then $m_{i+1} = 0$ and $H_{i+1} = o(x^{-N}e^{cx})$ by Lemma 3(a), while if $\operatorname{Re}(c - \lambda_{i+1}) = 0$, then $m_{i+1} = n_{i+1} + 1 \neq 0$ and

$$H_{i+1} = o(x^{m_{i+1}-1-N}e^{cx})$$

by Lemma 3(b). From these asymptotic relations follow the asymptotic relations of properties (S_1) and (S_2) for $k \leqq i \leqq n-1$.

For $0 \leqq i \leqq k - 1$, $I_k^i(x, B(x, \vec{y})) = \int_b^x L_i[G_k(x, t)](\int_t^\infty g_k(t, s)B(s, \vec{y})ds)dt$ is a linear combination of $\int_b^x \phi_j(x - t)(\int_t^\infty \zeta_\ell(t - s)B(s, \vec{y})ds)dt$, where $i+1 \leqq j \leqq k$ and $k+1 \leqq \ell \leqq n$. Thus it is enough to consider

$$H_{j,\ell}(x) = \int_b^x (x - t)^{n_j} e^{\operatorname{Re}\lambda_j(x-t)} H_\ell(t)dt, \; i+1 \leqq j \leqq k, \; k+1 \leqq \ell \leqq n.$$

Suppose first that $\operatorname{Re}(c - \lambda_{k+1}) < 0$. If $\operatorname{Re}(c - \lambda_{i+1}) = 0$, then

$$e^{\lambda_{i+1}x} \lll x^N e^{cx},$$

regardless of N. But then λ_{i+1} is included in the list $\{\lambda_{k+1}, \ldots, \lambda_n\}$, contradicting the assumption that $\operatorname{Re}(c - \lambda_{k+1}) < 0$. Thus, if $\operatorname{Re}(c - \lambda_{k+1}) < 0$, then $\operatorname{Re}(c - \lambda_{i+1}) > 0$ and $m_{i+1} = 0$ for $0 \leqq i \leqq k-1$.

We will use this last statement to show that

$$\lim_{x\to\infty} \frac{H_{j,\ell}(x)}{x^{-N}e^{cx}} = \frac{\int_b^x (x - t)^{n_j} e^{-\operatorname{Re}\lambda_j t} H_\ell(t)dt}{x^{-N}e^{\operatorname{Re}(c-\lambda_j)x}} = 0,$$

$$i+1 \leqq j \leqq k, \; k+1 \leqq \ell \leqq n.$$

If the numerator on the right hand side is bounded above, then the result follows from the fact that $\operatorname{Re}(c - \lambda_j) > 0$. If the numerator is unbounded, then we may apply l'Hôpital's rule and induction on n_j.

First, if $n_j = 0$, then from l'Hôpital's rule we get

$$\lim_{x\to\infty} \frac{H_{j,\ell}(x)}{x^{-N}e^{cx}} = \lim_{x\to\infty} \frac{e^{-\operatorname{Re}\lambda_j k} H_\ell(x)}{x^{-N}e^{\operatorname{Re}(c-\lambda_j)x}[\operatorname{Re}(c - \lambda_j) - N/x]} = 0,$$

since $H_\ell = o(x^{-N}e^{cx})$, $k+1 \leqq \ell \leqq n$. The inductive step is a similar application of l'Hôpital's rule.

Suppose now that $\mathrm{Re}(c - \lambda_{k+1}) = 0$. Then $m_{i+1} \geqq m_{k+1} > 0$ and we must show that

$$\lim_{x\to\infty} \frac{H_{j,\ell}(x)}{x^{m_{i+1}-1-N}e^{cx}} = \lim_{x\to\infty} \frac{\int_b^x (x-t)^{n_j} e^{-\mathrm{Re}\lambda_j t} H_\ell(t)\,dt}{x^{m_{i+1}-1-N} e^{\mathrm{Re}(c-\lambda_j)x}},$$

$$i+1 \leqq j \leqq k,\ k+1 \leqq \ell \leqq n.$$

If $\mathrm{Re}(c - \lambda_{i+1}) > 0$, then $\mathrm{Re}(c - \lambda_j) > 0$ and either the numerator is bounded above and the result follows, or we may employ l'Hôpital's rule in a proof by induction on n_j, as in the previous case, using here the fact that

$$H_\ell = o(x^{m_{i+1}-1-N}e^{cx}).$$

If $\mathrm{Re}(c - \lambda_{i+1}) = 0$, then $m_{k+1} - 1 - N = 0$, To see this, first note that $m_{k+1} - 1 - N \leqq 0$, since $\zeta_{k+1} \lll x^N e^{cx}$. Suppose, then, that $m_{k+1} - 1 - N \leqq -1$. It follows that $m_{k+1} \leqq N$ so that

$$x^{m_{k+1}} e^{\lambda_{i+1}x} \lll x^N e^{cx}$$

and λ_{i+1} is included at least $m_{k+1} + 1$ times in the list $\{\lambda_{k+1}, \ldots, \lambda_n\}$, a contradiction. Thus $m_{k+1} - 1 - N = 0$ and $m_{i+1} - 1 - N > 0$. It is this last fact which validates the use of l'Hôpital's rule in still another similar inductive argument showing that here, too,

$$\lim_{x\to\infty} H_{j,\ell}(x)/x^{m_{i+1}-1-N}e^{cx} = 0.$$

These arguments complete the proof of Theorem 2.

2.2. A bound for the nonlinear term and two examples. In Theorem 2, we have assumed the existence of some solution ζ of $L_n[y] = 0$ such that $\zeta \lll x^N e^{cx}$. The c which satisfies this requirement may be much larger than that required to satisfy (A). Also, if $F(x) = x^M e^{c'x}$, M a non-negative integer and c' a real number, Theorem 2 requires $c' - c < 0$, since negative N is not allowed there, while it is desirable to have $c' - c = 0$. Therefore, we shall study the case $F(x) = x^M e^{cx}$, and include the additional case $k = n$, in which $\{\zeta_{k+1}, \ldots, \zeta_n\}$ is empty. For this we require three additional lemmas. All three use induction, Lemma 4 also using integration by parts and Lemma 5 using l'Hôpital's rule.

LEMMA 4. *Let n and N be non-negative integers and α and λ complex numbers such that* $\mathrm{Re}\,(\alpha - \lambda) < 0$. *Then*

$$J_n^N(x) = \int_x^\infty (x-t)^n e^{\lambda(x-t)} t^N e^{\alpha t}\,dt = P_N(x)e^{\alpha x}$$

where $P_N(x)$ is a polynomial of degree N.

LEMMA 5. *Let n be a non-negative integer and N a positive integer and α and λ complex numbers. Let*

$$J_n^N(x) = \int_x^\infty ((x-t)^n/t^N)e^{\lambda(x-t)}e^{\alpha t}dt.$$

Then, (a) *if* $\mathrm{Re}(\alpha - \lambda) < 0$, *then* $\lim_{x\to\infty} J_n^N(x)/x^{-N}e^{\alpha x} = -n!/(\alpha-\lambda)^{n+1}$, *and* (b) *if* $\alpha - \lambda = 0$, *then for* $N \geqq n+2$, $\lim_{x\to\infty} J_n^N(x)/x^{n-N+1}e^{\alpha x} = (-1)^n(N-n-2)!n!/(N-1)!$

LEMMA 6. *Let p be a non-negative integer. Then for* $0 \leqq q \leqq p$ *there exist numbers* $a_q > 0$ *and* b_q *such that* $[x^p \log x]^{(q)} = x^{p-q}[a_q \log x + b_q)$. *Moreover,* $[x^p \log x]^{(p+1)} = a_p/x$.

With the ζ_i ordered as before, and $m_{n+1} = 0$, we can now prove the following result.

THEOREM 3. *Let* $B(x, u_0, u_1, \ldots, u_{n-1})$ *be continuous on* $D: 0 \leqq a \leqq x < \infty$, $-\infty < u_i < \infty$, $0 \leqq i \leqq n-1$. *Suppose that for some integer M, real number c and positive constant c_0,* $|B(x, \vec{u})| \leqq c_0 x^M e^{cx}$ *for each* $(x, \vec{u}) \in D$. *Let k be the smallest integer such that* $\zeta_{k+1} \lll x^{-M-2}e^{cx}$, *provided such a k exists. If no such k exists, let $k = n$. Let* m_{i+1}, $0 \leqq i \leqq n-1$, *be defined as before and let ψ be an arbitrary solution of* $L_n[y] = 0$.

Then for all $b \geqq a$, *equation* (4) *has a solution $y(x)$ defined at least on* $[b, \infty)$ *satisfying*

$$(S) \qquad y^{(i)}(b) = \psi^{(i)}(b),\ 0 \leqq i \leqq k-1,$$

and, if M is negative and $\mathrm{Re}(c - \lambda_{i+1}) = 0$ *and* $0 \leqq i \leqq k-1$, *then*

$$(S_1) \qquad L_i[y] = L_i[\psi] + O(x^{m_{i+1}+M}(\log x)e^{cx}),$$

while, if M is non-negative or $\mathrm{Re}(c - \lambda_{i+1}) > 0$ *or* $k \leqq i \leqq n$, *then*

$$(S_2) \qquad L_i[y] = L_i[\psi] + O(x^{m_{i+1}+M}e^{cx}).$$

PROOF. Because $\int_x^\infty t^{-M-2}e^{-ct}t^M e^{ct}dt < \infty$, the proof of Theorem 1 applies, as before, and the task is again reduced to estimating the asymptotic size of the $I_k^i(x, B(x, \vec{y}))$, $0 \leqq i \leqq n-1$. We shall omit the details involved in the remainder of the proof which, although somewhat intricate, are similar to those used in the proof of Theorem 2.

We now give an example which shows that in Theorem 3 the right hand sides of (S_1) and (S_2) are actually achieved.

Let n be a positive integer and suppose that M is an integer such that $M \geqq 0$ or $-M \geqq n+1$. Then $M + n \geqq 1$ or $M + n \leqq -1$ and the function $\psi_p = x^{M+n}e^{\lambda x}/(M+1)(M+2)\cdots(M+n)$ is a solution of the equation $(D-\lambda)^n[y] = x^M e^{\lambda x}$, $M \geqq 0$ or $M \leqq -n-1$.

If M is non-negative, then (S_2) applies. If $-M \geqq n + 1$, then $n - 1 \leqq -M-2$ so that $\zeta_1 \ll x^{-M-2}e^{\lambda x}$ and $k = 0$. Thus (S_2) applies again. m_{i+1} is the multiplicity of λ as a root of $(D - \lambda)^{n-i}$, that is, $m_{i+1} = n - i$. Taking $\phi = 0$, Theorem 3 predicts a solution $y(x)$ satisfying $(D - \lambda)^i[y] = O(x^{N+n-i}e^{\lambda x})$, $0 \leqq i \leqq n-1$, and ϕ_p clearly satisfies this prediction. Since $M + n > n - 1$ in case M is non-negative and $M + n \leqq -1$ in case $-M \geqq n + 1$, it is not possible to improve the asymptotic estimates of $(D - \lambda)^i[y(x)]$ by adding to ϕ_p a solution of the homogeneous equation $(D - \lambda)^n[z] = 0$.

Let us now suppose that n is a positive integer and suppose that M is an integer such that $1 \leqq -M \leqq n$. It can be shown that

$$\phi_q = (-1)^{-M-1}e^{\lambda x}x^{n+M} \log x/(-M - 1)!(n + M)!$$

is a solution of $(D - \lambda)^n[y] = x^M e^{\lambda x}$, $-n \leqq M \leqq -1$.

Again k is the smallest integer such that $\zeta_{k+1} = x^{n-k-1}\, e^{\lambda x} \ll x^{-M-2}e^{\lambda x}$. But then $n-k-1 = -M-2$, that is, $k = n + M + 1$. Thus (S_1) applies for $0 \leqq i \leqq n + M$ while (S_2) applies for $n + M + 1 \leqq i \leqq n$. Again $m_{i+1} = n - i$ and, if we take $\phi = 0$, Theorem 3 predicts a solution $y(x)$ satisfying

$$(D - \lambda)^i[y] = O(x^{n+M-i}(\log x)e^{\lambda x}),\ 0 \leqq i \leqq n + M,$$

and

$$(D - \lambda)^i[y] = O(x^{n+M-i}e^{\lambda x}),\ n + M + 1 \leqq i \leqq n,$$

and ϕ_q clearly satisfies this prediction. Since $(D - \lambda)^i[y]$ contains a term with factor $\log x$, $0 \leqq i \leqq n + M$, and since $n + M - i < 0$, $n + M + 1 \leqq i \leqq n$, it is not possible to improve the asymptotic estimates of $(D - \lambda)^i[y(x)]$ by adding to ϕ_q a solution of the homogeneous equation $(D - \lambda)^n[z] = 0$.

Next we give an example to show that in Theorem 3, Re $\lambda_{i+1} \neq c$ cannot be changed to $\lambda_{i+1} \neq c$. More specifically, we wish to exhibit an example in which M is negative, $\mathrm{Re}(c - \lambda_{i+1}) = 0$, but $c - \lambda_{i+1} \neq 0$, for some i, $0 \leqq i \leqq k - 1$, and yet (S_2) still does not hold.

Consider the equation

$$(D - (1 + i))(D - (1 - i))[y] = (D^2 - 2D + 2)[y] = e^t \sin t/t.$$

Since $M = -1$, we have $x^{-M-2}e^{cx} = x^{-1}e^x$ and $k = 2$. Then, taking $\phi = 0$, the solution given by Theorem 3, which in this case is just the solution given by the Method of Variation of Parameters, is $y(x) = -e^x \int_b^x (\sin t \sin(t - x)/t)dt$. Because $m_1 = 1$ our goal is to show that $y(x)$ does not satisfy $O(e^x)$. If we choose $b = 2\pi$, then $|y(2k\pi)| > (e^{2k\pi}/2)(1/3 + 1/4 + \cdots + 1/2k)$, which completes the example.

This example shows that for $\mathrm{Re}(c - \lambda_{i+1}) = 0$, $\lambda_{i+1} - c \neq 0$ we may not use the criterion: Let k be the smallest integer so that $\zeta_{k+1} \ll x^{-M-1}e^{cx}$ because here $x^{-M-1}e^{cx} = e^x$. Thus according to this criterion $k = 0$ yet, noting that $G_2(x, t) = g_0(x, t)$, we see that $\int_x^\infty g_0(x, t)B(t)dt$ does not exist.

2.3. A linear nonhomogeneous equation. We have the following corollary to the proof of Theorem 3.

COROLLARY 1. *Let M be an integer and c a real number, and consider the equation*

$$(E) \qquad L_n[y] = x^M e^{cx}.$$

Suppose that all the roots of the characteristic polynomial of $L_n[y]$ are real. Let k be the smallest integer such that $\zeta_{k+1} \ll x^{-M-2}e^{cx}$, provided such a k exists. If no such k exists, let $k = n$. Let m_{i+1}, $0 \leqq i \leqq n - 1$, be defined as before.

Then for all $b > 0$ the equation (E) has a solution $y(x)$ defined at least on $[b, \infty)$ satisfying

$$(S) \qquad y^{(i)}(b) = 0,\ 0 \leqq i \leqq k-1,$$

and, if M is negative and $\lambda_{i+1} = c$ and $0 \leqq i \leqq k-1$, then

$$(S_1) \qquad \lim_{x\to\infty} L_i[y]/x^{m_{i+1}+M}(\log x)e^{cx} \text{ exists}$$

and, if M is non-negative or $\lambda_{i+1} < c$ or $k \leqq i \leqq n$, then

$$(S_2) \qquad \lim_{x\to\infty} L_i[y]/x^{m_{i+1}+M}e^{cx} \text{ exists.}$$

PROOF. Here we are taking $\psi = 0$ in Theorem 3. Since $\mathrm{Re}\ \lambda_j = \lambda_j$, $1 \leqq j \leqq n$, and $|x - t|^{n_j} = \pm (x - t)^{n_j}$, $t \geqq x$, for any non-negative integer n_j, the limits established in the proof of Theorem 3 suffice for this corollary.

If M is non-negative, then the result is well-known from the Method of Undetermined Coefficients, even if either the roots of the characteristic polynomial of $L_n[y]$ are not real or c is not real.

III. The equation $y^{(n)} + B(x, y, y', \ldots, y^{(n-1)}) = 0$.

3.1. Two theorems of Svec and some examples. Let us consider the case when $L_n[y] = D^n[y]$ and take $N = n - k' - 1$. If we take $c = 0$, then m_{i+1} is just the multiplicity of 0 as a root of the characteristic polynomial of the operator D^{n-i}, $0 \leqq i \leqq n - 1$, namely $n - i$. Thus $m_{i+1} - 1 - N = (n - i) - 1 - (n - k' - 1) = k' - i$. Further, k is the smallest integer so that $\zeta_{k+1} = x^{n-k-1} \ll x^{n-k'-1}$ so that $k' = k$, provided $0 \leqq k \leqq n - 1$. Then we can obtain the following corollary to Theorem 2.

COROLLARY 2. *Let $B(x, u_0, u_1, \ldots, u_{n-1})$ be continuous on $D: a < x < \infty$, $-\infty < u_i < \infty$, $0 \leqq i \leqq n-1$. Let $F(x)$ be continuous on (a, ∞) so that $|B(x, \vec{u})| \leqq F(x)$ for each $(x, \vec{u}) \in D$ and let k be the smallest integer such that $0 \leqq k \leqq n-1$ and $\int^\infty t^{n-k-1}F(t)dt < \infty$. Let ψ be an arbitrary solution of the homogeneous equation $y^{(n)} = 0$.*

Then for all $b > a$ equation (4) *has a solution $y(x)$ defined at least on $[b, \infty)$ satisfying*

$$(S) \qquad y^{(i)}(b) = \psi^{(i)}(b), \ 0 \leqq i \leqq k-1,$$

and

$$(S\ ) \qquad y^{(i)}(x) = \psi^{(i)}(x) + o(x^{k-i}), \ 0 \leqq i \leqq n-1.$$

PROOF. Let us call the a of Theorem 2 by a'. Then given $b > a$, if we take $a' = b$, then Theorem 2 gives the result immediately.

Since ψ is arbitrary and k is chosen as small as possible, this corollary contains two results of Svec (Theorem 1 of [7] and Theorem 2 of [8]).

Corollary 2 guarantees one solution of (4) on $[b, \infty)$ with properties (S) and (S_1), for each solution ψ of $L_n[y] = 0$. In fact, we may find $n-k-1$ solutions of (4), linearly independent on $[b, \infty)$ and satisfying properties (S) and (S_1). To see this, define y_j, $k \leqq j \leqq n-1$, to be the result of using the above approximations with $\psi = x^j$. Suppose now that some linear combination of the y_j, say $y = \sum_{j=k}^{n-1} c_j y_j$ is identically zero on $[b, \infty)$. Then $\lim_{x\to\infty} y/x^{n-1} = c_{n-1}$ so that $c_{n-1} = 0$. Considering, in turn $\lim_{x\to\infty} y/x^j$, $j = n-2, n-3, \ldots, k$, we conclude that all the c_j are zero.

We now give an example which shows that Theorem 2 does not include Theorem 3 and vice versa.

Consider the equation $y^{(4)} = x^{-2+\varepsilon}$, $x \geqq 1$, $|\varepsilon| < 1$ (this was just to avoid zero denominators in the following expressions). If $\varepsilon \neq 0$, then it is easy to see that $\psi_\varepsilon = x^{2+\varepsilon}/(2+\varepsilon)(1+\varepsilon)\varepsilon(-1+\varepsilon)$ is a solution satisfying

$$D^i[\psi_\varepsilon] = x^{2-i+\varepsilon}/(2-i+\varepsilon)(1-i+\varepsilon)\cdots(-1+\varepsilon), \ 0 \leqq i \leqq 4.$$

If $\varepsilon = 0$, then $\psi_0 = -x^2 \log x/2!$ is a solution satisfying

$$D^i[\psi_0] = -x^{2-i}\log x/(2-i)!, \ i = 0, 1, 2$$

and

$$D^i[\psi_0] = (-1)^i x^{2-i}, \ i = 3, 4.$$

Let us see now what kind of asymptotic estimates are offered by Theorem 2 and 3 in each of the three cases $\varepsilon < 0$, $\varepsilon = 0$, and $\varepsilon > 0$. First observe that m_{i+1}, $0 \leqq i \leqq 3$, is the multiplicity of 0 as a root of D^{4-i}, namely $4-i$. Because $c = 0$ is the best possible choice in either Theorem 2 or 3, we will confine ourselves to determining the best possible choices, given $c = 0$, for N (in Theorem 2) and M (in Theorem 3) and the corresponding asymptotic estimates. For Theorem 2 we require a non-negative integer N such that $\int_1^\infty t^{N-2+\varepsilon}dt < \infty$, while for Theorem 3 we require an integer M such that $|x^{-2+\varepsilon}| \leqq c_0 x^M$, $x \geqq 1$, for some c_0. We take $\psi = 0$ throughout this example.

First suppose $\varepsilon < 0$. Then for Theorem 2, the best choice we can make for N is $N = 1$. Since (S_2) applies here, Theorem 2 predicts a solution y satisfying $D^i[y] = o(x^{2-i})$, $0 \leqq i \leqq 3$. On the other hand, for Theorem 3, the best choice we can make for M is $M = -2$. Here k is the smallest integer so that $\zeta_{k+1} = x^{3-k} \ll 1$. Thus $k = 3$ and Theorem 3 predicts a solution z satisfying $D^i[z] = O(x^{2-i}\log x)$, $i = 0, 1, 2$, and $D^i[z] = O(x^{2-i})$, $i = 3, 4$. Thus in this case Theorem 2 provides the best asymptotic estimates for $0 \leqq i \leqq 3$.

Next suppose that $\varepsilon = 0$. Then the best choice for N in Theorem 2 is $N = 0$. Again (S_2) applies and Theorem 2 predicts a solution satisfying $D^i[y] = o(x^{3-i})$, $0 \leqq i \leqq 3$. However, we may still take $M = -2$ for Theorem 2 so that $k = 3$ again and the prediction of Theorem 3, which is the same as in the case $\varepsilon < 0$, is best this time.

Finally, suppose that $\varepsilon > 0$. Then we may again take $N = 0$ in Theorem 2, whose prediction is thus the same as in the case $\varepsilon = 0$. But for Theorem 3 the best possible choice for M in this case is $M = -1$. Since there is no k, $0 \leqq k \leqq 3$, so that $\zeta_{k+1} = x^{3-k} \ll x^{-1}$, we have $k = 4$ and Theorem 3 predicts a solution z satisfying $D^i[z] = O(x^{3-i}\log x)$, $0 \leqq i \leqq 3$, and $D^4[y] = O(x^{-1})$. Hence, for $0 \leqq i \leqq 3$ the estimates of Theorem 2 are better in this case.

3.2. An integral condition with a monotone nonlinear term. The next theorem, which draws upon techniques used by Belohorec ([3]), uses a different condition on the function $B(x, y, y', \ldots, y^{(n-1)})$. While it offers a partial converse, we can no longer use an arbitrary solution of the homogeneous equation as our initial approximation and we have no knowledge of how large the left hand end-point of the solution's interval of existence might be.

We first require some preliminary definitions. We let $P_k(x)$, $0 \leqq k \leqq n-1$, denote a polynomial of degree at most k. If c_i, $0 \leqq i \leqq n-1$, are constants, then we let

$$\vec{c}_k(x) = (c_0 x^k, c_1 x^{k-1}{}_k, \ldots, c_{k-1}\, x, c_k, c_{k+1}, \ldots, c_{n-1})$$

and if $f(x)$ is a function with $n-1$ derivatives, then we let

$$\vec{f}(x) = (f(x), f'(x), \ldots, f^{(n-1)}(x)).$$

DEFINITION 2. Let $B(x, u_0, u_1, \ldots, u_{n-1})$ be non-negative on D: $a \leqq x < \infty$, $-\infty < u_i < \infty$, $0 \leqq i \leqq n-1$, and monotone in each u_i for each fixed x. For given constants c_i, $0 \leqq i \leqq n-1$, and polynomial $P_k(x)$ of degree at most k, suppose there exists a $b_0 > 0$ and an $\varepsilon > 0$ such that for all $x \geqq b_0$ one of the following conditions is satisfied: If B is non-decreasing in u_i, then $(c_i - \varepsilon)x^{k-i} \geqq P_k^{(i)}(x)$, $0 \leqq i \leqq k$, and $c_i - \varepsilon \geqq P_k^{(i)}(x) = 0$, $k+1 \leqq i \leqq n-1$, while if B is non-increasing in u_i, then $(c_i + \varepsilon)x^{k-i} \leqq P_k^{(i)}(x)$, $0 \leqq i \leqq k$, and $c_i + \varepsilon \leqq P_k^{(i)}(x) = 0$, $k+1 \leqq i \leqq n-1$. Then we say that $\vec{c}_k(x)$ is eventually a bound for $P_i(x)$ with respect to B because $B(x, \vec{c}_k) \geqq B(x, \vec{p}_k)$ for all $x \geqq b_0$.

We may, of course, interchange the roles of $\vec{c}_k(x)$ and $\vec{p}_k(x)$ to define the notion that $\vec{p}_k(x)$ is eventually a bound for $\vec{c}_k(x)$ with respect to B. Specifically, it suffices to interchange the words "non-decreasing" and "non-increasing" in Definition 2.

We now prove the following theorem.

THEOREM 4. *Let $B(x, u_0, u_1, \ldots, u_{n-1})$ be continuous and non-negative in D: $a \leqq x < \infty$, $-\infty < u_i < \infty$, $0 \leqq i \leqq n-1$, and monotone in u_i, $0 \leqq i \leqq n-1$, for each fixed x. Let k be an integer such that $0 \leqq k \leqq n-1$ and let $P_k(x)$ be a polynomial of degree at most k. Suppose there exist numbers c_i, $0 \leqq i \leqq n-1$, such that $\vec{c}_k(x)$ is eventually a bound for $\vec{p}_k(x)$ with respect to B and*

$$(15) \qquad \int^{\infty} t^{n-k-1}B(t, \vec{c}_k)dt < \infty.$$

Then there exists a $b^* > a$ such that for all $b \geqq b^*$ the equation

$$(16) \qquad y^{(n)} + B(x, y, y', \ldots, y^{(n-1)}) = 0$$

has a solution $y_k(x)$ defined at least on $[b. \infty)$ satisfying

$$(S) \qquad y_k^{(i)}(b) = P_k^{(i)}(b), 0 \leqq i \leqq k-1,$$

and

$$(S_1) \qquad y_k^{(i)}(x) = P_k^{(i)}(x) + o(x^{k-i}), 0 \leqq i \leqq n-1.$$

Conversely, if (16) *has such a solution and $\vec{p}_k(x)$ js eventually a bound for $\vec{c}_k(x)$ with respect to B, then* (15) *holds.*

PROOF. We first consider the equations

$$(17_k^i) \qquad y_k^{(i)}(x) = P_k^{(i)}(x) + I_k^i(x, B(\vec{y}_k)), 0 \leqq i \leqq n,$$

and show that (17_k^0) has a solution $\vec{y}_k(x)$ satisfying properties (S) and (S_1).

Since $\vec{c}_k(x)$ is eventually a bound for $\vec{p}_k(x)$, there exists an $\varepsilon > 0$ and $b_0 \geqq 0$ as required by Definition 2. Then, according to (15), there exists a $b^* \geqq b_0$ such that for all $b \geqq b^*$

$$\int_b^{\infty} t^{n-i-1}B(t, \vec{c}_k)dt \leqq \varepsilon/2, k \leqq i \leqq n-1.$$

Hence for $b \geqq b^*$ and $x \geqq b$

$$
\begin{aligned}
(18)\qquad |I_k^i(x, B(\vec{c}_k))| &\leqq (\varepsilon/2)\int_b^x ((x-t)^{k-i-1}/(k-i-1)!)dt \\
&\leqq (\varepsilon/2)x^{k-i},\ 0 \leqq i \leqq k-1,
\end{aligned}
$$

and

$$
(19)\qquad |I_k^i(x, B(\vec{c}_k))| < \varepsilon/2,\ k \leqq i \leqq n-1.
$$

We set $y_{k,0}^{(i)}(x) = P_k^{(i)}(x)$ and for $m = 1, 2, 3, \ldots$ we use equations (17_k^i) for successive approximations $y_{k,m+1}^{(i)}(x)$, $0 \leqq i \leqq n$.

We now show inductively that $B(x, \vec{y}_{k,m}) \leqq B(x, \vec{c}_k)$ for all $x \geqq b^*$. Since $y_{k,0}^{(i)}(x) = P_k^{(i)}(x)$, $0 \leqq i \leqq n-1$, the assertion is true for $m = 0$ by hypothesis. Now suppose it is true for $m \geqq 0$ and observe that $y_{k,m+1}^{(i)}(x) = P_k^{(i)}(x) + I_k^i(x, B(\vec{y}_{k,m}))$.

By the induction assumption, $B(x, \vec{c}_k) \geqq B(x, \vec{y}_{k,m})$ for $x \geqq b^* \geqq b_0$. Thus, from equations (18) and (19),

$$
|I_k^{(i)}(x, B(x, \vec{y}_{k,m}))| \leqq (\varepsilon/2)x^{k-i},\ 0 \leqq i \leqq k-1,
$$

and

$$
|I_k^{(i)}(x, B(x, \vec{y}_{k,m}))| \leqq \varepsilon/2,\ k \leqq i \leqq n-1.
$$

Thus, because $\vec{c}_k(x)$ is eventually a bound for $\vec{p}_k(x)$, we have for $x \geqq b^* \geqq b_0$ that if B is non-decreasing in u_i, then

$$
(c_i - \varepsilon/2)x^{k-i} \geqq P_k^{(i)}(x) + (\varepsilon/2)x^{k-i} \geqq y_{k,m+1}^{(i)}(x),\ 0 \leqq i \leqq k,
$$

and

$$
c_i - \varepsilon/2 \geqq P_k^{(i)}(x) + \varepsilon/2 \geqq y_{k,m+1}^{(i)}(x),\ k+1 \leqq i \leqq n-1,
$$

while if B is non-increasing in u_i, then

$$
(c_i + \varepsilon/2)x^{k-i} \leqq P_k^{(i)}(x) - (\varepsilon/2)x^{k-i} \leqq y_{k,m+1}^{(i)}(x),\ 0 \leqq i \leqq k,
$$

and

$$
c_i + \varepsilon/2 \leqq P_k^{(i)}(x) - \varepsilon/2 \leqq y_{k,m+1}^{(i)}(x),\ k+1 \leqq i \leqq n-1.
$$

This suffices for the induction.

The remainder of the proof of the convergence of the successive approximations to a solution of (16) on $[b, \infty)$ satisfying properties (S) is the same as in Theorem 1, except that $B(x, \vec{c}_k)$ replaces $F(x)$.

To show the properties (S_1) it suffices to apply l'Hôpital's rule to the ratio $I_k^i(x, B(x, \vec{c}_k))/x^{k-i}$, and use condition (15).

Conversely, suppose that $y_k(x)$ is a solution of (16) with properties (S_1) on $[b, \infty)$. Although we shall omit the details it can be shown, by induction on j, that for $1 \leqq j \leqq n-k$, where $0 \leqq k \leqq n-1$,

(20) $$y_k^{(n-j)}(x) = P_k^{(n-j)}(x) + \int_x^\infty ((x-t)^{j-1}/(j-1)!)B(t, \vec{y}_x)dt,\ x \geqq b.$$

Taking $j = n - k$ in (20), we now obtain

(21) $$y_k^{(k)}(x) = k!\, a_k + \int_x^\infty ((x-t)^{n-k-1}/(n-k-1)!)B(t, \vec{y}_k)dt.$$

If $\vec{p}_k(x)$ is eventually a bound for $\vec{c}_k(x)$ with respect to B, then there exists a $b_1 \geqq b^*$ such that for all $x \geqq b_1$ we have $B(x, \vec{y}_k) \geqq B(x, \vec{c}_k)$. Thus, for all $x > b_1$ and all $A \geqq x$

$$\left| \int_{b_1}^A ((b_1 - t)^{n-k-1}/(n-k-1)!)B(t, \vec{y}_k)dt \right|$$
$$> \left| \int_{b_1}^A ((b_1 - t)^{n-k-1}/(n-k-1)!)B(t, \vec{c}_k)dt \right|.$$

Since, by equation (21), the monotone increasing limit as $A \to \infty$ exists on the left, it also does on the right. It now follows that $\int^\infty t^{n-k-1}B(t, \vec{c}_k)dt < \infty$ (see Apostol [1], p. 431), completing the proof of the theorem.

Note that in the converse portion of this theorem, having obtained equation (21), we can, with the properties (S), obtain all of the equations (17_k^i), with y replaced by y_k, since $P_k(x) + I_k^0(x, B(\vec{y}_k))$ is the unique solution of $z^{(k)}(x) = k!a_k + I_k^k(x, B(\vec{y}_k))$ satisfying $z^{(i)}(b) = P_k^{(i)}(b)$, $0 \leqq i \leqq k-1$. Thus, y is a solution of (16) satisfying properties (S) and (S_1) if and only if y is a solution of $y = P_k(x) + I_k^0(x, B(\vec{y}))$.

Suppose the leading coefficient of $P_k(x)$ is a_k. Then a necessary and sufficient condition to assure that $\vec{c}_k(x)$ is eventually a bound for $\vec{p}_k(x)$ with respect to B is that if B is non-decreasing in u_i, then $c_i > k(k-1) \cdots (k-i+1)a_k$, $0 \leqq i \leqq k$, and $c_i > 0$, $k+1 \leqq i \leqq n-1$, while if B is non-increasing in u_i, then $c_i < k(k-1) \cdots (k-i+1)a_k$, $0 \leqq i \leqq k$, and $c_i < 0$, $k+1 \leqq i \leqq n-1$. A necessary and sufficient condition to ensure that $\vec{p}_k(x)$ is eventually a bound for $\vec{c}_k(x)$ with respect to B is obtained by interchanging the words "non-decreasing" and "non-increasing" in the preceding condition.

The function B needn't be continuous on all of $-\infty < u_i < \infty$, $k \leqq i \leqq n-1$, since the initial and successive approximations are all bounded above in absolute value. A finite interval suffices provided $B(x, \vec{c}_k(x))$ and $B(x, \vec{p}_k(x))$ are defined.

Further, instead of requiring B to be non-negative and monotone in each u_i, we may postulate a function $F(x, u_0, u_1, \ldots, u_{n-1})$ such that $|B(x, u)| \leqq F(x, \vec{u})$ for each $(x, \vec{u}) \in D$ and impose on F all those conditions that are imposed upon $B(x, \vec{u})$ in Theorem 4.

We also have the following result.

COROLLARY 3. *Let $B(x, u_0, u_1, \ldots, u_{n-1})$ be continuous on D: $a < x < \infty$,*

$-\infty < u_i < \infty$, $0 \leqq i \leqq n-1$. *Let* $F(x)$ *be condinuous on* (a, ∞) *so that* $|B(x, \vec{u})| \leqq F(x)$ *for each* $(x, \vec{u}) \in D$ *and let k be an integer such that* $0 \leqq k \leqq n-1$ *and* $\int^\infty t^{n-k-1}F(t)dt < \infty$. *Let* $P_k(x)$ *be a polynomial of degree at most* k.

Then for all $b > a$ *the equation* (16) *has a solution* $y_k(x)$ *defined at least on* $[b, \infty)$ *satisfying*

$$(S) \qquad y^{(i)}(b) = P_k^{(i)}(b),\ 0 \leqq i \leqq k-1,$$

and

$$(S_1) \qquad y^{(i)}(x) = P_k^{(i)}(x) + o(x^{k-i}),\ 0 \leqq i \leqq n-1.$$

Conversely, if (16) *has such a solution* $y_k(x)$, *then*

$$(A_1) \qquad \int^\infty t^{n-k-1}B(t, \vec{y}_k)dt < \infty.$$

PROOF. The proof of the convergence of the successive approximations to a solution of (16) on $[b, \infty)$ with properties (S) follows as in Theorem 1. The proof of the properties (S_1), which in Theorem 2 depended upon the partial order $\ll$, follow here from the corresponding arguments presented in Theorem 4, as does the proof of the partial converse.

Here the direct portion of the corollary also includes the two aforementioned theorems of Svec (Theorem 1 of [7] and Theorem 2 of [8]).

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